The Korean Communications in Statistics **Vol. 14** No. 2, 2007, pp. 345–353

Bayesian Estimation of the Nakagami-m Fading Parameter*

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Abstract

A Bayesian estimation of the Nakagami-m fading parameter is developed. Bayesian estimation is performed by Gibbs sampling, including adaptive rejection sampling. A Monte Carlo study shows that the Bayesian estimators proposed outperform any other estimators reported elsewhere in the sense of bias, variance, and root mean squared error.

Keywords: Nakagami-m fading parameter; Bayesian estimation; Gibbs sampling; adaptive rejection sampling.

1. Introduction

A useful statistical model for the envelope of the fading channel response is the Nakagami-m distribution of Nakagami (1960). Suzuki (1977) showed that the Nakagami-m distribution provides the best fit for data signals received in urban radio channels. The probability density function (*pdf*) for this distribution is given by

$$f_R(r) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m r^{2m-1} \exp\left(-\frac{mr^2}{\Omega}\right), \qquad r \geq 0,\ m \geq \frac{1}{2},\ \Omega \geq 0, \quad (1.1)$$

where the scale parameter Ω is defined as $E(R^2) = \Omega$ and the shape parameter m called the fading parameter is defined as $m = \left\{E(R^2)\right\}^2 / \mathrm{Var}(R^2)$.

* This study was financially supported by Chonnam National University.

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Extensive Monte Carlo simulations to assess the performance of several maximum likelihood (ML) based estimators and the method of moment(MM) based estimators for the fading parameter m have been shown in Adbi and Kaveh (2000), Cheng and Beaulieu (2001, 2002), and Zhang (2002).

Son and Oh (2006) proposed a Bayesian estimation of the two-parameter gamma distribution under the noninformative prior. The Bayesian estimator is obtained by Gibbs sampling of Gelfand and Smith (1990). The generation of the shape parameter in the Gibbs sampler is implemented using the adaptive rejection sampling (ARS) algorithm of Gilks and Wild (1992). Since a gamma random variable is the square of a Nakagami random variable, we can directly derive a Bayesian estimation of Nakagami-m fading parameter based on Son and Oh (2006). In this paper, we develop a Bayesian estimation of Nakagami-m fading parameter under a conjugate prior that reflects real world behaviour. Our Monte Carlo study shows that the Bayesian estimators proposed outperform all other estimators reported elsewhere with regards to bias, variance, and root mean square error.

2. Bayesian Estimation

Let R be a Nakagami random variable with a *pdf* (1.1). Then $Z = R^2$ is a gamma random variable with a *pdf*

$$f_Z(z \mid m, \beta) = \frac{1}{\Gamma(m)\beta^m}\, z^{m-1}\, e^{-z/\beta}, \qquad z \geq 0,\ m \geq \frac{1}{2},\ \beta = \frac{\Omega}{m} > 0.$$

If the random sample $\{R_1, R_2, \ldots, R_N\}$ is obtained from the Nakagami-m distribution, $\mathbf{Z} = \{Z_1, Z_2, \ldots, Z_N\}$, where $Z_j = R_j^2,\ j = 1, 2, \ldots, N$, is a random sample from the gamma distribution with parameters $m(\geq 1/2)$ and $\beta(> 0)$. The likelihood function, given an observed sample $\mathbf{z} = \{z_1, z_2, \ldots, z_N\}$, can be written as

$$L(m, \beta|\mathbf{z}) = \left\{\frac{1}{\Gamma(m)}\right\}^N \cdot \left\{\frac{1}{\beta^{Nm}}\right\} \cdot \left\{u^{m-1}\right\} \cdot \exp\left(-\frac{v}{\beta}\right), \tag{2.1}$$

where $u = \prod_{i=1}^N z_i$ and $v = \sum_{i=1}^N z_i$.

Bayesian inference is based on the posterior distribution obtained by combining the prior density of parameters and the likelihood function with the sample information observed. The conjugate prior assumption central in the Bayesian inference facilitates the derivation of the posterior distribution, since it makes the prior density and the likelihood function have the same functional form. Damsleth

(1975) developed a class of conjugate prior distributions for the unknown parameters of gamma distribution. A very general conjugate prior is obtained by the transformation of scale parameter from equation (3) of Miller (1980, p. 65) as follows

$$\pi(m,\beta)=\left\{\frac{1}{\Gamma(m)}\right\}^{N_1}\cdot\left\{\frac{1}{\beta^{N_2 m+1}}\right\}\cdot\left\{{u_1}^{m-1}\right\}\cdot\exp\left(-\frac{v_1}{\beta}\right), \tag{2.2}$$

where $N_1>0,\ N_2>0,\ u_1>0,\ v_1>0$ such that ${u_1}^{(1/N_1)}/(v_1/N_1)<1$. Selecting the values of hyperparameters, N_1, N_2, u_1 and v_1, leads to a wide variety of prior distribution and allows to take into account the prior beliefs on the studied data. The hyperparameter values may determined from historical data or expert's subjective knowledge. If the prior knowledge for the unknown parameters is very vague, $N_1=N_2=v_1=0$ and $u_1=1$ can be selected. This prior becomes the noninformative prior, $\pi(m,\beta)=1/\beta$.

The joint posterior *pdf* is proportional to the product of equation (2.1) and (2.2), that is,

$$p(m,\beta\mid\mathbf{z})\propto\left\{\frac{1}{\Gamma(m)}\right\}^{N_3}\cdot\left\{\frac{1}{\beta^{N_4 m+1}}\right\}\cdot\left\{{u_2}^{m-1}\right\}\cdot\exp\left(-\frac{v_2}{\beta}\right),$$

where $N_3=N+N_1,\ N_4=N+N_2,\ u_2=u\cdot u_1$ and $v_2=v+v_1$. Now, Bayesian parameter estimation is conducted using Gibbs sampling. The full conditional distributions to construct the Gibbs sampler are obtained as follows.

$$[\beta\mid m,\mathbf{z}]\sim IG\left(N_4 m,\ {v_2}^{-1}\right),$$

where $IG(a,b)$ denotes the inverse gamma distribution with parameters a and b, and its *pdf* is defined by $f_X(x\mid a,b)=\left(\Gamma(a)b^a\right)^{-1}x^{-(a+1)}\exp(-1/(bx)),\ x>0,\ a,b>0$, if $X\sim IG(a,b)$. In addition, the conditional *pdf* of m is

$$p(m\mid\beta,\mathbf{z})\propto g(m)=\left\{\frac{1}{\Gamma(m)}\right\}^{N_3}\cdot\left\{\frac{1}{\beta^{N_4 m}}\right\}\cdot{u_2}^{m}.$$

In the Gibbs sampler, while it is very straightforward to generate β from the inverse gamma distribution, to generate m from $p(m\mid\beta,\mathbf{z})$ can be difficult due to including the gamma function of m and an unknown normalizing constant.

When the probability density is defined only up to a normalizing constant, the general sampling method is a rejection sampling method in Ripley (1987). However, the rejection sampling method still needs the envelope function $g_u(m)$ such as $g(m) \leq g_u(m)$. The ARS method no longer requires a maximum of $g(m)$, only if the log-concavity of $p(m \mid \beta, \mathbf{z})$ is proven. Let $T_k = \{(m_0, m_1, \ldots, m_{k+1}) | \, m_0 \leq m_1 \leq \cdots \leq m_{k+1}\}$ be a set of abscissae in $D = \{m \mid m \geq 1/2\}$, where m_0 and m_{k+1} are the possibly infinite lower and upper limits of D. For $1 \leq i \leq j \leq k$ let $L_{ij}(m : T_k)$ denote the straight line through points $(m_i, \ln(g(m_i)))$ and $(m_j, \ln(g(m_j)))$ on a convex curve $\ln(g(m))$. Then a piecewise linear function $u_k(m)$ is defined by

$$u_k(m) = \min[L_{i-1,i}(m; T_k), L_{i+1,i+2}(m; T_k)], \quad m_i \leq m \leq m_{i+1}.$$

Thus due to the log-concavity of $p(m \mid \beta, \mathbf{z})$, $u_k(m)$ is an envelope for $\ln g(m)$, *i.e.* $g(m) \leq \exp\{u_k(m)\}$. Now the rejection sampling can be performed with the sampling distribution defined by

$$S_k(m) = \frac{\exp(u_k(m))}{\int_D \exp\{u_k(m)\}\, dm} \, .$$

If we use the derivation formula of the poly gamma function (Abramowitz and Stegun, 1972), we can easily show

$$\frac{\partial^2}{\partial m^2} \ln p(m \mid \beta, \mathbf{z}) = -N_3 \sum_{j=1}^{\infty} (m+j)^{-2} < 0 \, .$$

Hence, the conditional density, $p(m \mid \beta, \mathbf{z})$, of m is log-concave. Now, we can directly apply the ARS algorithm to generate m. The Gibbs sampler is constructed as follows.

Gibbs Sampling Algorithm

[STEP 1] *Initial Step* : Adopt $m^{(0)}$ as the initial value of m.

[STEP 2] *Repetition Step* : For $i = 0, 1, 2, \ldots, I$,

- Generate $\beta^{(i+1)}$ from $IG\left(N_4 m^{(i)}, v_2{}^{-1}\right)$.
- Perform the following ARS algorithm to generate $m^{(i+1)}$ from the conditional posterior distribution $p(m \mid \beta, \mathbf{z})$.

Adaptive Rejection Sampling Algorithm

[STEP 2.1] *Initial Step*

- Set k, the number of abscissae, and select T_k in D.

[STEP 2.2] *Sampling Step*

- Generate m^* from $S_k(m)$.
- Generate U from the uniform distribution $U(0,1)$.
- Set $m^{(i+1)} = m^*$ and go to STEP 2, if $U \leq g(m^*)/\exp\{u_k(m^*)\}$. Otherwise, go to STEP 2.3.

[STEP 2.3] *Updating Step*

- Include m^* in T_k to form T_{k+1}, and increment k.
- Return to STEP 2.2.

3. Numerical Results

We conducted Monte Carlo simulations with MATLAB (2002) to assess the performance of non Bayesian estimators and Bayesian estimators obtained using Gibbs sampling and ARS. The non Bayesian estimators compared with are the method of moment (MM) estimator, the noninteger-moment estimator (CB) of Cheng and Beaulieu (2002), the limiting member (WCB) in the class of MM estimator in Wines *et al.* (2003), as the MM based estimators, and the approximate maximum likelihood estimator (THOM) of Thom (1958) , another approximate estimator (GD) of Greenwood and Durand (1960), and a recursive solution (BS) to the ML equation of Bowman and Shenton (1988), as the ML based estimators. The estimators, MM, CB, THOM, GD, BS, are already well defined in Zhang (2002), respectively.

All of the experiments are performed for two sample sizes, $n = 30$ or $n = 100$. The 1,000 Nakagami random samples are generated for $\Omega = 1$, without loss of generality, and $m = 0.5j$, $j = 1, 2, \ldots, 30$. The mean of 100 m's generated by 100 Gibbs samples is used as a Bayesian estimator (BAYES) of m under the conjugate or the noninformative prior. The hyperparameters, v_1 and u_1, are determined from the arithmetic mean and the geometric mean of a prior sample in hypothetically historical experiment generated with a prior sample of size $N_1 = N_2 = 30$ or 100, respectively. The estimators, MM and THOM, are used as starting values required for the estimators, BS and BAYES, respectively. The performance of the estimator, CB, for large integer p values approaches that of the ML estimator. Therefore, we set $p = 100$ as a large value p of CB.

Figure 3.1 and Figure 3.3 show simulated bias, variance, and root mean square error(RMSE) of each estimator based on 1,000 repetitions of samples of size 30 and 100, respectively. It is shown from two figures that non Bayesian estimators have positive biases, whereas the Bayesian estimators are relatively unbiased.

When we inspect closely numerical results of Figure 3.1 and Figure 3.3, we can observe that the differences between all non Bayesian estimators are indistinguishable when compared with the Bayesian estimators, although the estimator, WCB, is better in the bias than the remainder except the Bayesian estimators. The MM estimator that gave the worst results among all of them was excluded from the figure. Since the simulation curves of three Bayesian estimators are very low compared with others, it is not easy to distinguish between them. Therefore, we plotted these three Bayesian estimators in one figure, Figure 3.2 for sample size $n = 30$ or Figure 3.4 for sample size $n = 100$. We note that for sample size $n = 30$, the scale of y-axis in RMSE of Figure 3.1 is from 0 to 4.890, while that in Figure 3.2 is from 0 to 0.217, and that for sample size $n = 100$, the scale of y-axis in RMSE of Figure 3.3 is from 0 to 1.873, while that in Figure 3.4 is from 0 to 0.138. As results, the Bayesian estimators surpass others, with regards to bias, variance, and RMSE. Also, Figure 3.2 and Figure 3.4 show that the Bayesian estimators under the conjugate prior with the greater prior sample size, $N_1 = 100$, are better than others in the sense of variance and RMSE. In conclusion, our simulation study shows that the number of repetitions in ARS algorithm is more as the shape parameter m is larger. So, the time required in the Bayesian estimation very increases. We guess the reason is the lackness of log-concavity due to the flatness of $p(m \mid \beta, \mathbf{z})$ for the very large shape parameter.

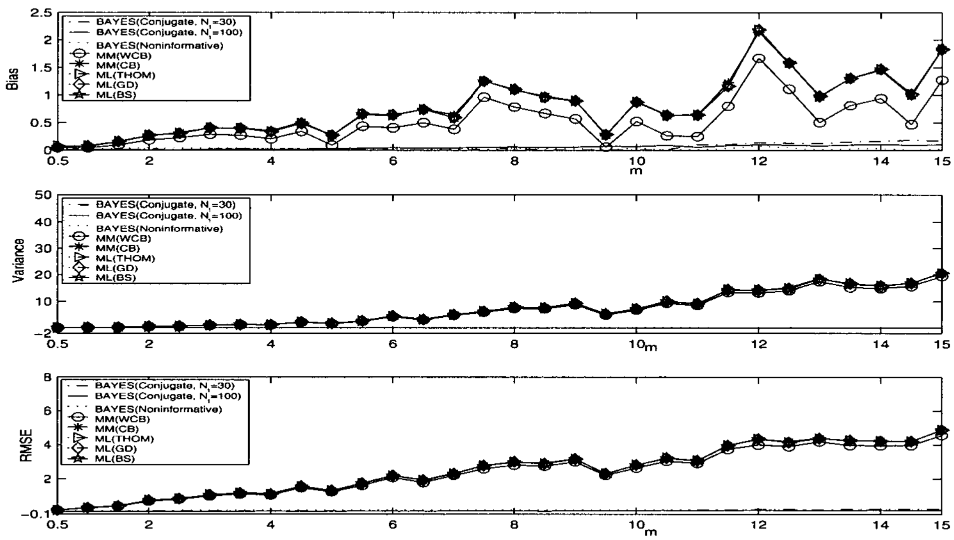

Figure 3.1: Simulated bias, variance and root mean square error (RMSE), $n = 30$

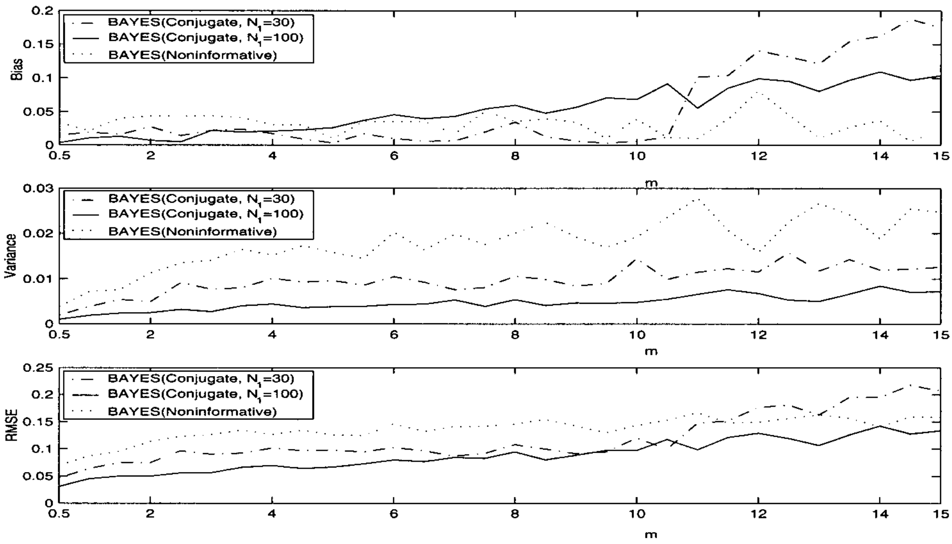

Figure 3.2: Simulated bias, variance and root mean square error (RMSE) of Bayesian estimators, $n = 30$

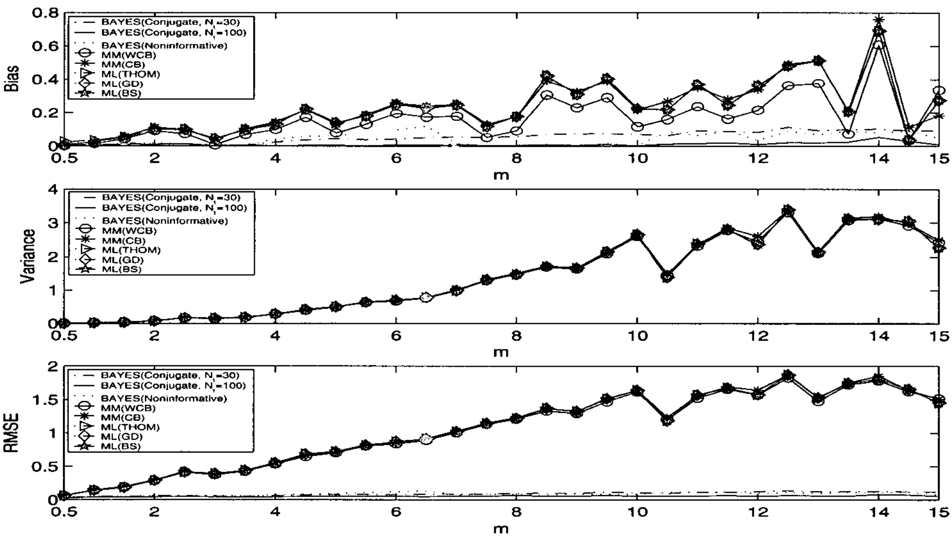

Figure 3.3: Simulated bias, variance and root mean square error (RMSE), $n = 100$

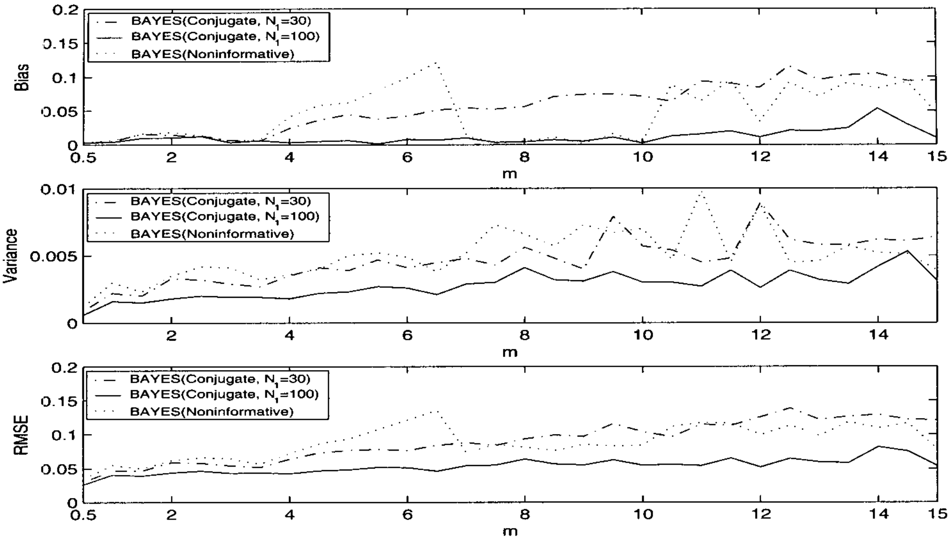

Figure 3.4: Simulated bias, variance and root mean square error (RMSE) of Bayesian estimators, $n = 100$

4. Conclusion

In this paper, the Bayesian estimation for the Nakagami-m parameter was proposed. The Bayesian estimation was performed by operating the Gibbs sampler with the adaptive rejection sampling algorithm. A numerical study shows that Bayesian estimators are superior over all other non Bayesian estimators. Though it is generally expected that Bayesian inference with noninformative prior and classical inference using likelihood theory agree, we think that the reason the Bayesian estimator with vague prior information outperforms the ML based estimators is that the likelihood function is not be exactly optimized and the ML based estimators are obtained by approximation. The precision of the Bayesian estimators proposed makes up for the computational complexity requiring computer programming skills, compared to non Bayesian estimators.

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[Received March 2007, Accepted May 2007]