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ON THE STABILITY OF A PEXIDERIZED MIXED TYPE QUADRATIC FUNCTIONAL EQUATION II

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Abstract. In this paper, we establish the generalized Hyers-Ulam-Rassias stability of the Pexider type quadratic equation $f_1(x+y+z)+f_2(x-y)+f_3(x-z)-f_4(x-y-z)-f_5(x+y)-f_6(x+z)=0$ and its general solution.

1. Introduction

In 1940, S. M. Ulam [23] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, D. H. Hyers [6] proved that if $f : V \to X$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in V$, where V and X are Banach spaces and ε is a given positive number, then there exists a unique additive mapping $T : V \to X$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in V$. In 1978, Th. M. Rassias [17] gave a significant generalization of the Hyers' result. Th. M. Rassias [20] during the 27th International Symposium on Functional Equations, that took place in Bielsko-Biala, Poland, in 1990, asked the question whether such a theorem can also be proved for a more general setting. Z. Gadja [4] following Th. M. Rassias's approach ([17]) gave an affirmative solution to the question. Recently, P. Găvruta [5] obtained a further generalization of Rassias' theorem, the so-called generalized Hyers-Ulam-Rassias stability (see also [7, 18-21]). Lee and Jun [15,16] also obtained the Hyers-Ulam-Rassias stability of the Pexider equation of $f(x+y) = g(x) + h(y)$.

A stability problem for the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

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was proved by F. Skof [22] for a function $f : V \to X$, where V is a normed space and X a Banach space. P. W. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain V is replaced by an Abelian group. S. Czerwik [3] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Jun and Lee [8-12, 14] proved the Hyers-Ulam-Rassias stability of the Pexider type quadratic equation

$$f(x+y) + g(x-y) = 2h(x) + 2k(y).$$

In [13], the authors investigated the following Pexider type quadratic functional equation

$$(1.1)\quad f_1(x+y+z)+f_2(x-y)+f_3(z-x)-f_4(x-y-z)-f_5(x+y)-f_6(x+z) = 0$$

in the punctured domain. In this paper, we establish the generalized Hyers-Ulam-Rassias stability for the equation (1.1) and its general solution in the whole domain. Throughout this paper, let V and X be a normed space and a Banach space, respectively. For convenience, we employ the operators as follows; for a function $\varphi : V^3 \to [0,\infty)$, let $\varphi', \varphi_e, \varphi_e' : V^3 \to [0,\infty)$, $M, M', M_e, M_e' : V \to [0,\infty)$ be functions defined by

$$\begin{aligned}
\varphi'(x,y,z) &:= \frac{1}{2}[\varphi(x,y,z) + \varphi(-x,y,z)],\\
\varphi_e(x,y,z) &:= \frac{1}{2}[\varphi(x,y,z) + \varphi(-x,-y,-z)],\\
\varphi_e'(x,y,z) &:= \frac{1}{4}[\varphi(x,y,z) + \varphi(-x,y,z) + \varphi(-x,-y,-z) + \varphi(x,-y,-z)],\\
M(x) &:= \varphi'(\frac{x}{2}, \frac{3x}{2}, -\frac{x}{2}) + 2\varphi'(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}) + \varphi'(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}),\\
M'(x) &:= \varphi'(\frac{x}{2}, \frac{x}{2}, -\frac{3x}{2}) + 2\varphi'(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}) + \varphi'(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}),\\
M_e(x) &:= \varphi_e'(\frac{x}{2}, \frac{3x}{2}, -\frac{x}{2}) + 2\varphi_e'(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}) + \varphi_e'(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}),\\
M_e'(x) &:= \varphi_e'(\frac{x}{2}, \frac{x}{2}, -\frac{3x}{2}) + 2\varphi_e'(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}) + \varphi_e'(\frac{x}{2}, \frac{x}{2}, \frac{x}{2})
\end{aligned}$$

for all $x, y, z \in V$.

2. Stability of the equation (1.1)

The authors [13] obtained the following lemma.

Lemma 2.1. *Let a be a positive real number. Let $\Phi : V \to [0,\infty)$ be a map such that*

$$(*)\qquad \tilde{\Phi}(x) := \sum_{l=0}^{\infty} \frac{1}{a^{l+1}} \Phi(2^l x) < \infty \text{ for all } x \in V$$

or

$$(**) \qquad \tilde{\Phi}(x) := \sum_{l=0}^{\infty} a^l \Phi(\frac{x}{2^{l+1}}) < \infty \text{ for all } x \in V.$$

Suppose that the function $f : V \to X$ *satisfies the inequality*

$$\|f(x) - \frac{f(2x)}{a}\| \leq \frac{\Phi(x)}{a}$$

for all $x \in V$. *Then there exists exactly one function* $F : V \to X$ *satisfying*

$$\|f(x) - F(x)\| \leq \tilde{\Phi}(x) \text{ for all } x \in V \quad \text{and} \quad aF(x) = F(2x) \text{ for all } x \in V.$$

Moreover, the function F *is given by*

$$F(x) = \begin{cases} \lim_{n\to\infty} \frac{f(2^n x)}{a^n} & \text{if } \Phi \text{ satisfies } (*), \\ \lim_{n\to\infty} a^n f(2^{-n}x) & \text{if } \Phi \text{ satisfies } (**) \end{cases}$$

for all $x \in V$.

We establish the stability results for the even functions in the following Theorem 2.1 and Theorem 2.2.

Theorem 2.1. *Let* $\varphi : V^3 \to [0, \infty)$ *be a function such that*

$$(a) \qquad \tilde{\varphi}(x, y, z) := \sum_{l=0}^{\infty} \frac{1}{4^{l+1}} \varphi(2^l x, 2^l y, 2^l z) < \infty$$

holds for all $x, y, z \in V$. *If the even functions* $f_1, f_2, f_3, f_4, f_5, f_6 : V \to X$ *satisfy the inequality*

$$(2.1) \qquad \begin{aligned} &\|f_1(x+y+z) + f_2(x-y) + f_3(x-z) \\ &- f_4(x-y-z) - f_5(x+y) - f_6(x+z)\| \leq \varphi(x, y, z) \end{aligned}$$

for all $x, y, z \in V$, *then there exists exactly one quadratic function* $Q : V \to X$ *satisfying the inequalities*

(2.2)

$$\|f_1(x) - f_1(0) - Q(x)\| \leq \frac{\varphi'(x,0,0) + \varphi'(0,x,-x)}{2} + \tilde{M}(x) + \varphi'(\frac{x}{2}, \frac{x}{2}, 0),$$

$$\|f_2(x) - f_2(0) - Q(x)\| \leq \tilde{M}(x) + \frac{\varphi'(x,0,0) + \varphi'(0,0,x)}{2},$$

$$\|f_3(x) - f_3(0) - Q(x)\| \leq \tilde{M}'(x) + \frac{\varphi'(x,0,0) + \varphi'(0,x,0)}{2},$$

$$\|f_4(x) - f_4(0) - Q(x)\| \leq \frac{\varphi'(x,0,0) + \varphi'(0,x,-x)}{2} + \tilde{M}(x) + \varphi'(\frac{x}{2}, \frac{x}{2}, 0),$$

$$\|f_5(x) - f_5(0) - Q(x)\| \leq \tilde{M}(x) + \frac{\varphi'(x,0,0) + \varphi'(0,0,x)}{2},$$

$$\|f_6(x) - f_6(0) - Q(x)\| \leq \tilde{M}'(x) + \frac{\varphi'(x,0,0) + \varphi'(0,x,0)}{2}$$

for all $x \in V$, *where*

$$\tilde{M}(x) := \sum_{l=0}^{\infty} \frac{1}{4^{l+1}} M(2^l x), \quad \tilde{M'}(x) := \sum_{l=0}^{\infty} \frac{1}{4^{l+1}} M'(2^l x)$$

for all $x \in V$. *Moreover, the function* Q *is given by*

$$Q(x) = \lim_{n\to\infty} \frac{f_k(2^n x)}{4^n}$$

for all $x \in V$ *and for* $k = 1, 2, 3, 4, 5, 6$.

Proof. Replace x by $-x$ in (2.1) to get

$$\begin{aligned} \|f_1(x-y-z) + f_2(x+y) + f_3(x+z) - f_4(x+y+z) \\ - f_5(x-y) - f_6(x-z)\| \le \varphi(-x, y, z) \end{aligned} \tag{2.3}$$

for all $x, y, z \in V$. It follows from (2.1) and (2.3) that

$$\begin{aligned} &\|F(x+y+z) + G(x-y) + H(x-z) - F(x-y-z) \\ &- G(x+y) - H(x+z)\| \le \varphi'(x, y, z) \end{aligned} \tag{2.4}$$

for all $x, y, z \in V$ and $H(0) = G(0) = H(0) = 0$, where the functions $F, G, H : V \to X$, are defined by

$$\begin{aligned} F(x) &:= \frac{1}{2}[f_1(x) + f_4(x) - f_1(0) - f_4(0)], \\ G(x) &:= \frac{1}{2}[f_2(x) + f_5(x) - f_2(0) - f_5(0)], \\ H(x) &:= \frac{1}{2}[f_3(x) + f_6(x) - f_3(0) - f_6(0)] \end{aligned}$$

for all $x, y, z \in V$. Replace y and z by $x/2$ and $-x/2$ in (2.4), to get

$$\|H(x) - G(x)\| \le \varphi'(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}) \tag{2.5}$$

for all $x \in V$. It follows from (2.4) and (2.5) that

$$\begin{aligned} &\|G(x) - \frac{G(2x)}{4}\| \\ &\le \frac{1}{4}\|F(\frac{3x}{2}) - F(\frac{x}{2}) - G(x) - H(x)\| + \frac{1}{2}\|H(x) - G(x)\| \\ &\quad + \frac{1}{4}\| - F(\frac{3x}{2}) + F(\frac{x}{2}) + G(2x) - G(x) - H(x)\| \\ &\le \frac{1}{4}\varphi'(\frac{x}{2}, \frac{3x}{2}, -\frac{x}{2}) + \frac{1}{2}\varphi'(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}) + \frac{1}{4}\varphi'(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}) = \frac{1}{4}M(x) \end{aligned}$$

for all $x \in V$. By Lemma 2.1, we can define

$$Q(x) := \lim_{n\to\infty} \frac{G(2^n x)}{4^n} \tag{2.6}$$

for all $x \in V$ and

$$\|G(x) - Q(x)\| \le \tilde{M}(x) \tag{2.7}$$

for all $x \in V$. By the similar method in obtaining the inequality (2.7), we get

$$\|H(x) - \lim_{n\to\infty} \frac{H(2^n x)}{4^n}\| \le \tilde{M}'(x) \tag{2.8}$$

for all $x \in V$. It follows from (2.5) and (2.6) that

$$Q(x) = \lim_{n\to\infty} \frac{H(2^n x)}{4^n} \tag{2.9}$$

for all $x \in V$. It follows from (2.4) and (2.7) that

$$\|F(x)-Q(x)\| \le \|G(x)-Q(x)\|+\|F(x)-G(x)\| \le \tilde{M}(x)+\varphi'(\frac{x}{2},\frac{x}{2},0) \tag{2.10}$$

for all $x \in V$. Replacing x by $2^n x$, dividing by 4^n in the above inequality and taking the limit in the resulted inequality as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{F(2^n x)}{4^n} = Q(x)$$

for all $x \in V$. Using (2.4), (2.6), (2.9) and the above equality, we obtain

$$Q(x+y+z)+Q(x-y)+Q(z-x)-Q(x-y-z)-Q(x+y)-Q(x+z) = 0 \tag{2.11}$$

for all $x, y, z \in V$. Replace x and z by $\frac{x}{2}$ in (2.11) to have

$$Q(x+y) + Q(\frac{x}{2} - y) - Q(-y) - Q(\frac{x}{2} + y) - Q(x) = 0 \tag{2.12}$$

for all $x, y \in V$. Replace x and z by $\frac{x}{2}$ and $\frac{-x}{2}$ in (2.11), we have

$$Q(y) + Q(\frac{x}{2} - y) + Q(x) - Q(x-y) - Q(\frac{x}{2} + y) = 0.$$

for all $x, y \in V$. Subtracting the above equality from (2.12) and using the evenness of Q, we lead to

$$Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = 0$$

for all $x, y \in V$.

On the other hand, it follows from (2.1) and (2.3) that

$$\begin{aligned} \|F'(x+y+z) + G'(x-y) + H'(x-z) &+ F'(x-y-z) \\ &+ G'(x+y) + H'(x+z)\| \le \varphi'(x,y,z) \end{aligned} \tag{2.13}$$

for all $x, y, z \in V$, where the functions $F', G', H' : V \to X$ are defined by

$$F'(x) = \frac{1}{2}[f_1(x) - f_4(x)], \quad G'(x) = \frac{1}{2}[f_2(x) - f_5(x)], \quad H'(x) = \frac{1}{2}[f_3(x) - f_6(x)]$$

for all $x, y, z \in V$. It follows from that (2.13) that

$$\|F'(x) + G'(x) + H'(x)\| \le \frac{\varphi'(x,0,0)}{2}$$
$$\|F'(x) + H'(x) + G'(0)\| \le \frac{\varphi'(0,0,x)}{2}$$
$$\|F'(x) + H'(0) + G'(x)\| \le \frac{\varphi'(0,x,0)}{2}$$
$$\|F'(0) + H'(x) + G'(x)\| \le \frac{\varphi'(0,x,-x)}{2}$$

for all $x, y, z \in V$. From the above inequalities, we have

$$\|G'(x) - G'(0)\| \le \frac{\varphi'(x,0,0) + \varphi'(0,0,x)}{2}$$
$$\|H'(x) - H'(0)\| \le \frac{\varphi'(x,0,0) + \varphi'(0,x,0)}{2}$$
$$\|F'(x) - F'(0)\| \le \frac{\varphi'(x,0,0) + \varphi'(0,x,-x)}{2}$$

for all $x \in V$. By using (2.7), (2.8), (2.9), (2.10), the above inequalities and the definition of F, G, H, F', G', H', we have

$$\begin{aligned}\|f_1(x) - f_1(0) - Q(x)\| &\le \|F(x) - Q(x)\| + \|F'(x) - F'(0)\| \\ &\le \frac{\varphi'(x,0,0) + \varphi'(0,x,-x)}{2} + \tilde{M}(x) + \varphi'(\frac{x}{2}, \frac{x}{2}, 0), \\ \|f_2(x) - f_2(0) - Q(x)\| &= \|G(x) + G'(x) - G'(0) - Q(x)\| \\ &\le \tilde{M}(x) + \frac{\varphi'(x,0,0) + \varphi'(0,0,x)}{2}\end{aligned}$$

for all $x \in V$. The rest of inequalities in (2.2) can be shown similarly. Also the uniqueness of Q follows from Lemma 2.1. □

Theorem 2.2. *Let $\varphi : V^3 \to [0, \infty)$ be a function such that*

$$(a') \qquad \tilde{\varphi}(x,y,z) := \sum_{l=0}^{\infty} 4^l \varphi(\frac{x}{2^{l+1}}, \frac{y}{2^{l+1}}, \frac{z}{2^{l+1}}) < \infty$$

holds for all $x, y, z \in V$. If the even functions $f_1, f_2, f_3, f_4, f_5, f_6 : V \to X$ satisfy the inequality (2.1) for all $x, y, z \in V$, then there exist exactly one quadratic function $Q : V \to X$ satisfying the inequalities (2.2) for all $x \in V$, where

$$\tilde{M}(x) := \sum_{l=0}^{\infty} 4^l M(\frac{x}{2^{l+1}}), \quad \tilde{M}'(x) := \sum_{l=0}^{\infty} 4^l M'(\frac{x}{2^{l+1}})$$

for all $x \in V$. Moreover, the function Q is given by

$$Q(x) = \lim_{n\to\infty} 4^n (f_k(2^{-n}x) - f_k(0))$$

for all $x \in V$ and for $k = 1, 2, 3, 4, 5, 6$.

Proof. The proof is similar to that of Theorem 2.1. □

Applying Theorem 2.1 and Theorem 2.2, we get the following corollary in the sense of Rassias inequality.

Corollary 2.1. *Let $p \neq 2$ be a positive real number and $\varepsilon > 0$. If the even functions $f_i : V \to X$, $i = 1, 2, \ldots, 6$, satisfy*

$$\|f_1(x+y+z) + f_2(x-y) + f_3(x-z) - f_4(x-y-z) - f_5(x+y) - f_6(x+z)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in V$, then there exist exactly one quadratic function $Q : V \to X$ satisfying

$$\|f_k(x) - f_k(0) - Q(x)\| \leq \begin{cases} [\frac{3}{2} + \frac{2}{2^p} + \frac{3^p+11}{2^p|2^p-4|}]\varepsilon \cdot \|x\|^p, & \text{if } k = 1, 4 \\ [1 + \frac{3^p+11}{2^p|2^p-4|}]\varepsilon \cdot \|x\|^p & \text{if } k = 2, 3, 5, 6 \end{cases}$$

for all $x \in V$. Moreover, the function Q is given by

$$Q(x) = \begin{cases} \lim_{n\to\infty} \frac{f_k(2^n x)}{4^n} & \text{if } p < 2, \\ \lim_{n\to\infty} 4^n(f_k(2^{-n}x) - f_k(0)) & \text{if } p > 2 \end{cases}$$

for all $x \in V$ and $k = 1, 2, 3, 4, 5, 6$.

We establish the following Theorem 2.3 and Theorem 2.4 for the odd functions.

Theorem 2.3. *Let $\varphi : V^3 \to [0, \infty)$ be a function such that*

$$(b) \qquad \hat{\varphi}(x, y, z) := \sum_{l=0}^{\infty} \frac{1}{2^{l+1}} \varphi(2^l x, 2^l y, 2^l z) < \infty$$

holds for all $x, y, z \in V$. If the odd functions $f_1, f_2, f_3, f_4, f_5, f_6 : V \to X$ satisfy

$$(2.14) \qquad \|f_1(x+y+z) + f_2(x-y) + f_3(x-z) - f_4(x-y-z) - f_5(x+y) - f_6(x+z)\| \leq \varphi(x, y, z)$$

for all $x, y, z \in V$, then there exist exactly three additive functions $A, A_1, A_2 : V \to X$ satisfying

$$(2.15) \qquad \begin{aligned} &\|f_1(x) - A(x) + A_1(x) + A_2(x)\| \leq L_1(x), \\ &\|f_2(x) - A(x) - A_1(x)\| \leq L_2(x), \\ &\|f_3(x) - A(x) - A_2(x)\| \leq L_3(x), \\ &\|f_4(x) - A(x) - A_1(x) - A_2(x)\| \leq L_1(x), \\ &\|f_5(x) - A(x) + A_1(x)\| \leq L_2(x), \\ &\|f_6(x) - A(x) + A_2(x)\| \leq L_3(x) \end{aligned}$$

for all $x \in V$, where

$$\hat{\varphi}'(x,y,z) := \sum_{i=0}^{\infty} \frac{\varphi'(2^i x, 2^i y, 2^i z)}{2^{i+1}},$$

$$L_1(x) := \frac{1}{2}[\hat{\varphi}'(0,x,0) + \hat{\varphi}'(0,0,x) + \hat{\varphi}'(2x,0,0) + \hat{\varphi}'(0,x,x))] + \hat{\varphi}'(x,x,-x),$$

$$L_2(x) := \min(\frac{1}{2}[\hat{\varphi}'(0,2x,0) + \hat{\varphi}'(0,x,-x)], \frac{1}{2}[\hat{\varphi}'(0,x,0) + \hat{\varphi}'(0,0,x) + \varphi'(0,x,0)]) + \frac{1}{2}[\hat{\varphi}'(2x,0,0) + \hat{\varphi}'(0,x,x))] + \hat{\varphi}'(x,0,x),$$

$$L_3(x) := \min(\frac{1}{2}[\hat{\varphi}'(0,0,2x) + \hat{\varphi}'(0,x,-x)], \frac{1}{2}[\hat{\varphi}'(0,x,0) + \hat{\varphi}'(0,0,x) + \varphi'(0,0,x)]) + \frac{1}{2}[\hat{\varphi}'(2x,0,0) + \hat{\varphi}'(0,x,x))] + \hat{\varphi}'(x,x,0).$$

Moreover, the function A, A_1, A_2 are given by

$$A(x) = \lim_{n\to\infty} \frac{f_1(2^n x) + f_4(2^n x)}{2^{n+1}},$$
$$A_1(x) = \lim_{n\to\infty} \frac{f_2(2^n x) - f_5(2^n x)}{2^{n+1}},$$
$$A_2(x) = \lim_{n\to\infty} \frac{f_3(2^n x) - f_5(2^n x)}{2^{n+1}}$$

for all $x \in V$.

Proof. Replace x by $-x$ in (2.14) to obtain

$$\|-f_1(x-y-z) - f_2(x+y) - f_3(x+z) + f_4(x+y+z) + f_5(x-y) + f_6(x-z)\| \le \varphi(-x,y,z) \tag{2.16}$$

for all $x, y, z \in V$. Let the functions $F, G, H : V \to X$ be defined by

$$F(x) := \frac{1}{2}[f_1(x) + f_4(x)], \quad G(x) := \frac{1}{2}[f_2(x) + f_5(x)], \quad H(x) := \frac{1}{2}[f_3(x) + f_6(x)]$$

for all $x, y, z \in V$. From (2.14) and (2.16), we get

$$\|F(x+y+z) + G(x-y) + H(x-z) - F(x-y-z) - G(x+y) - H(x+z)\| \le \varphi'(x,y,z) \tag{2.17}$$

for all $x, y, z \in V$. From (2.17), we have

$$\|F(x) - G(x)\| \le \frac{1}{2}\varphi'(0,x,0), \tag{2.18}$$

$$\|F(x) - H(x)\| \le \frac{1}{2}\varphi'(0,0,x) \tag{2.19}$$

for all $x \in V$. It follows from (2.17) that

$$\begin{aligned}\|F(x) - \frac{F(2x)}{2}\| &= \frac{1}{2}[\|G(x) - F(x)\| + \|F(2x) - G(x) - H(x)\| \\ &\quad + \|H(x) - F(x)\|] \\ &\le \frac{1}{4}\varphi'(0,x,0) + \frac{1}{4}\varphi'(0,x,x) + \frac{1}{4}\varphi'(0,0,x),\end{aligned}$$

$$\begin{aligned}\|G(x) - \frac{G(2x)}{2}\| &= \frac{1}{2}[\|G(2x) - F(2x)\| + \|F(2x) - G(x) - H(x)\| \\ &\quad + \|H(x) - G(x)\|] \\ &\le \frac{1}{4}\varphi'(0,2x,0) + \frac{1}{4}\varphi'(0,x,x) + \frac{1}{4}\varphi'(0,x,-x)\end{aligned}$$

and

$$\begin{aligned}\|H(x) - \frac{H(2x)}{2}\| &= \frac{1}{2}[\|H(2x) - F(2x)\| + \|F(2x) - G(x) - H(x)\| \\ &\quad + \| - H(x) + G(x)\|] \\ &\le \frac{1}{4}\varphi'(0,0,2x) + \frac{1}{4}\varphi'(0,x,x) + \frac{1}{4}\varphi'(0,x,-x)\end{aligned}$$

for all $x \in V$. Applying Lemma 2.1, we obtain

$$\|F(x) - \lim_{n\to\infty} \frac{F(2^n x)}{2^n}\| \le \frac{1}{2}\hat{\varphi}'(0,x,0) + \frac{1}{2}\hat{\varphi}'(0,x,x) + \frac{1}{2}\hat{\varphi}'(0,0,x), \tag{2.20}$$

$$\|G(x) - \lim_{n\to\infty} \frac{G(2^n x)}{2^n}\| \le \frac{1}{2}\hat{\varphi}'(0,2x,0) + \frac{1}{2}\hat{\varphi}'(0,x,x) + \frac{1}{2}\hat{\varphi}'(0,x,-x) \tag{2.21}$$

and

$$\|H(x) - \lim_{n\to\infty} \frac{H(2^n x)}{2^n}\| \le \frac{1}{2}\hat{\varphi}'(0,0,2x) + \frac{1}{2}\hat{\varphi}'(0,x,x) + \frac{1}{2}\hat{\varphi}'(0,x,-x) \tag{2.22}$$

for all $x \in V$. From (2.18) and (2.19), we easily obtain

$$\lim_{n\to\infty} \frac{F(2^n x)}{2^n} = \lim_{n\to\infty} \frac{G(2^n x)}{2^n} = \lim_{n\to\infty} \frac{H(2^n x)}{2^n}$$

for all $x \in V$ and we can define a function $A : V \to X$ by

$$A(x) = \lim_{n\to\infty} \frac{F(2^n x)}{2^n} = \lim_{n\to\infty} \frac{G(2^n x)}{2^n} = \lim_{n\to\infty} \frac{H(2^n x)}{2^n} \tag{2.23}$$

for all $x \in V$. On the other hand, it follows from (2.18),(2.19) and (2.20) that
(2.24)

$$\begin{aligned}\|G(x) - A(x)\| &\le \|G(x) - F(x)\| + \|F(x) - A(x)\| \\ &\le \frac{1}{2}\varphi'(0,x,0) + \frac{1}{2}\hat{\varphi}'(0,x,0) + \frac{1}{2}\hat{\varphi}'(0,x,x) + \frac{1}{2}\hat{\varphi}'(0,0,x)\end{aligned}$$

and
(2.25)
$$\begin{aligned}\|H(x)-A(x)\| &\le \|H(x)-F(x)\|+\|F(x)-A(x)\|\\ &\le \frac{1}{2}\varphi'(0,0,x)+\frac{1}{2}\hat{\varphi}'(0,x,0)+\frac{1}{2}\hat{\varphi}'(0,x,x)+\frac{1}{2}\hat{\varphi}'(0,0,x)\end{aligned}$$

for all $x \in V$. Replacing x by $2^n x$, dividing 2^n in (2.17) and taking the limit in the resulted inequality as $n \to \infty$, we obtain

$$A(x+y+z)+A(x-y)+A(x-z)-A(x-y-z)-A(x+y)-A(x+z)=0$$

for all $x, y, z \in V$. Replace x, z by $0, x$ in the above equality to have

$$2A(x+y)-2A(x)-2A(y)=0$$

for all $x, y \in V$. Hence, A is an additive function.

Let the functions $F', G', H' : V \to X$ be defined by

$$F'(x)=\frac{1}{2}[f_1(x)-f_4(x)],\quad G'(x)=\frac{1}{2}[f_2(x)-f_5(x)],\quad H'(x)=\frac{1}{2}[f_3(x)-f_6(x)]$$

for all $x \in V$. From (2.14) and (2.16), we have

(2.26)
$$\begin{aligned}&\|F'(x+y+z)+G'(x-y)+H'(x-z)\\ &\quad+F'(x-y-z)+G'(x+y)+H'(x+z)\| \le \varphi'(x,y,z)\end{aligned}$$

for all $x, y, z \in V$. It follows from (2.26) that

(2.27)
$$\begin{aligned}\|G'(x)-\frac{G'(2x)}{2}\| &\le \frac{1}{2}[\|F'(2x)+G'(2x)+H'(2x)\|\\ &\quad+\|-F'(2x)-2G'(x)-H'(2x)\|]\\ &\le \frac{1}{4}\varphi'(2x,0,0)+\frac{1}{2}\varphi'(x,0,x),\\ \|H'(x)-\frac{H'(2x)}{2}\| &\le \frac{1}{2}[\|F'(2x)+G'(2x)+H'(2x)\|\\ &\quad+\|-F'(2x)-G'(2x)-2H'(x)\|]\\ &\le \frac{1}{4}\varphi'(2x,0,0)+\frac{1}{2}\varphi'(x,x,0),\\ \|F'(x)-\frac{F'(2x)}{2}\| &\le \frac{1}{2}[\|F'(2x)+G'(2x)+H'(2x)\|\\ &\quad+\|-2F'(x)-G'(2x)-H'(2x)\|]\\ &\le \frac{1}{4}\varphi'(2x,0,0)+\frac{1}{2}\varphi'(x,x,-x)\end{aligned}$$

for all $x \in V$. Applying Lemma 2.1, we obtain an odd functions $A_1, A_2, A_3 : V \to X$ satisfying

(2.28)
$$\|G'(x)-A_1(x)\| \le \frac{1}{2}\hat{\varphi}'(2x,0,0)+\hat{\varphi}'(x,0,x),$$

$$\|H'(x) - A_2(x)\| \leq \frac{1}{2}\hat{\varphi}'(2x, 0, 0) + \hat{\varphi}'(x, x, 0) \tag{2.29}$$

for all $x \in V$, where

$$A_1(x) := \lim_{n\to\infty} \frac{G'(2^n x)}{2^n}, \tag{2.30}$$

$$A_2(x) := \lim_{n\to\infty} \frac{H'(2^n x)}{2^n} \tag{2.31}$$

for all $x \in V$. Replacing x, y, z by $2^n x, 0, 0$ and dividing by 2^{n+1} in (2.26), we obtain

$$\|\frac{F'(2^n x) + G'(2^n x) + H'(2^n x)}{2^n}\| \leq \frac{\varphi'(2^n x, 0, 0)}{2^{n+1}}$$

for all $x \in V$. Taking the limit in the above inequality as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{F'(2^n x)}{2^n} = -A_1(x) - A_2(x) \tag{2.32}$$

for all $x \in V$. By Lemma 2.1, (2.27) and (2.32), we have

$$\|F'(x) + A_1(x) + A_2(x)\| \leq \frac{1}{2}\hat{\varphi}'(2x, 0, 0) + \hat{\varphi}'(x, x, -x) \tag{2.33}$$

for all $x \in V$. From (2.30), (2.31) and (2.32), we have

(2.34)
$$\begin{aligned} -A_1(x+y+z) - A_2(x+y+z) + A_1(x-y) + A_2(x-z) - A_1(x-y-z) \\ - A_2(x-y-z) + A_1(x+y) + A_2(x+z) = 0 \end{aligned}$$

for all $x, y, z \in V$. Replace x, y, z by $\frac{x+y}{2}, 0, \frac{x-y}{2}$ in (2.34) to get

$$-A_1(x) + A_1(x+y) - A_1(y) = 0$$

for all $x, y \in V$. Replace x, y, z by $\frac{x+y}{2}, \frac{x-y}{2}, 0$ in (2.34) to get

$$-A_2(x) + A_2(x+y) - A_2(y) = 0$$

for all $x, y \in V$. Hence A_1 and A_2 are additive. From (2.20), (2.23), (2.33) and the definition of F, F', we have

$$\begin{aligned} \|f_1(x) - A(x) + A_1(x) + A_2(x)\| \leq& \|F(x) - A(x)\| + \|F'(x) + A_1(x) + A_2(x)\| \\ \leq& \frac{1}{2}\hat{\varphi}'(0, x, 0) + \frac{1}{2}\hat{\varphi}'(0, x, x) + \frac{1}{2}\hat{\varphi}'(0, 0, x) \\ &+ \frac{1}{2}\hat{\varphi}'(2x, 0, 0) + \hat{\varphi}'(x, x, -x) \end{aligned}$$

for all $x \in V$. The rest of inequalities in (2.15) can be shown by the similar method. □

Theorem 2.4. *Let $\varphi : V^3 \to [0, \infty)$ be a function such that*

$$\hat{\varphi}(x, y, z) := \sum_{i=0}^{\infty} 2^l \varphi(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}, \frac{z}{2^{i+1}}) < \infty \tag{b'}$$

holds for all $x, y, z \in V$. If the odd functions $f_1, f_2, f_3, f_4, f_5, f_6 : V \to X$ satisfy the inequalities (2.14) for all $x, y, z \in V$, then there exist exactly three additive functions $A, A_1, A_2 : V \to X$ satisfying the inequalities (2.15) for all $x \in V$, where

$$\hat{\varphi}'(x, y, z) := \sum_{i=0}^{\infty} 2^i \varphi'(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}, \frac{z}{2^{i+1}}).$$

Moreover, the function A, A_1, A_2 are given by

$$\begin{aligned} A(x) &= \lim_{n\to\infty} 2^{n-1}(f_k(2^{-n}x) + f_{k+3}(2^{-n}x)), \\ A_1(x) &= \lim_{n\to\infty} 2^{n-1}(f_2(2^{-n}x) - f_5(2^{-n}x)), \\ A_2(x) &= \lim_{n\to\infty} 2^{n-1}(f_3(2^{-n}x) - f_6(2^{-n}x)) \end{aligned}$$

for all $x \in V$ and for $k = 1, 2, 3$.

Proof. The proof is similar to that of Theorem 2.3. □

Applying Theorem 2.3 and Theorem 2.4, we get the following corollary in the sense of Rassias inequality.

Corollary 2.2. *Let $p \neq 1$ be a nonnegative real number and $\varepsilon > 0$. If the odd functions $f_k : V \to X$, $k = 1, 2, \ldots, 6$, satisfy*

$$\begin{aligned} &\|f_1(x+y+z) + f_2(x-y) + f_3(x-z) - f_4(x-y-z) \\ &\quad - f_5(x+y) - f_6(x+z)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned}$$

for all $x, y, z \in V$, then there exist exactly three additive functions $A, A_1, A_2 : V \to X$ satisfying

$$\begin{aligned} \|f_1(x) - A(x) + A_1(x) + A_2(x)\| &\leq \frac{10 + 2^p}{2|2^p - 2|}\varepsilon \cdot \|x\|^p, \\ \|f_2(x) - A(x) - A_1(x)\| &\leq (\frac{8 + 2^p}{2|2^p - 2|} + \frac{1}{2})\varepsilon \cdot \|x\|^p, \\ \|f_3(x) - A(x) - A_2(x)\| &\leq (\frac{8 + 2^p}{2|2^p - 2|} + \frac{1}{2})\varepsilon \cdot \|x\|^p, \\ \|f_4(x) - A(x) - A_1(x) - A_2(x)\| &\leq \frac{10 + 2^p}{2|2^p - 2|}\varepsilon \cdot \|x\|^p, \\ \|f_5(x) - A(x) + A_1(x)\| &\leq (\frac{8 + 2^p}{2|2^p - 2|} + \frac{1}{2})\varepsilon \cdot \|x\|^p, \\ \|f_6(x) - A(x) + A_2(x)\| &\leq (\frac{8 + 2^p}{2|2^p - 2|} + \frac{1}{2})\varepsilon \cdot \|x\|^p \end{aligned}$$

for all $x \in V$. *Moreover, the functions* A, A_1, A_2 *are given by*

$$A(x) = \begin{cases} \lim_{n\to\infty} \frac{f_1(2^n x)+f_4(2^n x)-f_1(-2^n x)-f_4(-2^n x)}{2^{n+2}} & \text{if } p < 1, \\ \lim_{n\to\infty} 2^{n-2}(f_1(\frac{x}{2^n}) + f_4(\frac{x}{2^n}) - f_1(-\frac{x}{2^n}) - f_4(-\frac{x}{2^n})) & \text{if } p > 1, \end{cases}$$

$$A_1(x) = \begin{cases} \lim_{n\to\infty} \frac{f_2(2^n x)-f_5(2^n x)-f_2(-2^n x)+f_5(-2^n x)}{2^{n+2}} & \text{if } p < 1, \\ \lim_{n\to\infty} 2^{n-2}(f_2(\frac{x}{2^n}) - f_5(\frac{x}{2^n}) - f_2(-\frac{x}{2^n}) + f_5(-\frac{x}{2^n})) & \text{if } p > 1, \end{cases}$$

$$A_2(x) = \begin{cases} \lim_{n\to\infty} \frac{f_3(2^n x)-f_6(2^n x)-f_3(-2^n x)+f_6(-2^n x)}{2^{n+2}} & \text{if } p < 1, \\ \lim_{n\to\infty} 2^{n-2}(f_3(\frac{x}{2^n}) - f_6(\frac{x}{2^n}) - f_3(-\frac{x}{2^n}) + f_6(-\frac{x}{2^n})) & \text{if } p > 1 \end{cases}$$

for all $x \in V$.

We establish the following theorem from Theorem 2.1 and Theorem 2.3.

Theorem 2.5. *Let* $\varphi : V^3 \to [0, \infty)$ *be a function that satisfies the condition* (*a*) *and* (*b*). *Suppose the functions* $f_k : V \to X$, $k = 1, 2, \ldots, 6$ *satisfy the inequality*

$$\begin{aligned} &\|f_1(x+y+z) + f_2(x-y) + f_3(x-z) - f_4(x-y-z) \\ &- f_5(x+y) - f_6(x+z)\| \le \varphi(x, y, z) \end{aligned} \tag{2.34}$$

for all $x, y, z \in V$. *Then there exist exactly one quadratic function* $Q : V \to X$ *and exactly three additive functions* $A, A_1, A_2 : V \to X$ *satisfying*

$$\begin{aligned} &\|f_1(x) - f_1(0) - Q(x) - A(x) + A_1(x) + A_2(x)\| \le K_1(x), \\ &\|f_2(x) - f_2(0) - Q(x) - A(x) - A_1(x)\| \le K_2(x), \\ &\|f_3(x) - f_3(0) - Q(x) - A(x) - A_2(x)\| \le K_3(x), \\ &\|f_4(x) - f_4(0) - Q(x) - A(x) - A_1(x) - A_2(x)\| \le K_1(x), \\ &\|f_5(x) - f_5(0) - Q(x) - A(x) + A_1(x)\| \le K_2(x), \\ &\|f_6(x) - f_6(0) - Q(x) - A(x) + A_2(x)\| \le K_3(x) \end{aligned} \tag{2.35}$$

for all $x \in V$, *where* $K_1(x), K_2(x), K_3(x), \tilde{\varphi}'_e, \hat{\varphi}'_e$ *are given by*

$$\begin{aligned} K_1(x) := {} & \tilde{\varphi}'_e(\frac{x}{2}, \frac{3x}{2}, -\frac{x}{2}) + 2\tilde{\varphi}'_e(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}) + \tilde{\varphi}'_e(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}), \\ & + \frac{1}{2}[\hat{\varphi}'_e(0, x, 0) + \hat{\varphi}'_e(0, 0, x) + \hat{\varphi}'_e(2x, 0, 0) + \hat{\varphi}'_e(0, x, x))] \\ & + \hat{\varphi}'_e(x, x, -x) + \varphi'_e(\frac{x}{2}, \frac{x}{2}, 0) + \frac{\varphi'_e(x, 0, 0) + \varphi'_e(0, x, -x)}{2} \\ K_2(x) := {} & \tilde{\varphi}'_e(\frac{x}{2}, \frac{3x}{2}, -\frac{x}{2}) + 2\tilde{\varphi}'_e(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}) + \tilde{\varphi}'_e(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}), \end{aligned}$$

$$+\min(\frac{1}{2}[\hat{\varphi}'_e(0,2x,0)+\hat{\varphi}'_e(0,x,-x)],$$
$$\frac{1}{2}[\hat{\varphi}'_e(0,x,0)+\hat{\varphi}'_e(0,0,x)+\varphi'_e(0,x,0)])$$
$$+\frac{1}{2}[\hat{\varphi}'_e(2x,0,0)+\hat{\varphi}'_e(0,x,x))]$$
$$+\hat{\varphi}'_e(x,0,x)+\frac{\varphi'_e(x,0,0)+\varphi'_e(0,0,x)}{2},$$
$$K_3(x):=\tilde{\varphi}'_e(\frac{x}{2},\frac{x}{2},-\frac{3x}{2})+2\tilde{\varphi}'_e(\frac{x}{2},\frac{x}{2},-\frac{x}{2})+\tilde{\varphi}'_e(\frac{x}{2},\frac{x}{2},\frac{x}{2})$$
$$+\min(\frac{1}{2}[\hat{\varphi}'_e(0,0,2x)+\hat{\varphi}'_e(0,x,-x)],$$
$$\frac{1}{2}[\hat{\varphi}'_e(0,x,0)+\hat{\varphi}'_e(0,0,x)+\varphi'_e(0,0,x)])$$
$$+\frac{1}{2}[\hat{\varphi}'_e(2x,0,0)+\hat{\varphi}'_e(0,x,x))]+\hat{\varphi}'_e(x,x,0)$$
$$+\frac{\varphi'_e(x,0,0)+\varphi'_e(0,x,0)}{2},$$
$$\tilde{\varphi}'_e(x,y,z):=\sum_{i=0}^{\infty}\frac{\varphi'_e(2^ix,2^iy,2^iz)}{4^{i+1}},$$
$$\hat{\varphi}'_e(x,y,z):=\sum_{i=0}^{\infty}\frac{\varphi'_e(2^ix,2^iy,2^iz)}{2^{i+1}}$$

for all $x,y,z\in V$. Moreover, the functions Q,A,A_1,A_2 are given by

$$Q(x)=\lim_{n\to\infty}\frac{f_k(2^nx)+f_k(-2^nx)}{2\cdot 4^n},$$
$$A(x)=\lim_{n\to\infty}\frac{f_1(2^nx)-f_1(-2^nx)+f_4(2^nx)-f_4(-2^nx)}{2^{n+2}},$$
$$A_1(x)=\lim_{n\to\infty}\frac{f_2(2^nx)-f_2(-2^nx)-f_5(2^nx)+f_5(-2^nx)}{2^{n+2}},$$
$$A_2(x)=\lim_{n\to\infty}\frac{f_3(2^nx)-f_3(-2^nx)-f_6(2^nx)+f_6(-2^nx)}{2^{n+2}}$$

for all $x\in V$ and $k=1,2,3,4,5,6$.

Proof. From (2.34), we obtain

$$\|f_1(-x-y-z)+f_2(-x+y)+f_3(-x+z)-f_4(-x+y+z)$$
$$-f_5(-x-y)-f_6(-x-z)\|\le\varphi(-x,-y,-z)$$

for all $x,y,z\in V$. From (2.34) and the above inequality, one gets

$$\|f_{1e}(x+y+z)+f_{2e}(x-y)+f_{3e}(x-z)$$
$$-f_{4e}(x-y-z)-f_{5e}(x+y)-f_{6e}(x+z)\|$$

$$\le \frac{\varphi(x,y,z)+\varphi(-x,-y,-z)}{2}$$

$$\|f_{1o}(x+y+z)+f_{2o}(x-y)+f_{3o}(x-z)-f_{4o}(x-y-z)$$
$$-f_{5o}(x+y)-f_{6o}(x+z)\|$$
$$\le \frac{\varphi(x,y,z)+\varphi(-x,-y,-z)}{2}$$

for all $x,y,z \in V$, where $f_{ke}(x) = \frac{f_k(x)+f_k(-x)}{2}$, $f_{ko}(x) = \frac{f_k(x)-f_k(-x)}{2}$ for all $x \in V$, $k = 1,2,3,4,5,6$. Since f_{ke} is an even function, f_{ko} is an odd function and $f_k = f_{ke} + f_{ko}$, we can apply Theorem 2.1 and Theorem 2.3 to get the desired result. □

We establish the following theorem from Theorem 2.1 and Theorem 2.4.

Theorem 2.6. *Let $\varphi : V^3 \to [0,\infty)$ be a function that satisfies the condition (a) and (b′). If the functions $f_1, f_2, f_3, f_4, f_5, f_6 : V \to X$ satisfy the inequality (2.34) for all $x,y,z \in V$, then there exist exactly one quadratic function $Q : V \to X$ and exactly three additive functions $A, A_1, A_2 : V \to X$ satisfying (2.35) for all $x \in V$, where $\tilde{\varphi}'_e, \hat{\varphi}'_e$ are given by*

$$\tilde{\varphi}'_e(x,y,z) := \sum_{i=0}^{\infty} \frac{\varphi'_e(2^i x, 2^i y, 2^i z)}{4^{i+1}}, \quad \hat{\varphi}'_e(x,y,z) := \sum_{i=0}^{\infty} 2^i \varphi'_e(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}, \frac{z}{2^{i+1}})$$

for all $x,y,z \in V$. Moreover, the function Q is given by the equality in Theorem 2.5 and A, A_1, A_2 are given by

$$A(x) = \lim_{n\to\infty} 2^{n-2}(f_1(\frac{x}{2^n}) + f_4(\frac{x}{2^n}) - f_1(-\frac{x}{2^n}) - f_4(-\frac{x}{2^n}))$$
$$A_1(x) = \lim_{n\to\infty} 2^{n-2}(f_2(\frac{x}{2^n}) - f_5(\frac{x}{2^n}) - f_2(-\frac{x}{2^n}) + f_5(-\frac{x}{2^n}))$$
$$A_2(x) = \lim_{n\to\infty} 2^{n-2}(f_3(\frac{x}{2^n}) - f_6(\frac{x}{2^n}) - f_3(-\frac{x}{2^n}) + f_6(-\frac{x}{2^n}))$$

for all $x \in V$.

We establish the following theorem from Theorem 2.2 and Theorem 2.4.

Theorem 2.7. *Let $\varphi : V^3 \to [0,\infty)$ be a function that satisfies the condition (a′) and (b′). If the functions $f_1, f_2, f_3, f_4, f_5, f_6 : V \to X$ satisfy the inequality (2.34) for all $x,y,z \in V$, then there exist exactly one quadratic function $Q : V \to X$ and exactly three additive functions $A, A_1, A_2 : V \to X$ satisfying (2.35) for all $x \in V$, where $\tilde{\varphi}'_e, \hat{\varphi}'_e$ are given by*

$$\tilde{\varphi}'_e(x,y,z) := \sum_{i=0}^{\infty} 4^i \varphi'_e(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}, \frac{z}{2^{i+1}}),$$
$$\hat{\varphi}'_e(x,y,z) := \sum_{i=0}^{\infty} 2^i \varphi'_e(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}, \frac{z}{2^{i+1}})$$

Moreover, the function Q is given by

$$Q(x) = \lim_{n\to\infty} 4^n \left(\frac{f_k(\frac{x}{2^n}) + f_k(-\frac{x}{2^n})}{2} - f_k(0)\right)$$

for $k = 1,2,3,4,5,6$ and A, A_1, A_2 are given by the equalities in Theorem 2.6.

Corollary 2.3. *Let $p \neq 1,2$ be a positive real number and $\varepsilon > 0$. Suppose that the functions $f_k : V \to X$, $k = 1,2,\ldots,6$, satisfy*

$$\begin{aligned}&\|f_1(x+y+z) + f_2(x-y) + f_3(x-z) - f_4(x-y-z)\\ &\quad - f_5(x+y) - f_6(x+z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)\end{aligned}$$

for all $x, y, z \in V$. Then there exist exactly one quadratic function $Q : V \to X$ and three additive functions $A, A_1, A_2 : V \to X$ satisfying

$$\begin{aligned}
&\|f_1(x) - f_1(0) - Q(x) - A(x) + A_1(x) + A_2(x)\|\\
&\qquad \le \left[\frac{3^p+11}{2^p|2^p-4|} + \left(\frac{2}{2^p} + \frac{3}{2}\right) + \frac{10+2^p}{2|2^p-2|}\right]\varepsilon \cdot \|x\|^p,\\
&\|f_2(x) - f_2(0) - Q(x) - A(x) - A_1(x)\| \le \left[\frac{(3^p+11)}{2^p|2^p-4|} + \frac{3}{2} + \frac{8+2^p}{2|2^p-2|}\right]\varepsilon \cdot \|x\|^p,\\
&\|f_3(x) - f_3(0) - Q(x) - A(x) - A_2(x)\| \le \left[\frac{(3^p+11)}{2^p|2^p-4|} + \frac{3}{2} + \frac{8+2^p}{2|2^p-2|}\right]\varepsilon \cdot \|x\|^p,\\
&\|f_4(x) - f_4(0) - Q(x) - A(x) - A_1(x) - A_2(x)\|\\
&\qquad \le \left[\frac{3^p+11}{2^p|2^p-4|} + \left(\frac{2}{2^p} + \frac{3}{2}\right) + \frac{10+2^p}{2|2^p-2|}\right]\varepsilon \cdot \|x\|^p,\\
&\|f_5(x) - f_5(0) - Q(x) - A(x) + A_1(x)\| \le \left[\frac{(3^p+11)}{2^p|2^p-4|} + \frac{3}{2} + \frac{8+2^p}{2|2^p-2|}\right]\varepsilon \cdot \|x\|^p,\\
&\|f_6(x) - f_6(0) - Q(x) - A(x) + A_2(x)\| \le \left[\frac{(3^p+11)}{2^p|2^p-4|} + \frac{3}{2} + \frac{8+2^p}{2|2^p-2|}\right]\varepsilon \cdot \|x\|^p
\end{aligned}$$

for all $x \in V$. Moreover, the function Q is given by

$$Q(x) = \begin{cases} \lim_{n\to\infty} \frac{f_k(2^n x) + f_k(-2^n x)}{2\cdot 4^n} & \text{if } p < 2,\\ \lim_{n\to\infty} 4^n\left(\frac{f_k(\frac{x}{2^n}) + f_k(-\frac{x}{2^n})}{2} - f_k(0)\right) & \text{if } p > 2\end{cases}$$

for $k = 1,2,3,4,5,6$ and the functions A, A_1, A_2 are given by

$$A(x) = \begin{cases} \lim_{n\to\infty} \frac{f_1(2^n x) + f_4(2^n x) - f_1(-2^n x) - f_4(-2^n x)}{2^{n+2}} & \text{if } p < 1,\\ \lim_{n\to\infty} 2^{n-2}\left(f_1(\frac{x}{2^n}) + f_4(\frac{x}{2^n}) - f_1(-\frac{x}{2^n}) - f_4(-\frac{x}{2^n})\right) & \text{if } p > 1,\end{cases}$$

$$A_1(x) = \begin{cases} \lim_{n\to\infty} \frac{f_2(2^n x) - f_5(2^n x) - f_2(-2^n x) + f_5(-2^n x)}{2^{n+2}} & \text{if } p < 1,\\ \lim_{n\to\infty} 2^{n-2}\left(f_2(\frac{x}{2^n}) - f_5(\frac{x}{2^n}) - f_2(-\frac{x}{2^n}) + f_5(-\frac{x}{2^n})\right) & \text{if } p > 1,\end{cases}$$

$$A_2(x) = \begin{cases} \lim_{n\to\infty} \frac{f_3(2^n x) - f_6(2^n x) - f_3(-2^n x) + f_6(-2^n x)}{2^{n+2}} & \text{if } p < 1,\\ \lim_{n\to\infty} 2^{n-2}\left(f_3(\frac{x}{2^n}) - f_6(\frac{x}{2^n}) - f_3(-\frac{x}{2^n}) + f_6(-\frac{x}{2^n})\right) & \text{if } p > 1\end{cases}$$

for all $x \in V$.

Corollary 2.4. *Let* $\varepsilon > 0$ *be a fixed real number. Suppose that the functions* $f_k : V \to X$, $k = 1, 2, \ldots, 6$, *satisfy*

$$\|f_1(x+y+z)+f_2(x-y)+f_3(x-z)-f_4(x-y-z)-f_5(x+y)-f_6(x+z)\| \le \varepsilon$$

for all $x, y, z \in V$. *Then there exist exactly one quadratic function* $Q : V \to X$ *and three additive functions* $A, A_1, A_2 : V \to X$ *satisfying*

$$\begin{aligned}
&\|f_1(x)-f_1(0) - Q(x) - A(x) + A_1(x) + A_2(x)\| \le \frac{19}{3}\varepsilon,\\
&\|f_2(x)-f_2(0) - Q(x) - A(x) - A_1(x)\| \le \frac{16}{3}\varepsilon,\\
&\|f_3(x)-f_3(0) - Q(x) - A(x) + A_1(x)\| \le \frac{16}{3}\varepsilon,\\
&\|f_4(x)-f_4(0) - Q(x) - A(x) - A_1(x) - A_2(x)\| \le \frac{19}{3}\varepsilon,\\
&\|f_5(x)-f_5(0) - Q(x) - A(x) - A_2(x)\| \le \frac{16}{3}\varepsilon,\\
&\|f_6(x)-f_6(0) - Q(x) - A(x) + A_2(x)\| \le \frac{16}{3}\varepsilon
\end{aligned}$$

for all $x \in V$.

Now we obtain the general solution of the equation (1.1).

Corollary 2.5. *Suppose that the functions* $f_k : V \to X$, $k = 1, 2, \ldots, 6$, *satisfy*

$$f_1(x+y+z)+f_2(x-y)+f_3(x-z)-f_4(x-y-z)-f_5(x+y)-f_6(x+z) = 0$$

for all $x, y, z \in V$.

Then there exist exactly one quadratic function $Q : V \to X$ *and three additive functions* $A, A_1, A_2 : V \to X$ *satisfying*

$$\begin{aligned}
f_1(x) &= Q(x) + A(x) - A_1(x) - A_2(x) + f_1(0),\\
f_2(x) &= Q(x) + A(x) + A_1(x) + f_2(0),\\
f_3(x) &= Q(x) + A(x) + A_2(x) + f_3(0),\\
f_4(x) &= Q(x) + A(x) + A_1(x) + A_2(x) + f_4(0),\\
f_5(x) &= Q(x) + A(x) - A_1(x) + f_5(0),\\
f_6(x) &= Q(x) + A(x) - A_2(x) + f_6(0)
\end{aligned}$$

for all $x \in V$. *Moreover, the functions* Q, A, A_1, A_2 *are given by*

$$Q(x) = \frac{f_k(x) + f_k(-x)}{2} - f_k(0),$$

$$A(x) = \frac{f_1(x) + f_4(x) - f_1(-x) - f_4(-x)}{4},$$
$$A_1(x) = \frac{f_2(x) - f_5(x) - f_2(-x) + f_5(-x)}{4},$$
$$A_2(x) = \frac{f_3(x) - f_6(x) - f_3(-x) + f_6(-x)}{4}$$

for $x \in V$ *and for* $k = 1, 2, 3, 4, 5, 6$.

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