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TEMPORAL AND SPATIAL DECAY RATES OF NAVIER-STOKES SOLUTIONS IN EXTERIOR DOMAINS

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ABSTRACT. We obtain spatial-temporal decay rates of weak solutions of incompressible flows in exterior domains. When a domain has a boundary, the pressure term yields difficulties since we do not have enough information on the pressure term near the boundary. For our calculations we provide an idea which does not require any pressure information. We also estimated the spatial and temporal asymptotic behavior for strong solutions.

1. Introduction

Let Ω be the exterior of a simply connected set with C^2 boundary in $\mathbb{R}^3$ which contains the origin and is contained in a unit ball. Let $\mathbf{u}$ and p be the velocity and the pressure, respectively, of the incompressible fluid in the exterior domain. We consider the Navier-Stokes equations described in Ω:

$$\begin{array}{c}\frac{\partial}{\partial t}\mathbf{u} - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0 \\ \nabla\cdot\mathbf{u} = 0\end{array} \quad \text{in } (x,t) \in \Omega \times (0,\infty) \tag{1.1}$$

with no slip boundary condition

$$\mathbf{u}(x,t) = 0 \text{ for } (x,t) \in \partial\Omega \times (0,\infty),$$

zero velocity condition at space infinity, and the initial data

$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \text{ for } x \in \Omega.$$

For the stability and asymptotic analysis, we have been interested in estimating decay rates of solutions. Temporal decay estimates have been considered by Miyakawa and Schonbek [27], Miyakawa [25], Schonbek [29], Wiegner [32], etc, for the whole space; Miyakawa and Fujigaki [13], Bae and Choe [4], Bae [1, 3] for the half space; Miyakawa [26], Iwashita[20], Kozono [21], etc, for the exterior domain. Spatial decay estimates has been considered by He [15], He and Xin [18], Takahashi [31], Brandolese [9], Bae and Jin [5, 6] for the whole

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space; Bae [2] for the half space; Farwig and Sohr [11, 12], He and Xin [17] for the exterior domain.

While the temporal decay rate of solutions is almost completely well known for the whole space, half space, and the exterior domain, the spatial decay estimate is not yet well studied for the domain with nonempty boundary such as exterior domain problem. In this paper, we intend to derive spatial-temporal decay estimates of weak and strong solutions of Navier-Stokes flow in exterior domains.

Weak solutions globally in time have been constructed by several mathematicians such as Leray [24], Hopf [19], etc. The uniqueness or the existence of a strong solution has been known only for small data or for a large viscosity (which is not described here). For the partial regularity, the stability estimate, and the localization, an idea of the suitably weak solution is useful: constructed initially by Scheffer [28], by Caffarelli, Kohn and Nirenberg [10] for bounded domains or $\mathbb{R}^3$, and by Galdi and Maremonti [14] for an exterior domain. (See also Seregin [30].) In this paper, we follow the construction in [14] for the weak solution. Throughout this paper, a weak solution means a suitably weak solution.

If we try to have spatial decay estimates for the weak solution via the energy method, we might meet the following integral identity:

$$\int_\Omega \phi^2(\mathbf{u}\cdot\nabla)p\,dx = -\int_\Omega p(\mathbf{u}\cdot\nabla)\phi^2\,dx, \tag{1.2}$$

where ϕ is a weight function $(1+|x|^2)^\alpha$.

As it is seen in (1.2), the pressure term must be treated. When the whole space $\mathbb{R}^3$ is concerned, a pressure representation in terms of the velocity function is useful. From the pressure representation, we have seen that the effect of the pressure p is almost the same as the square of the velocity $|\mathbf{u}|^2$. The situation is not simple when a domain with nonempty boundary is involved. Unfortunately, the pressure has non-local property and we don't have enough information on the pressure near the boundary. This fact makes it difficult to derive norm estimates when the boundary is involved.

In this paper, we suggest an idea treating energy estimates for the domain with a nonempty boundary, and we avoid the computations involved with the pressure term.

Our main idea is explained as follows: Suppose ϕ be a weight function vanishing near the boundary. We introduce an auxiliary vector field $\mathbf{v}$ defined by $\mathbf{v} := \int_\Omega \frac{1}{4\pi|x-y|}[\phi\nabla\times\mathbf{u}](y)dy$. Instead of taking inner product by $\phi\mathbf{u}$ to the pressure term, we take inner product by $\nabla\times(\phi\mathbf{v})$ to get the identity

$$\int_\Omega \nabla\times(\phi\mathbf{v})\cdot\nabla p\,dx = -\int_\Omega p\,\mathrm{div}\nabla\times(\phi\mathbf{v})\,dx = 0.$$

In section 2.2, we will see $\nabla\times(\phi\mathbf{v})$ behaves almost like $\phi^2\mathbf{u}$.

In Section 2, we have used the energy method after removing the pressure term by a special form of a test function as above, and then obtain the temporal decay of $\||x|\mathbf{u}(t)\|_{L^2(\Omega)}$ for the weak solutions.

In Section 3, we obtain the decay estimates $\||x|^2\mathbf{u}(t)\|_{L^p}$, $p \geq 3$, for strong solutions. We are indebted to He and Xin [17] in the sense that the main difficulties in Section 3 have been overcome by removing the pressure term with a slight modification of the idea in [17].

We state our main theorems below, of which proofs are the main objectives of the subsequent sections.

Theorem 1.1. *Let $\mathbf{u}_0 \in L^r(\Omega) \cap L^2(\Omega)$ with $1 < r < 6/5$, and $|x|\mathbf{u}_0 \in L^{6/5}(\Omega)$, $|x|^2\mathbf{u}_0 \in L^2(\Omega)$ and $\nabla \cdot \mathbf{u}_0 = 0$. Then there is a weak solution $\mathbf{u}$ of the Navier-Stokes equation* (1.1) *satisfying the following asymptotic property: for any $\delta > 0$ there is a positive constant c_δ independent of t such that*

$$\||x|\mathbf{u}(t)\|_{L^2(\Omega)} \leq c_\delta(1+t)^{\frac{5}{4}-\frac{3}{2r}+\delta}.$$

Here, c_δ depends on $\delta > 0$ and also on the initial velocity.

Theorem 1.2. *Let $1 < r < 6/5$ and let $3 < p < \infty$. Let $\mathbf{u}_0 \in L^r \cap L^3$ with $\nabla \cdot \mathbf{u}_0 = 0$ and $|x|^2\mathbf{u}_0$, $|x|\mathbf{u}_0 \in L^r$ $|x|\mathbf{u}_0 \in L^{\frac{6}{5}}, |x|^2\mathbf{u}_0 \in L^2(\Omega)$. Suppose that $\mathbf{u}$ is a strong solution of the Navier-Stokes equations* (1.1). *Then we have the following spatial-temporal decay rates: for any $\delta > 0$ there is c_δ so that*

$$\||x|^2\mathbf{u}\|_{L^p} \leq c_\delta t^{1-\frac{3}{2}(\frac{1}{r}-\frac{1}{p})+\delta} \quad \text{for large } t.$$

Throughout this paper, the constants c, c_1,c_a or c_δ, etc, depend on $\mathbf{u}_0$ as well as the subscripts because all the previous results for the temporal decay have been obtained concerning to $\mathbf{u}_0$.

Remark 1.3. In (1.3), the inequality holds for $r = 1$, but in this case $\mathbf{u}_0$ must have some differentiability (refer to [21], [7, 8]), where the optimal decay rate is obtained by $\|\mathbf{u}(t)\|_{L^2} = O(t^{-\frac{3}{4}})$ for $\mathbf{u}_0 \in L^1 \cap L^2 \cap D(A_2^\epsilon)$. Refer to [21] for the definition of $D(A_2^\epsilon)$. Thereby our Theorem 1.1 and Theorem 1.2 could be true for $r = 1$, if $\mathbf{u}_0 \in L^1 \cap L^2 \cap D(A_2^\epsilon)$.

As far as an exterior domain is concerned, the temporal decay rates of the solution (weak for $1 < p \leq 2$, strong for $3 < p \leq \infty$) of the Navier-Stokes equation are well known.

If $\mathbf{u}_0 \in L^r \cap L^2$, with $1 < r < 2$, then for $r \leq q \leq 2$, weak solutions satisfy

$$\|\mathbf{u}(t)\|_{L^q(\Omega)} \leq c(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}. \tag{1.3}$$

For details, refer to [7, 8, 21]. In Section 3, the above result for the weak solution will be used for our estimate.

In [16, 20, 22, 32], it is shown that strong solutions exist in L^q for all times provided that $\mathbf{u}_0$ is small in L^3. Furthermore, it is also shown that for $\mathbf{u}_0 \in L^3$

$$\|\mathbf{u}(t)\|_{L^q} \leq ct^{-\frac{3}{2}(\frac{1}{3}-\frac{1}{q})} \quad \text{for} \quad 3 \leq q \leq \infty, \quad t > 0;$$

for $\mathbf{u}_0 \in L^3 \cap L^r$

$$\|\mathbf{u}(t)\|_{L^q} \leq ct^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})} \quad \text{for } 1 < r \leq q \leq \infty, \quad t \geq 1.$$

The above two estimate can be written at a time as follows:

$$(1.4)\quad \|\mathbf{u}(t)\|_{L^q} \leq ct^{-\frac{3}{2}(\frac{1}{3}-\frac{1}{q})}(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{3})} \quad \text{for } 1 < r \leq 3 \leq q \leq \infty, \quad t > 0.$$

In Section 3, the above result for the strong solution will be used for our estimate .

Throughout this paper this paper, we use the notation $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$ for short.

2. Proof of Theorem 1.1

In this section we consider the decay rates of weak solutions with weight $(1+|x|^2)^{1/2}$ of the Navier-Stokes equations.

2.1. Suitable weak solutions

We consider suitable weak solutions for our estimates, and for the definition we refer to [14] or [30]. We consider the approximate solutions $\mathbf{u}^M$, $M = 1, 2, \ldots$, of (1.1) with initial data $\mathbf{u}_0 \in L^r \cap L^2$, $1 \leq r < 6/5$, and $\operatorname{div} \mathbf{u}_0 = 0$, of the following equations:

$$(2.1)\quad \begin{aligned} \frac{\partial}{\partial t}\mathbf{u}^M - \Delta \mathbf{u}^M + (U^M \cdot \nabla)\mathbf{u}^M + \nabla p^M &= 0, \qquad t > 0, \\ \nabla \cdot \mathbf{u}^M &= 0, \\ \mathbf{u}^M(0) &= \mathbf{u}_0, \end{aligned}$$

where

$$U^M(x,t) = \int_{\mathbb{R}^3} J_{1/M}(y)\widetilde{\mathbf{u}}^M(x-y,t)dy$$

is a (spatial) mollification of $\mathbf{u}^M$ in [14]. The mollifier J_ϵ is defined by $J_\epsilon(x) = \epsilon^{-3}J(x/\epsilon)$, where $J(x) = C\exp(1/(|x|^2-1))$ if $|x| < 1$, and $J(x) = 0$ otherwise. The constant C is selected so that $\int_{\mathbb{R}^3} J(x)dx = 1$. The zero extension of the function $\mathbf{u}^M$ is denoted by $\widetilde{\mathbf{u}}^M$. The solution $\mathbf{u}^M$ has the following properties:

(a) $\mathbf{u}^M$ exist uniquely in $L^2(0,T; W^{1,2}(\Omega)) \cap L^\infty(0,T; L^2_\sigma(\Omega))$ for all $T > 0$.

(b) U^M and their derivatives are continuous and bounded (for each fixed $M \in \mathbf{N}$, but not uniformly in M) on $\mathbb{R}^3 \times [0,T]$, $T > 0$, and satisfy $\nabla \cdot U^M = 0$ and

$$\|U^M(t)\|_p \leq \|\mathbf{u}^M(t)\|_p \qquad \text{for all } p \geq 1,$$

where the generalized Minkowski's inequality is used.

(c) There is a subsequence of $\mathbf{u}^M$ which converges in $L^2_{loc}(\Omega \times [0,\infty))$ to a weak solution of the Navier–Stokes equations (1.1).

(d) For all $t > 0$,

$$\|\mathbf{u}^M(t)\|_2^2 + 2\int_0^t \|\nabla \mathbf{u}^M(\tau)\|_2^2 d\tau = \|\mathbf{u}_0\|_2^2.$$

We recall that for the cases of bounded domains or $\mathbb{R}^3$ in [10], U^M is the retarded mollification U^M of $\mathbf{u}^M$ defined by

$$U^M(x,t) = \delta^{-4} \iint \psi(y/\delta, s/\delta)\widetilde{\mathbf{u}}^M(x-y, t-s)dyds, \qquad \delta = M^{-1},$$

where ψ is a smooth function with $\psi \geq 0$,

$$\iint \psi dxdt = 1, \qquad \text{and} \qquad \operatorname{supp}\psi \subset \{(x,t) : |x|^2 \leq t,\ 1 < t < 2\},$$

and $\widetilde{\mathbf{u}}^M$ is the zero-extension of the function $\mathbf{u}^M$ which is originally defined for $t \geq 0$.

In the following, we write $U = U^M$, $\mathbf{u} = \mathbf{u}^M$ and $p = p^M$ for simplicity. The estimates derived below are uniform in M, hence the desired results are obtained through passage to the limit $M \to \infty$.

2.2. Preliminary estimates

Let N be the fundamental function of $-\Delta$, that is, $N := N(x-y) := \frac{1}{4\pi}\frac{1}{|x-y|}$. Set $\phi_R(x) := |x|^2\chi^2(|x|)[1-\chi_R(|x|)]^2$, where χ is a nonnegative cut-off function with $\chi \in C^\infty[0,\infty)$, $\chi(s) = 0$ for $s \leq 1$, $\chi(s) = 1$ for $s \geq 2$, and we define $\chi_R(s) = \chi(\frac{s}{R})$.

Observe that $\phi_R = O(|x|^2)$ as $|x| \to \infty$ uniformly in R, and that $|\nabla\phi_R| \leq c\phi_R^{1/2}$, $\nabla^2\phi_R = O(1)$ as $|x| \to \infty$. For $|\beta| \geq 3$, $\nabla^\beta\phi_R$ is compactly supported in the annulus $D_1 \cup D_R$, where $D_1 = B_2\backslash B_1$ and $D_R = B_{2R}\backslash B_R$, moreover, $|\nabla^3\phi_R| \leq \frac{c}{R}$ on D_R. Here, B_i means the ball of radius i centered at the origin.

We introduce an auxiliary vector field $\mathbf{v}_R$ defined by

$$\mathbf{v}_R(x) := \int_{\mathbb{R}^3} N(x-y)[\phi_R(y)(\nabla\times\mathbf{u})(y)]dy = N * [\phi_R\nabla\times\mathbf{u}].$$

By the definition of $\mathbf{v}_R$ we have $-\Delta\mathbf{v}_R = \phi_R\nabla\times\mathbf{u}$. Moreover,

$$\nabla\times\mathbf{v}_R = \int_\Omega N(x-y)\nabla\times[\phi_R(\nabla\times\mathbf{u})](y)dy = \phi_R\mathbf{u} + \mathbf{R}_{0,R}, \tag{2.2}$$

where

$$\mathbf{R}_{0,R} := \nabla N * [(\mathbf{u}\cdot\nabla)\phi_R] - \nabla\times N * [(\nabla\phi_R)\times\mathbf{u}]. \tag{2.3}$$

The above identity comes from the following observations:

$$\begin{aligned}\nabla\times\big[\phi_R(y)(\nabla\times\mathbf{u})(y)\big] &= \nabla\times[\nabla\times(\phi_R\mathbf{u})] - \nabla\times[(\nabla\phi_R)\times\mathbf{u}]\\ &= -\Delta(\phi_R\mathbf{u}) + \nabla[(\mathbf{u}\cdot\nabla)\phi_R] - \nabla\times[(\nabla\phi_R)\times\mathbf{u}].\end{aligned}$$

By Gagliardo-Nirenberg-Sobolev inequality, we have for $1 < p < \frac{3}{2}$

$$\|\mathbf{v}_R\|_{3p/(3-2p)} \leq c\|\nabla\mathbf{v}_R\|_{3p/(3-p)} \leq c\|\nabla^2\mathbf{v}_R\|_p,$$

and for $1 < p < 3$

$$\|\mathbf{R}_{0,R}\|_{3p/(3-p)} \le c\|\nabla \mathbf{R}_{0,R}\|_p.$$

On the other hand, by the Calderon-Zygmund inequality,

$$\|\nabla^2 N * f\|_p \le c\|f\|_p, \ 1 < p < \infty.$$

Combining the Gagliardo-Nirenberg-Sobolev inequality and the Calderon-Zygmund inequality for $\mathbf{v}_R$ and $\mathbf{R}_{0,R}$, we have the following lemmas.

Lemma 2.1. *For* $1 < p < \frac{3}{2}$,

$$\|\mathbf{v}_R\|_{3p/(3-2p)} \le c\|\phi_R \nabla \mathbf{u}\|_p.$$

Lemma 2.2. *For* $1 < p < 3$,

$$\|\mathbf{R}_{0,R}\|_{3p/(3-p)} \le c\|\mathbf{u}\phi_R^{1/2}\|_p.$$

Lemma 2.3.

$$\|\mathbf{u}\phi_R^{1/2}\|_2^2 \le c\|\mathbf{u}\|_2\|\text{curl } \mathbf{v}_R\|_{L^2} + c\|\mathbf{u}\|_{\frac{6}{5}}^2. \tag{2.4}$$

Furthermore, we also have

$$\|\mathbf{u}\phi_R^{1/2}\|_2^2 \le c\|\phi_R\nabla \mathbf{u}\|_2\|\mathbf{u}\|_{6/5} + c\|\mathbf{u}\|_{6/5}^2. \tag{2.5}$$

Proof. Since $\mathbf{u}\phi_R = \text{curl } \mathbf{v}_R - \mathbf{R}_{0,R}$, we have

$$\begin{aligned}\|\mathbf{u}\phi_R^{1/2}\|_2^2 &= \|\mathbf{u}\cdot(\mathbf{u}\phi_R)\|_1 \le c\|\mathbf{u}\|_2\|\text{curl } \mathbf{v}_R\|_2 + \|\mathbf{u}\|_{6/5}\|\mathbf{R}_{0,R}\|_6 \\ &\le c\|\mathbf{u}\|_2\|\text{curl } \mathbf{v}_R\|_2 + c\|\mathbf{u}\|_{6/5}\|\mathbf{u}\phi_R^{\frac{1}{2}}\|_2.\end{aligned}$$

By Young's inequality there is c_ϵ so that

$$c\|\mathbf{u}\|_{6/5}\|\mathbf{u}\phi_R^{1/2}\|_2 \le \epsilon\|\mathbf{u}\phi_R^{1/2}\|_2^2 + c_\epsilon\|\mathbf{u}\|_{\frac{6}{5}}^2.$$

Taking ϵ small enough we complete our proof of (2.4).

For (2.5), we apply Hölder's inequality to get

$$\begin{aligned}\|\mathbf{u}\phi_R^{1/2}\|_2^2 &\le \|\phi_R\mathbf{u}\|_6\|\mathbf{u}\|_{6/5} \le c\|\nabla(\phi_R\mathbf{u})\|_2\|\mathbf{u}\|_{6/5} \\ &\le c\|\phi_R\nabla\mathbf{u}\|_2\|\mathbf{u}\|_{6/5} + c\|\phi_R^{1/2}\mathbf{u}\|_2\|\mathbf{u}\|_{6/5} \\ &\le \epsilon\|\phi_R^{1/2}\mathbf{u}\|_2^2 + c\|\phi_R\nabla\mathbf{u}\|_2\|\mathbf{u}\|_{6/5} + c_\epsilon\|\mathbf{u}\|_{6/5}^2\end{aligned}$$

for some constant c_ϵ depending on ϵ. Now take $\epsilon < 1/2$. □

2.3. Spatial decay estimates of weak solution

We now ready to show the decay estimates; for any $\delta > 0$, there is c_δ so that

$$(1+t)^{-\frac{1}{2}+\frac{3}{2}(\frac{1}{r}-\frac{1}{2})-\delta}\|\,|x|\mathbf{u}\|_2 \le c_\delta \text{ uniformly in } t > 0.$$

Equivalently, we will show

$$(1+t)^{-\frac{1}{2}+\frac{3}{2}(\frac{1}{r}-\frac{1}{2})-\delta}\|\,\phi_R^{1/2}\mathbf{u}\|_2 \le c_\delta \text{ uniformly in } R,\ t,$$

where $\phi_R(x) = |x|^2\chi^2(x)[1-\chi(\frac{x}{R})]^2$.

Consider $\mathbf{v}_R$ defined in the previous subsection.

Proposition 2.4.

$$(1+t)^{-\frac{7}{4}+\frac{3}{2r}-\delta}\|\text{curl }\mathbf{v}_R(t)\|_2 \le c_\delta \textit{ uniformly in } R, t.$$

By combining (2.4) in Lemma 2.3 and Proposition 2.4, and by applying the well known temporal decays to $\|\mathbf{u}\|_{6/5}$, we obtain our main estimates for the weak solutions in Theorem 1.1.

From now on in this subsection, we concentrate ourselves on the proof of Proposition 2.4. In order to show this, multiply $(2.1)_1$ by $\nabla \times (\phi_R \mathbf{v}_R)$. Since

$$\int_\Omega (\nabla p)\cdot[\nabla\times(\phi_R\mathbf{v}_R)]dx = 0,$$

we have

$$\int_\Omega \partial_t\mathbf{u}\cdot\big[\nabla\times(\phi_R\mathbf{v}_R)\big]dx + \int_\Omega \big[(U\cdot\nabla)\mathbf{u} - \Delta\mathbf{u}\big]\cdot\big[\nabla\times(\phi_R\mathbf{v}_R)\big]dx = 0.$$

Before proceeding further, we'd rather mention our strategy that

(1) from the above identity, by tedious and long calculations we will finally obtain the inequality (2.7), and
(2) we apply Gronwall's inequality to get our estimates for $\|\nabla\mathbf{v}_R\|_2$ at the end of this section.

Now we return to our proof. Integrating by parts, we first observe that

$$\int \partial_t\mathbf{u}\cdot[\nabla\times(\phi_R\mathbf{v}_R)]dx = \frac{d}{dt}\int \mathbf{u}\cdot[\nabla\times(\phi_R\mathbf{v}_R)]dx - \int \mathbf{u}\cdot[\nabla\times(\phi_R\partial_t\mathbf{v}_R)]dx.$$

Note that

$$\begin{aligned}
&\int_\Omega \mathbf{u}\cdot[\nabla\times(\phi_R\partial_t\mathbf{v}_R)]dx = \int_\Omega (\phi_R\nabla\times\mathbf{u})\cdot\partial_t\mathbf{v}_R dx\\
&= \int(\phi_R\nabla\times\mathbf{u})\cdot\Big(\int N(x-y)[\phi_R(y)\nabla\times\partial_t\mathbf{u}(y)]dy\Big)dx\\
&= \int(\phi_R\nabla\times\mathbf{u})\cdot\Big(\int N(x-y)[\phi_R\nabla\times(\Delta\mathbf{u}-(U\cdot\nabla)\mathbf{u})]dy\Big)dx\\
&= \int\Big(\int N(x-y)[\phi_R\nabla\times\mathbf{u}(x)]dx\Big)\cdot\Big(\phi_R\nabla\times(\Delta\mathbf{u}-(U\cdot\nabla)\mathbf{u})\Big)dy\\
&= \int \mathbf{v}_R\cdot[\phi_R\nabla\times(\Delta\mathbf{u}-(U\cdot\nabla)\mathbf{u}]dy = \int[\nabla\times(\phi_R\mathbf{v}_R)]\cdot(\Delta\mathbf{u}-(U\cdot\nabla)\mathbf{u})dx.
\end{aligned}$$

With the previous calculations, we have

$$\frac{d}{dt}\int \mathbf{u}\cdot[\nabla\times(\phi_R\mathbf{v}_R)]dx + 2\int[(U\cdot\nabla)\mathbf{u}]\cdot\nabla\times(\phi_R\mathbf{v}_R)dx - 2\int(\Delta\mathbf{u})\cdot\nabla\times(\phi_R\mathbf{v}_R)dx = 0. \tag{2.6}$$

Define $X(t)$ and $Y(t)$ by

$$X(t) = \int_\Omega \mathbf{u}\cdot[\nabla\times(\phi_R\mathbf{v}_R)]dx, \quad Y(t) = \int_\Omega \phi_R^2|\nabla\mathbf{u}|^2dx.$$

Note that $\mathbf{v}_R(t)$ is defined in $\mathbb{R}^3$ and $X(t) = \|\nabla \mathbf{v}_R(t)\|^2_{L^2(\mathbb{R}^3)}$ since

$$X(t) = \int \mathbf{u} \cdot \nabla \times (\phi_R \mathbf{v}_R) dx = \int (\phi_R \nabla \times \mathbf{u}) \cdot \mathbf{v}_R dx = - \int (\Delta \mathbf{v}_R) \cdot \mathbf{v}_R dx.$$

Lemma 2.5.

$$\begin{aligned} \frac{d}{dt} X(t) + Y(t) \le\ & c\big(\|\mathbf{u}\|_2^2 + \|\nabla \mathbf{u}\|_2^2 + \|\mathbf{u}\|_2^{2a}\big) X(t) + c\|\mathbf{u}\|_2^{2(1-a)} \\ & + c\|\mathbf{u}\|_2^2 + c\|\mathbf{u}\|_2^4 + c\|\mathbf{u}\|_2^6 + c\|\mathbf{u}\|_{6/5}^2 + c\|\nabla \mathbf{u}\|_2^2 \\ & + c\|\nabla \mathbf{u}\|_2 \|\mathbf{u}\|_2 \|\mathbf{u}\|_{6/5}^2. \end{aligned} \tag{2.7}$$

Proof. Considering the identity

$$\nabla \times (\phi_R \mathbf{v}_R) = \phi_R^2 \mathbf{u} + \phi_R \mathbf{R}_{0,R} + (\nabla \phi_R) \times \mathbf{v}_R,$$

the above identity (2.6) becomes the following identity

$$\begin{aligned} \frac{d}{dt} X(t) + 2Y(t) =& \int |\mathbf{u}|^2 \Delta \phi_R^2 dx + \int |\mathbf{u}|^2 (U \cdot \nabla) \phi_R^2 dx \\ & + 2 \int \mathbf{u} \cdot \Delta\Big(\phi_R \mathbf{R}_{0,R} + (\nabla \phi_R) \times \mathbf{v}_R\Big) dx \\ & + 2 \int \mathbf{u} \cdot \Big((U \cdot \nabla)[\phi_R \mathbf{R}_{0,R} + (\nabla \phi_R) \times \mathbf{v}_R]\Big) dx \\ =& I + I\!I + I\!I\!I + I\!V. \end{aligned} \tag{2.8}$$

Recall $-\Delta \mathbf{v}_R = \phi_R \nabla \times \mathbf{u}$ and $-\Delta \mathbf{R}_{0,R} = \nabla[(\mathbf{u} \cdot \nabla)\phi_R] - \nabla \times [(\nabla \phi_R) \times \mathbf{u}]$ to get

$$\begin{aligned} \Delta(\phi_R \mathbf{R}_{0,R}) =& (\Delta \phi_R) \mathbf{R}_{0,R} + 2(\partial_i \phi_R)(\partial_i \mathbf{R}_{0,R}) \\ & - \phi_R \Big(\nabla[(\mathbf{u} \cdot \nabla)\phi_R] - \nabla \times [(\nabla \phi_R) \times \mathbf{u}]\Big), \end{aligned}$$

$$\Delta[(\nabla \phi_R) \times \mathbf{v}_R] = (\nabla \Delta \phi_R) \times \mathbf{v}_R + 2(\nabla \partial_i \phi_R) \times (\partial_i \mathbf{v}_R) - \phi_R (\nabla \phi_R) \times (\nabla \times \mathbf{u}).$$

So, $I\!I\!I$ can be replaced by the following terms:

$$\begin{aligned} I\!I\!I =& -2 \int \phi_R \mathbf{u} \cdot \Big(\nabla[(\mathbf{u} \cdot \nabla)\phi_R] - \nabla \times [(\nabla \phi_R) \times \mathbf{u}] + (\nabla \phi_R) \times (\nabla \times \mathbf{u})\Big) dx \\ & + 2 \int \mathbf{u} \cdot \Big((\Delta \phi_R) \mathbf{R}_{0,R} + 2(\partial_i \phi_R)(\partial_i \mathbf{R}_{0,R}) + 2(\nabla \partial_i \phi_R) \times (\partial_i \mathbf{v}_R)\Big) dx \\ & + 2 \int \mathbf{u} \cdot [(\nabla \Delta \phi_R) \times \mathbf{v}_R] dx = I\!I\!I_1 + I\!I\!I_2 + I\!I\!I_3. \end{aligned}$$

By integrations by parts, $I\!I\!I_1$ can be rewritten by

$$I\!I\!I_1 = 2 \int \Big((\mathbf{u} \cdot \nabla \phi_R)^2 + [(\nabla \phi_R) \times \mathbf{u}]^2 + 2\phi_R (\nabla \phi_R) \cdot [\mathbf{u} \times (\nabla \times \mathbf{u})]\Big) dx.$$

From the vector identity

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2} \nabla |\mathbf{u}|^2 - (\mathbf{u} \cdot \nabla)\mathbf{u},$$

the last term of $I\!I\!I_1$ is equal to

$$\begin{aligned}\int 2\phi_R(\nabla\phi_R)\cdot[\mathbf{u}\times(\nabla\times\mathbf{u})]dx =& \int(\nabla\phi_R^2)\cdot\Big[\frac{1}{2}\nabla|\mathbf{u}|^2-(\mathbf{u}\cdot\nabla)\mathbf{u}\Big]dx\\ =& -\int\frac{1}{2}|\mathbf{u}|^2\Delta\phi_R^2dx+\int\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\nabla\phi_R^2dx.\end{aligned}$$

Hence $I+I\!I\!I_1$ is equal to

$$I+I\!I\!I_1=2\int\Big((\mathbf{u}\cdot\nabla\phi_R)^2+[(\nabla\phi_R)\times\mathbf{u}]^2+\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\nabla\phi_R^2\Big)dx.$$

Therefore, $I\!I$ and $I+I\!I\!I_1$ are bounded by

$$\begin{aligned}|I+I\!I\!I_1|&\le c\|\phi_R^{1/2}\mathbf{u}\|_2^2,\\ |I\!I|&\le c\|\phi_R\mathbf{u}\|_6^{3/2}\|\mathbf{u}\|_2^{1/2}\|U\|_2\le c\|\nabla(\phi_R\mathbf{u})\|_2^{3/2}\|\mathbf{u}\|_2^{3/2}\\ &\le c\|\phi_R\nabla\mathbf{u}\|_2^{3/2}\|\mathbf{u}\|_2^{3/2}+c\|\phi_R^{1/2}\mathbf{u}\|_2^{3/2}\|\mathbf{u}\|_2^{3/2}.\end{aligned}$$

Recalling $\nabla\Delta\phi_R$ has a compact support in the set $D_1\cup D_R$, where $D_1=B_2\backslash B_1$ and $D_R=B_{2R}\backslash B_R$, with $|\nabla^3\phi_R|\le\frac{c}{R}$ on D_R, we have

$$\begin{aligned}|I\!I\!I_3|&\le c\|\mathbf{v}_R\|_6\|\mathbf{u}\|_{L^{6/5}(D_1)}+\frac{c}{R}\|\mathbf{v}_R\|_6\|\mathbf{u}\|_{L^{\frac{6}{5}}(D_R)}\\ &\le c\|\nabla\mathbf{v}_R\|_2\|\mathbf{u}\|_{L^2(D_1)}+c\|\nabla\mathbf{v}_R\|_2\|\mathbf{u}\|_{L^2(D_R)}\le cX(t)^{\frac{1}{2}}\|\mathbf{u}\|_2.\end{aligned}$$

Applying the estimates for $\mathbf{v}_R$, $\mathbf{R}_{0,R}$ in Lemma 2.1, 2.2 to $I\!I\!I_2$, we have

$$\begin{aligned}|I\!I\!I_2|&\le c\|\mathbf{u}\|_{6/5}\|\mathbf{R}_{0,R}\|_6+c\|\phi_R^{1/2}\mathbf{u}\|_2\|\nabla\mathbf{R}_{0,R}\|_2+c\|\mathbf{u}\|_{6/5}\|\nabla\mathbf{v}_R\|_6\\ &\le c\|\mathbf{u}\|_{6/5}\|\phi_R^{1/2}\mathbf{u}\|_2+c\|\phi_R^{1/2}\mathbf{u}\|_2^2+c\|\mathbf{u}\|_{6/5}\|\phi_R\nabla\mathbf{u}\|_2.\end{aligned}$$

Since

$$\begin{aligned}&(U\cdot\nabla)[\phi_R\mathbf{R}_{0,R}+(\nabla\phi_R)\times\mathbf{v}_R]\\ =&\,\mathbf{R}_{0,R}(U\cdot\nabla)\phi_R+\phi_R(U\cdot\nabla)\mathbf{R}_{0,R}\\ &+[(U\cdot\nabla)\nabla\phi_R]\times\mathbf{v}_R+(\nabla\phi_R)\times[(U\cdot\nabla)\mathbf{v}_R],\end{aligned}$$

$I\!V$ can be rewritten by

$$\begin{aligned}I\!V=&2\int\mathbf{u}\cdot\Big(\mathbf{R}_{0,R}(U\cdot\nabla)\phi_R+\phi_R(U\cdot\nabla)\mathbf{R}_{0,R}\Big)dx\\ &+2\int\mathbf{u}\cdot\Big((\nabla\phi_R)\times[(U\cdot\nabla)\mathbf{v}_R]\Big)dx\\ &+2\int\mathbf{u}\cdot\Big([(U\cdot\nabla)\nabla\phi_R]\times\mathbf{v}_R\Big)dx=I\!V_1+I\!V_2+I\!V_3.\end{aligned}$$

Applying the estimates for $\mathbf{v}_R$, $\mathbf{R}_{0,R}$ in Lemma 2.1, 2.2 to IV_1, IV_2, we have

$$\begin{aligned}
|IV_1| &\le c\|\phi_R\mathbf{u}\|_6^{1/2}\|\mathbf{u}\|_2^{1/2}\|U\|_2\|\mathbf{R}_{0,R}\|_6 + c\|\phi_R\mathbf{u}\|_6\|U\|_6^{1/2}\|U\|_2^{1/2}\|\nabla\mathbf{R}_{0,R}\|_2\\
&\le c\|\nabla(\phi_R\mathbf{u})\|_2^{1/2}\|\mathbf{u}\|_2^{3/2}\|\phi_R^{1/2}\mathbf{u}\|_2 + c\|\nabla(\phi_R\mathbf{u})\|_2\|\nabla\mathbf{u}\|_2^{1/2}\|\mathbf{u}\|_2^{1/2}\|\phi_R^{1/2}\mathbf{u}\|_2\\
&\le c\|\phi_R^{1/2}\mathbf{u}\|_2^{3/2}\|\mathbf{u}\|_2^{3/2} + c\|\phi_R\nabla\mathbf{u}\|_2^{1/2}\|\mathbf{u}\|_2^{3/2}\|\phi_R^{1/2}\mathbf{u}\|_2\\
&\quad + c\|\phi_R^{1/2}\mathbf{u}\|_2^2\|\nabla\mathbf{u}\|_2^{1/2}\|\mathbf{u}\|_2^{1/2} + c\|\phi_R\nabla\mathbf{u}\|_2\|\nabla\mathbf{u}\|_2^{1/2}\|\mathbf{u}\|_2^{1/2}\|\phi_R^{1/2}\mathbf{u}\|_2,\\
|IV_2| &\le c\|\phi_R\mathbf{u}\|_6^{1/2}\|\mathbf{u}\|_2^{1/2}\|U\|_2\|\nabla\mathbf{v}_R\|_6 \le c\|\nabla(\phi_R\mathbf{u})\|_2^{1/2}\|\mathbf{u}\|_2^{3/2}\|\phi_R\nabla\mathbf{u}\|_2\\
&\le \|\phi_R^{1/2}\mathbf{u}\|_2^{1/2}\|\mathbf{u}\|_2^{3/2}\|\phi_R\nabla\mathbf{u}\|_2 + c\|\phi_R\nabla\mathbf{u}\|_2^{3/2}\|\mathbf{u}\|_2^{3/2},\\
|IV_3| &\le c\|\mathbf{u}\|_6^{1/2}\|\mathbf{u}\|_2^{3/2}\|\mathbf{v}_R\|_6 \le c\|\nabla\mathbf{u}\|_2^{1/2}\|\mathbf{u}\|_2^{3/2}\|\nabla\mathbf{v}_R\|_2\\
&\le c\|\nabla\mathbf{u}\|_2^{1/2}\|\mathbf{u}\|_2^{3/2}X(t)^{\frac{1}{2}}.
\end{aligned}$$

In the above, we used the fact $\|U\|_p \le c\|\mathbf{u}\|_p$ since U is a mollification of $\mathbf{u}$.

Therefore, combining the estimates of I, II, III and IV, and recalling $Y = \|\phi_R\nabla\mathbf{u}\|_2^2$, our identity (2.8) is transformed to the inequality

$$\begin{aligned}
&\frac{d}{dt}X(t) + 2Y(t)\\
&\le c\|\mathbf{u}\|_2^{3/2}Y^{3/4}(t) + c\|\mathbf{u}\|_2^{3/2}\|\phi_R^{1/2}\mathbf{u}\|_2Y^{1/4}(t)\\
&\quad + c\Big(\|\mathbf{u}\|_{6/5} + \|\mathbf{u}\|_2^{3/2}\|\phi_R^{1/2}\mathbf{u}\|_2^{1/2} + \|\mathbf{u}\|_2^{1/2}\|\nabla\mathbf{u}\|_2^{1/2}\|\phi_R^{1/2}\mathbf{u}\|_2\Big)Y^{1/2}(t)\\
&\quad + c\|\phi_R^{1/2}\mathbf{u}\|_2^2 + c\|\mathbf{u}\|_2^{3/2}\|\phi_R^{1/2}\mathbf{u}\|_2^{3/2} + c\|\mathbf{u}\|_{6/5}\|\phi_R^{1/2}\mathbf{u}\|_2\\
&\quad + c\|\nabla\mathbf{u}\|_2^{1/2}\|\mathbf{u}\|_2^{1/2}\|\phi_R^{1/2}\mathbf{u}\|_2^2 + c\|\nabla\mathbf{u}\|_2^{1/2}\|\mathbf{u}\|_2^{3/2}X(t)^{\frac{1}{2}} + c\|\mathbf{u}\|_2X(t)^{\frac{1}{2}}.
\end{aligned}$$

By Young's inequality we have

$$\begin{aligned}
&\|\mathbf{u}\|_2^{3/2}Y^{3/4}(t) \le \epsilon Y(t) + c_\epsilon\|\mathbf{u}\|_2^6,\\
&\|\mathbf{u}\|_2^{3/2}\|\phi_R^{1/2}\mathbf{u}\|_2Y^{1/4}(t) \le \epsilon Y(t) + c_\epsilon\big(\|\phi_R^{1/2}\mathbf{u}\|_2^2 + \|\mathbf{u}\|_2^6\big),\\
&\Big(\|\mathbf{u}\|_{6/5} + \|\mathbf{u}\|_2^{3/2}\|\phi_R^{1/2}\mathbf{u}\|_2^{1/2} + \|\mathbf{u}\|_2^{1/2}\|\nabla\mathbf{u}\|_2^{1/2}\|\phi_R^{1/2}\mathbf{u}\|_2\Big)Y^{1/2}(t)\\
&\quad \le \epsilon Y(t) + c_\epsilon\Big(\|\mathbf{u}\|_{6/5}^2 + \|\phi_R^{1/2}\mathbf{u}\|_2^2 + \|\mathbf{u}\|_2^6 + \|\mathbf{u}\|_2\|\nabla\mathbf{u}\|_2\|\phi_R^{1/2}\mathbf{u}\|_2^2\Big),
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla\mathbf{u}\|_2^{1/2}\|\mathbf{u}\|_2^{1/2}\|\phi_R^{1/2}\mathbf{u}\|_2^2 &\le \|\phi_R^{1/2}\mathbf{u}\|_2^2 + c\|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2\|\phi_R^{1/2}\mathbf{u}\|_2^2,\\
\|\nabla\mathbf{u}\|_2^{1/2}\|\mathbf{u}\|_2^{3/2}X(t)^{\frac{1}{2}} &\le c\|\mathbf{u}\|_2^2X(t) + c\|\mathbf{u}\|_2^2 + c\|\nabla\mathbf{u}\|_2^2,\\
\|\mathbf{u}\|_2X(t)^{\frac{1}{2}} &\le \|\mathbf{u}\|_2^{2a}X(t) + \|\mathbf{u}\|_2^{2(1-a)}
\end{aligned}$$

for any $0 < a < 1$. Recalling (2.4) in Lemma 2.3 we have the inequality

$$\begin{aligned}
\|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2\|\phi_R^{1/2}\mathbf{u}\|_2^2 &\le c\|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2\big(X(t)^{1/2}\|\mathbf{u}\|_2 + \|\mathbf{u}\|_{6/5}^2\big)\\
&\le c\|\nabla\mathbf{u}\|_2^2X(t) + c\|\mathbf{u}\|_2^4 + c\|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2\|\mathbf{u}\|_{6/5}^2,
\end{aligned}$$

and by (2.5) we also have that

$$\|\phi_R^{1/2}\mathbf{u}\|_2^2 \le Y(t)^{1/2}\|\mathbf{u}\|_{6/5} + \|\mathbf{u}\|_{6/5}^2 \le \epsilon Y(t) + c\|\mathbf{u}\|_{6/5}^2,$$

Hence we have that for any $0 < a < 1$,

$$\begin{aligned}\frac{d}{dt}X(t) + Y(t) \le c\big(\|\mathbf{u}\|_2^2 + \|\nabla\mathbf{u}\|_2^2 + \|\mathbf{u}\|_2^{2a}\big)X(t) + c\|\mathbf{u}\|_2^{2(1-a)}\\ + c\|\mathbf{u}\|_2^2 + c\|\mathbf{u}\|_2^4 + c\|\mathbf{u}\|_2^6 + c\|\mathbf{u}\|_{6/5}^2 + c\|\nabla\mathbf{u}\|_2^2\\ + c\|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2\|\mathbf{u}\|_{6/5}^2.\end{aligned}$$

□

Now, we try to apply the well-known Gronwall's inequality to (2.7). We remind Gronwall's inequality:

Lemma 2.6. *Suppose $X(t)$ satisfies the following inequality*

$$\frac{d}{dt}X(t) \le A(t) + B(t)X(t), \qquad t > 0.$$

Then one has

$$X(t) \le e^{\int_0^t B(s)ds}X(0) + \int_0^t A(s)e^{\int_s^t B(\tau)d\tau}ds, \qquad t > 0.$$

Now we set

$$\begin{aligned}A(t) = c\|\mathbf{u}\|_2^{2(1-a)} + c\|\mathbf{u}\|_2^2 + c\|\mathbf{u}\|_2^4 + c\|\mathbf{u}\|_2^6 + c\|\mathbf{u}\|_{6/5}^2 + c\|\nabla\mathbf{u}\|_2^2\\ + c\|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2\|\mathbf{u}\|_{6/5}^2\end{aligned}$$

and

$$B(t) = \|\mathbf{u}\|_2^2 + \|\nabla\mathbf{u}\|_2^2 + \|\mathbf{u}\|_2^{2a}.$$

By (1.3), it is clear that

$$\int_0^t \|\mathbf{u}\|_2^2 + \|\nabla\mathbf{u}\|_2^2 ds \le c,$$

and note that

$$\int_0^t \|\mathbf{u}(s)\|_2^{2a}ds = c_a\Big[1 - (1+t)^{1-3a(\frac{1}{r}-\frac{1}{2})}\Big] \le c_a$$

if $\frac{2r}{3(2-r)} < a < 1$ for $1 \le r < \frac{6}{5}$. Therefore, we have

$$\int_0^t B(s)ds \le c_a \tag{2.9}$$

for some positive constant c_a depending on a.

By (1.3), we again note that

$$\int_0^t \|\mathbf{u}\|_2^2 + \|\mathbf{u}\|_2^4 + \|\mathbf{u}\|_2^6 + \|\nabla\mathbf{u}\|_2^2 ds \le c,$$

$$\int_0^t \|\nabla \mathbf{u}\|_2 \|\mathbf{u}\|_2 \|\mathbf{u}\|_{\frac{6}{5}}^2 ds \le c \int_0^t \|\nabla \mathbf{u}\|_2^2 + \|\mathbf{u}\|_2^2 \|\mathbf{u}\|_{\frac{6}{5}}^4 ds \le c,$$

and that

$$\int_0^t \|\mathbf{u}\|_{\frac{6}{5}}^2 ds \le c(1+t)^{\frac{7}{2}-\frac{3}{r}}.$$

We also note that

$$\int_0^t \|\mathbf{u}\|_2^{2(1-a)} ds \le c_a \Big[(1+t)^{\frac{5}{2}-\frac{3}{r}+\frac{3(2-r)a}{2r}} - 1\Big] \le c_a (1+t)^{\frac{5}{2}-\frac{3}{r}+\frac{3(2-r)a}{2r}},$$

since $\frac{5}{2}-\frac{3}{r}+\frac{3(2-r)a}{2r} > 0$ for $1 \le r < \frac{6}{5}$ and $1 > a > \frac{2r}{3(2-r)}$.

Let $\delta > 0$ be given any small number. Since $\frac{3(2-r)a}{2r} > 1$ is arbitrary, we can take a such that $\frac{3(2-r)a}{2r} = 1+\delta$, so that

$$\int_0^t \|\mathbf{u}\|_2^{2(1-a)} ds \le c_\delta (1+t)^{\frac{7}{2}-\frac{3}{r}+\delta}.$$

Therefore, for any small $\delta > 0$ there is a constant c_δ depending on δ and independent of t such that

$$\int_0^t A(s) ds \le c_\delta (1+t)^{\frac{7}{2}-\frac{3}{r}+\delta}. \tag{2.10}$$

Applying Gronwall's inequality of Lemma 2.6 to (2.7), and by (2.9) and (2.10), we obtain that

$$X(t) \le c_\delta (1+t)^{\frac{7}{2}-\frac{3}{r}+\delta}$$

for all $t > 0$. Here, we need the boundedness assumption on $X(0)$, that is, $\||x|^2\mathbf{u}_0\|_2 + \||x|\mathbf{u}_0\|_{\frac{6}{5}} < \infty$ by (2.2). We note that the above estimate is uniform in R. This implies that

$$\|\nabla \mathbf{v}_R(t)\|_2^2 \le c_\delta (1+t)^{\frac{7}{2}-\frac{3}{r}+\delta} \text{ uniformly in } R,$$

which completes the proof of Proposition 2.4.

By taking $R \to \infty$ and with the lower semicontinuity of the norm, we complete the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Let $1 < r < \frac{6}{5}$ and $3 < p < \infty$. In this section, based on the previous estimate and modifying the idea in [17] we obtain the decay rates for strong solutions: for any $\delta > 0$, there is c_δ so that

$$\||x|^2 \mathbf{u}(t)\|_p \le c_\delta t^{1-\frac{3}{2}(\frac{1}{r}-\frac{1}{p})+\delta} \quad \text{for large } t.$$

Equivalently, we will show

$$\|\phi \mathbf{u}(t)\|_p \le c_\delta t^{1-\frac{3}{2}(\frac{1}{r}-\frac{1}{p})+\delta} \quad \text{for large } t.$$

Here, $\phi(x) := |x|^2 \chi(x)$.

3.1. Preliminaries

Lemma 3.1. *For $p > 3$,*

$$\|\mathbf{u}\phi\|_p \le c\|\mathrm{curl}\,\mathbf{v}\|_p + c\|\mathbf{u}\|_{\frac{3p}{3+2p}}.$$

Proof. Like (2.3), define $\mathbf{R}_0$ by $\mathbf{R}_0 := \nabla N * [(\mathbf{u}\cdot\nabla)\phi] - \nabla \times N * [(\nabla\phi)\times\mathbf{u}]$. Since $\mathbf{u}\phi = \mathrm{curl}\,\mathbf{v} - \mathbf{R}_0$, we have

$$\|\mathbf{u}\phi\|_p \le c\|\mathrm{curl}\,\mathbf{v}\|_p + \|\mathbf{R}_0\|_p \le c\|\mathrm{curl}\ \mathbf{v}\|_p + c\|\mathbf{u}\phi^{\frac{1}{2}}\|_{\frac{3p}{p+3}}.$$

By Hölder's and Young's inequalities, there is c_ϵ so that

$$\|\mathbf{u}\phi^{\frac{1}{2}}\|_{\frac{3p}{p+3}} \le c\|\mathbf{u}\phi\|_p^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{3p}{3+2p}}^{\frac{1}{2}} \le \epsilon\|\mathbf{u}\phi\|_p + c_\epsilon\|\mathbf{u}\|_{3p/(3+2p)}.$$

Taking ϵ small enough, we complete the proof. □

Owing to (1.4) and by Lemma 3.1, it is enough to show the following.

Proposition 3.2. *Let $2 \le s < \infty$ and $1 < r < \frac{6}{5}$. For any $\delta > 0$ there is c_δ so that*

$$\|\nabla\times\mathbf{v}\|_s \le c_\delta t^{1+\frac{3}{2s}-\frac{3}{2r}+\delta} \quad \textit{for large } t.$$

Following estimate for the Beta-type function will be useful to the proof of our proposition 3.2.

Lemma 3.3. *Let $d < 1,\ b < 1$ and $a > 0$. Then we have*

$$\int_0^t (1+s)^{-a}s^{-d}(t-s)^{-b}ds \le \begin{cases} t^{\max\{1-a-b-d,-b\}} & \textit{for } t \ge 2, \\ t^{1-b-d} & \textit{for } t \le 2. \end{cases}$$

Proof. For $t > 2$,

$$\begin{aligned}
&\int_0^t (1+s)^{-a}s^{-d}(t-s)^{-b}ds \\
&\le ct^{-a-d}\int_{\frac{t}{2}}^t (t-s)^{-b}ds + ct^{-b}\int_1^{\frac{t}{2}} s^{-a-d}ds + ct^{-b}\int_0^1 s^{-d}ds \\
&= ct^{1-a-b-d} + ct^{-b}[t^{1-a-d}-1] + ct^{-b} \le ct^{1-a-b-d} + t^{-b}.
\end{aligned}$$

For $t \le 2$,

$$\int_0^t (1+s)^{-a}s^{-d}(t-s)^{-b}ds \le \int_0^t s^{-d}(t-s)^{-b}ds \le ct^{1-b-d}.$$

□

Remark 3.4. In the proof of Lemma 3.1, we obtained the inequality

$$\|\mathbf{u}(t)\phi\|_p \le c\|\ \mathrm{curl}\ \mathbf{v}(t)\|_p + c\|\mathbf{u}(t)\|_{\frac{3p}{3+2p}}.$$

On the other hand, the temporal decay rate for $\|\mathbf{u}(t)\|_{\frac{3p}{3+2p}}$ is known for $\frac{3p}{3+2p} > 1$, and therefore, we restricted $p > 3$.

We compare this observation with the estimate in Theorem 1.1. If we proceed the same argument to the estimate of $\|\mathbf{u}\phi\|_2$, we have

$$\|\mathbf{u}(t)\phi\|_2 \le c\|\mathrm{curl}\ \mathbf{v}(t)\|_2 + c\|\mathbf{u}(t)\|_{\frac{6}{7}}.$$

However, we do not have any previous result for the temporal decay of $\|\mathbf{u}(t)\|_{\frac{6}{7}}$. This is why, only for $p > 3$, the estimate of $\|\mathrm{curl}\ \mathbf{v}(t)\|_p$ implies directly the decay estimate of $\|\mathbf{u}(t)|x|^2\|_p$.

3.2. Integral representation without the pressure term

We consider the fundamental solution for the nonstationary Stokes equation, $(\mathbb{V}, \mathbf{Q})$, $\mathbb{V} = (\mathbf{V}^1, \mathbf{V}^2, \mathbf{V}^3)$, $\mathbf{Q} = (Q^1, Q^2, Q^3)$ written by

$$\mathbf{V}^i_t(x) := \mathbf{V}^i(x,t) := \Gamma_t(x)\mathbf{e}^i + \nabla\frac{\partial}{\partial x_i}(N * \Gamma_t)(x),\ \ i = 1,2,3,$$

and $Q^i(x,t) := -\delta(t)\frac{\partial}{\partial x_i}N$. Here, $\Gamma_t(x) := \Gamma(x,t) := (4\pi t)^{-\frac{3}{2}}e^{-|x|^2/4t}$, $N(x) := \frac{1}{4\pi}\frac{1}{|x|}$, and $\mathbf{e}^i$ is a standard unit vector of which i-th component is 1. (See Chapter 4 of [23] for the integral representation by the fundamental solution of the Stokes equations.)

Set $\omega^i_t(x) = \omega^i(x,t) = (N * \Gamma_t)(x)\mathbf{e}^i$. We notice that

$$\nabla\times\nabla\times\omega^i = -\Delta\omega^i + \nabla\mathrm{div}\omega^i = \Gamma_t\mathbf{e}^i + \nabla\frac{\partial}{\partial x_i}(N*\Gamma_t) = \mathbf{V}^i,$$

hence, we have the identity

$$\nabla_y \times [\phi(y)\nabla_y \times \omega^i(x-y,t-\tau)] = \phi(y)V^i(x-y,t-\tau) + \mathbf{R}^i_1(x,y,t-\tau),$$

where

$$\mathbf{R}^i_1(x,y,t-\tau) = \nabla\phi\times\nabla_y\times\omega^i(x-y,t-\tau).$$

It is easy to check that $\mathbf{R}^i_1 = 0$ on $\partial\Omega$.

We multiply $\nabla_y \times [\phi(y)\nabla_y\times\omega^i(x-y,t-\tau)]$ to (1.1), and integrate over $\Omega\times(0,t-\epsilon)$, and then we have

(3.1)
$$\begin{aligned}&\int_0^{t-\epsilon}\int_\Omega\Big(\frac{\partial\mathbf{u}}{\partial\tau} - \Delta\mathbf{u} + (u\cdot\nabla)\mathbf{u}\Big)\cdot\nabla_y\times[\phi(y)\nabla_y\times\omega^i(x-y,t-\tau)]dyd\tau\\ &= -\int_0^{t-\epsilon}\int_\Omega \nabla p(y)\cdot\nabla_y\times[\phi(y)\nabla_y\times\omega^i(x-y,t-\tau)]dyd\tau = 0.\end{aligned}$$

Let

$$\mathbf{R}^i_2(x,y,t-\tau) = -2(\nabla\phi\cdot\nabla)\mathbf{V}^i - (\Delta\phi)\mathbf{V}^i.$$

Taking integration by parts and observing $(-\partial_\tau - \Delta_y)\mathbf{V}^i = 0$, the identity (3.1) becomes

(3.2)

$$\begin{aligned}
&\int_0^{t-\epsilon}\int_\Omega \mathbf{u}(y,\tau)\cdot\Big(\mathbf{R}_2^i(x,y,t-\tau) - (\frac{\partial}{\partial\tau} + \Delta_y)\mathbf{R}_1^i(x,y,t-\tau)\Big)dyd\tau \\
&+ \int_\Omega \mathbf{u}(y,t-\epsilon)\cdot\nabla_y\times\big\{\phi(y)\nabla_y\times[\omega^i(x-y,\epsilon)]\big\}dy \\
&- \int_\Omega \mathbf{u}_0(y)\cdot\nabla_y\times\big\{\phi(y)\nabla_y\times[\omega^i(x-y,t)]\big\}dy \\
= &- \int_0^{t-\epsilon}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}(y,\tau)\cdot\Big(\phi(y)\mathbf{V}^i(x-y,t-\tau) + \mathbf{R}_1^i(x,y,t-\tau)\Big)dyd\tau.
\end{aligned}$$

We observe that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_\Omega \mathbf{u}(y,t-\epsilon)\cdot\nabla_y\times\big\{\phi(y)\nabla_y\times[\omega^i(x-y,\epsilon)]\big\}dy \\
= &-\nabla_x\times\big\{N*[\phi(y)\nabla_y\times\mathbf{u}]\big\}\cdot\mathbf{e}^i \equiv -(\nabla_x\times\mathbf{v})\cdot\mathbf{e}^i = -(\nabla_x\times\mathbf{v})_i
\end{aligned}$$

and

$$\begin{aligned}
&\int \mathbf{u}_0(y)\cdot\nabla_y\times\{\phi(y)\nabla_y\times\omega^i(x-y,t)\}dy \\
= &-\Big(\nabla_x\times\big[N*\{\phi(y)\nabla_y\times\mathbf{u}_0\}\big]\cdot\mathbf{e}^i\Big)*\Gamma_t = -(\nabla_x\times\mathbf{v}_0)_i*\Gamma_t,
\end{aligned}$$

where $\mathbf{v}_0 = N*[\phi(y)\nabla_y\times\mathbf{u}_0]$, and $\mathbf{v} = N*[\phi(y)\nabla_y\times\mathbf{u}]$. From (3.2) we obtain

$$\begin{aligned}
(\nabla_x\times\mathbf{v})_i = &-\int_0^t (u_ju_i\phi)(\tau)*\partial_{y_j}\Gamma_{t-\tau}d\tau - \int_0^t (u_ju_i\partial_{y_j}\phi)(\tau)*\Gamma_{t-\tau}d\tau \\
&- \int_0^t (u_ju_k\phi)(\tau)*\partial_{y_i}\partial_{y_j}\partial_{y_k}(N*\Gamma_{t-\tau})d\tau \\
&- \int_0^t (u_ju_k\partial_{y_j}\phi)(\tau)*\partial_{y_i}\partial_{y_k}(N*\Gamma_{t-\tau})d\tau \\
&- \int_0^t\int (u_j\mathbf{u}(y,\tau))\cdot\partial_{y_j}\mathbf{R}_1^i(x,y,t-\tau)dyd\tau + (\nabla_x\times\mathbf{v}_0)_i*\Gamma_t \\
&- \int_0^t\int \mathbf{u}(y,\tau)\cdot(\partial_\tau + \Delta_y)\mathbf{R}_1^i(x,y,t-\tau)dyd\tau \\
&+ \int_0^t\int \mathbf{u}(y,\tau)\cdot\mathbf{R}_2^i(x,y,t-\tau)dyd\tau \\
= &\, I_1^i + I_2^i + \cdots + I_8^i.
\end{aligned}$$

From straightforward calculations we have that

$$\|\partial^{2+\beta}(N*\Gamma_{t-\tau})\|_s \le c\|\partial^\beta\Gamma_{t-\tau}\|_s \le c(t-\tau)^{-\frac{|\beta|}{2}-\frac{3}{2}(1-\frac{1}{s})}$$

for $1 \lesssim s < \infty$. This estimate will be used in the proof of proposition 3.2.

3.3. Proof of Proposition 3.2 for $2 \le s < 6$

Recall the results in the previous section: for any small $1 \gg \delta > 0$ there is a constant c_δ such that

$$\|\mathrm{curl}\,\mathbf{v}(t)\|_2 \le c_\delta(1+t)^{\frac{7}{4}-\frac{3}{2r}+\delta}, \quad \||x|\mathbf{u}(t)\|_2 \le c_\delta(1+t)^{\frac{5}{4}-\frac{3}{2r}+\delta}.$$

(Of course, the above estimates holds for $\mathbf{u}_0$ satisfying the hypothesis of theorem 1.1.)

By the help of the generalized Minkowski's and Young's convolution inequalities, and (1.4), we obtain the estimate for $I_2 + I_4$:

$$\begin{aligned} \|I_2+I_4\|_s &\le \int_0^t \|u_j u_i \partial_{y_j}\phi\|_{2s/(s+2)}\|\Gamma_{t-\tau}\|_2 \\ &\quad + \|u_j u_k \partial_{y_j}\phi\|_{2s/(s+2)}\|\nabla^2(N*\Gamma_{t-\tau})\|_2 d\tau \\ &\le c\int_0^t \|\mathbf{u}\phi^{\frac12}\|_2\|\mathbf{u}\|_s(t-\tau)^{-\frac34}d\tau \\ &\le c_\delta \int_0^t (1+\tau)^{\frac54-\frac{3}{2r}+\delta}\tau^{-\frac12+\frac{3}{2s}}(1+\tau)^{-\frac{3}{2r}+\frac12}(t-\tau)^{-\frac34}d\tau \\ &\le c_\delta t^{\max\{\frac32-\frac3r+\frac{3}{2s}+\delta,-\frac34\}}, \qquad t\ge 2, \end{aligned} \tag{3.3}$$

for $2 \le s < \infty$.

For the estimate of $I_1 + I_3$, we observe the following inequality by (2.2) and (1.4)

$$\begin{aligned} \|u_j u_k \phi\|_{\frac{18s}{12+7s}} &= \|u_j(\mathrm{curl}\,\mathbf{v}-\mathbf{R}_0)_k\|_{\frac{18s}{12+7s}} \\ &\le c\|\mathbf{u}\|_{\frac{9s}{6-s}}\|\mathrm{curl}\,\mathbf{v}\|_2 + c\|\mathbf{u}\|_{\frac{9s}{6+2s}}\|\mathbf{R}_0\|_6 \\ &\le c\|\mathbf{u}\|_{\frac{9s}{6-s}}\|\mathrm{curl}\,\mathbf{v}\|_2 + c\|\mathbf{u}\|_{\frac{9s}{6+2s}}\|\mathbf{u}\phi^{1/2}\|_2 \\ &\le c_\delta t^{-\frac23+\frac1s}(1+t)^{-\frac{3}{2r}+\frac12}(1+t)^{-\frac{3}{2r}+\frac74+\delta} \\ &\quad + c_\delta t^{-\frac16+\frac1s}(1+t)^{-\frac{3}{2r}+\frac12}(1+t)^{-\frac{3}{2r}+\frac54+\delta} \end{aligned}$$

for $2 \le s < 6$. Making use of the above inequality and by the help of the generalized Minkowski's and Young's convolution inequalities, we obtain the following estimates: for $2 \le s < 6$

$$\begin{aligned} &\|I_1+I_3\|_s \\ &\le \int_0^t \|u_j u_i \phi\|_{\frac{18s}{12+7s}}\|\nabla\Gamma_{t-\tau}\|_{\frac{18s}{11s+6}}d\tau \\ &\le c_\delta \int_0^t \Big[\tau^{-\frac23+\frac1s}(1+\tau)^{-\frac3r+\frac94+\delta} + \tau^{-\frac16+\frac1s}(1+\tau)^{-\frac3r+\frac74+\delta}\Big](t-\tau)^{-\frac{13}{12}+\frac{1}{2s}}d\tau \\ &\le c_\delta t^{\max\{\frac32-\frac3r+\frac{3}{2s}+\delta,-\frac{13}{12}+\frac{1}{2s}\}},\ t\ge 2. \end{aligned} \tag{3.4}$$

Since $\partial_{y_l}\mathbf{R}_1^i = \nabla\partial_l\phi \times \nabla_y \times \omega^i + \nabla\phi \times \nabla_y \times \partial_l\omega^i$, by the generalized Minkowski's inequality we have

$$\|I_5\|_s \le \int_0^t \|(u_j\mathbf{u} \times \nabla\partial_j\phi) * \nabla_y \times \omega_{t-\tau}^i\|_s d\tau + \int_0^t \|(u_j\mathbf{u} \times \nabla\phi) * \nabla_y \times \partial_j\omega_{t-\tau}^i\|_s d\tau = J_1 + J_2$$

for $1 < s < \infty$. By Young's convolution inequality, J_1 is estimated as follows

$$\begin{aligned} J_1 &\le \int_0^t \|u_j u_k \nabla\partial_j\phi\|_1 \|\nabla_y\omega_{t-\tau}^i\|_s d\tau \le \int_0^t \|\mathbf{u}\|_2^2 \|\Gamma_{t-\tau}\|_{3s/(3+s)} d\tau \\ &\le \int_0^t (1+\tau)^{-3(\frac{1}{r}-\frac{1}{2})}(t-\tau)^{-\frac{3}{2}(1-\frac{3+s}{3s})} d\tau \qquad (3.5) \\ &\le ct^{\max\{\frac{3}{2}-\frac{3}{r}+\frac{3}{2s},-1+\frac{3}{2s}\}}, \ t \ge 2 \end{aligned}$$

for $2 \le s < \infty$.

By the similar reasoning as in I_1, I_2, I_3 and I_4, J_2 is estimated by

(3.6)
$$\begin{aligned} J_2 &\le \int_0^t \|u_j u_k \nabla\phi\|_{2s/(2+s)} \|\nabla^2\omega_{t-\tau}^i\|_2 d\tau \le c\int_0^t \|\mathbf{u}\phi^{1/2}\|_2 \|\mathbf{u}\|_s \|\Gamma_{t-\tau}\|_2 d\tau \\ &\le c_\delta \int_0^t (1+\tau)^{\frac{5}{4}-\frac{3}{2r}+\delta}\tau^{-\frac{1}{2}+\frac{3}{2s}}(1+\tau)^{-\frac{3}{2r}+\frac{1}{2}}(t-\tau)^{-\frac{3}{4}} d\tau \\ &\le ct^{\max\{\frac{3}{2}-\frac{3}{r}+\frac{3}{2s}+\delta,-\frac{3}{4}\}}, \ t \ge 2, \end{aligned}$$

for $2 \le s < \infty$.

Applying Young's convolution inequality to I_6, we obtain that

$$\begin{aligned} \|(\nabla \times \mathbf{v}_0) * \Gamma_t\|_s &\le \|\mathbf{u}_0\phi\|_r \|\Gamma_t\|_{\frac{sr}{sr+r-s}} + \|\nabla N * [\mathbf{u}_0\nabla\phi]\|_{\frac{3r}{3-r}} \|\Gamma_t\|_{\frac{3sr}{4sr+3r-3s}} \\ &\le ct^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{s})}\|\phi\mathbf{u}_0\|_r + ct^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{s})+\frac{1}{2}}\|\mathbf{u}_0\nabla\phi\|_r. \end{aligned}$$

for $r \le s < \infty$. (Here we note that $\frac{sr}{sr+r-s}$ if $s \ge r$ and $\frac{3sr}{4sr+3r-3s} \ge 1$ if $r \le \frac{3s}{s+3}$.) Hence, if $|x|^2\mathbf{u}_0 \in L^r$ and $|x|\mathbf{u}_0 \in L^r$, then

$$\|I_6\|_s \le ct^{\max\{-\frac{3}{2}(\frac{1}{r}-\frac{1}{s}),-\frac{3}{2}(\frac{1}{r}-\frac{1}{s})+\frac{1}{2}\}}, \ t \ge 2, \qquad (3.7)$$

for $r \le \frac{3s}{3+s}$, $r \le s < \infty$.

Since $\partial_t\Gamma - \Delta_x\Gamma = 0$, we have that

$$\partial_\tau\omega^i(x-y,t-\tau) + \Delta_y\omega^i(x-y,t-\tau) = N * (-\partial_t\Gamma + \Delta_x\Gamma)(x-y,t-\tau) = 0.$$

Hence

$$\begin{aligned} (\partial_\tau + \Delta_y)R_1^i(x,y,t-\tau) &= \nabla_y\phi(y) \times \nabla_y \times (\partial_\tau\omega^i + \Delta_y\omega^i)(x-y,t-\tau) \\ &\quad + 2\nabla_y\partial_{y_k}\phi \times \partial_{y_k}\nabla_y \times \omega^i(x-y,t-\tau) + \nabla_y\Delta_y\phi \times \nabla_y \times \omega^i(x-y,t-\tau) \\ &= 2\partial_{y_k}(\nabla_y\phi) \times \partial_{y_k}\nabla_y \times \omega^i(x-y,t-\tau) + \nabla_y\Delta_y\phi \times \nabla_y \times \omega^i(x-y,t-\tau). \end{aligned}$$

Recall $\nabla\Delta\phi$ is compactly supported in the ball $D = \{|x| \le 2\}$. Hence, I_7 is bounded by

$$\begin{aligned}
\|I_7\|_s &\le c\int_0^t \|(\mathbf{u}\times\nabla\partial_k\phi)*\partial_k\nabla\times\omega^i\|_s + \|(\mathbf{u}\times\nabla\Delta\phi)*\nabla\times\omega^i\|_s d\tau \\
&\le c\int_0^t \|\mathbf{u}\times\nabla\partial_k\phi\|_{\frac{3}{2}}\|\nabla^2\omega^i\|_{\frac{3s}{3+s}} + \|\mathbf{u}\times\nabla\Delta\phi\|_{L^1(D)}\|\nabla\omega^i\|_s d\tau \\
&\le c\int_0^t \|\mathbf{u}\|_{\frac{3}{2}}\|\Gamma_{t-\tau}\|_{\frac{3s}{3+s}} + c\|\mathbf{u}\|_{L^{\frac{3}{2}}(D)}\|\Gamma_{t-\tau}\|_{\frac{3s}{3+s}} d\tau \qquad (3.8)\\
&\le c\int_0^t \tau^{\frac{1}{2}}(1+\tau)^{\frac{1}{2}-\frac{3}{2r}}(t-\tau)^{-1+\frac{3}{2s}} d\tau \\
&\le ct^{\max\{1-\frac{3}{2r}+\frac{3}{2s},-1+\frac{3}{2s}\}},\ t\ge 2,
\end{aligned}$$

for $2 \le s < \infty$.

Now we consider the estimate of I_8. Observe that

$$\|I_8\|_s \le \int_0^t \|(\mathbf{u}\partial_j\phi)*\partial_j\mathbf{V}^i\|_s + \|(\mathbf{u}\Delta\phi)*\mathbf{V}^i\|_s d\tau = P_1 + P_2.$$

It is easy to see

$$\begin{aligned}
P_2 &\le \int_0^t \|\mathbf{u}\|_{\frac{3}{2}}\|\mathbf{V}^i\|_{\frac{3s}{3+s}} d\tau \le \int_0^t \|\mathbf{u}\|_{\frac{3}{2}}\|\nabla^2\omega^i_{t-\tau}\|_{\frac{3s}{3+s}} d\tau \qquad (3.9)\\
&\le ct^{\max\{1-\frac{3}{2r}+\frac{3}{2s},-1+\frac{3}{2s}\}},\ t\ge 2,
\end{aligned}$$

for $2 \le s < \infty$. For the estimate of P_1 we observe

$$\begin{aligned}
\|\mathbf{u}\phi^{1/2}\|^2_{4s/(4+s)} = \|\mathbf{u}\cdot(\mathbf{u}\phi)\|_{2s/(s+4)} &= \|\mathbf{u}\cdot(\text{curl }\mathbf{v}-\mathbf{R}_0)\|_{2s/(s+4)} \\
&\le c\|\mathbf{u}\|_{s/2}\|\text{curl }\mathbf{v}\|_2 + c\|\mathbf{R}_0\|_6\|\mathbf{u}\|_{3s/(s+6)} \\
&\le c\|\mathbf{u}\|_{s/2}\|\text{curl }\mathbf{v}\|_2 + c\|\mathbf{u}\phi^{1/2}\|_2\|\mathbf{u}\|_{3s/(s+6)}.
\end{aligned}$$

for $3 < s < \infty$. Observe that

$$\begin{aligned}
P_1 &\le \int_0^t c\|\mathbf{u}\partial_j\phi\|_{4s/(4+s)}\|\nabla\Gamma\|_{4/3} d\tau \\
&\le \int_0^t \left\{\|\mathbf{u}\|^{1/2}_{s/2}\|\text{curl}\,\mathbf{v}\|^{1/2}_2 + \|\mathbf{u}\phi^{1/2}\|^{1/2}_2\|\mathbf{u}\|^{1/2}_{3s/(s+6)}\right\}(t-\tau)^{-\frac{7}{8}} d\tau \qquad (3.10)\\
&\le c_\delta\int_0^t \Big[\tau^{-\frac{1}{4}+\frac{3}{4}\frac{2}{s}}(1+\tau)^{-\frac{3}{4r}+\frac{1}{4}}(1+\tau)^{\frac{7}{8}-\frac{3}{4r}+\delta} \\
&\qquad\qquad + \tau^{-\frac{1}{4}+\frac{3}{4}\frac{s+6}{3s}}(1+\tau)^{-\frac{3}{4r}+\frac{1}{4}}(1+\tau)^{\frac{5}{8}-\frac{3}{4r}+\delta}\Big](t-\tau)^{-\frac{7}{8}} d\tau \\
&\le ct^{\max\{1-\frac{3}{2r}+\frac{3}{2s}+\delta,-\frac{7}{8}\}},\ t\ge 2,
\end{aligned}$$

for $3 < s < \infty$.

We consider the inequalities (3.3)-(3.10);

$$(3.3) + (3.5) + \cdots + (3.10) \le c_\delta t^{1+\frac{3}{2s}-\frac{3}{2r}+\delta} \text{ for } t \ge 2,$$

when $r \le \frac{6}{5}$, $2 \le s < \infty$. The above inequality holds for (3.4) when $r \le \frac{6}{5}$, $2 \le s < 6$. Therefore, we have

$$\|\nabla \times \mathbf{v}\|_s \le c_\delta t^{1+\frac{3}{2s}-\frac{3}{2r}+\delta},\ t \ge 2, \tag{3.11}$$

when $r \le \frac{6}{5}$, $2 \le s < 6$. This completes the proof of Proposition 3.2.

Remark 3.5. As a matter of fact, we can derive the above inequality even for $t \le 2$. We omit the details, but we will use this fact in the next section:

$$\|\nabla \times \mathbf{v}\|_s \le c_\delta t^{1+\frac{3}{2s}-\frac{3}{2r}+\delta},\ t > 0,$$

when $r < \frac{6}{5}$, $2 \le s < 6$.

3.4. Proof of Proposition 3.2 for $p \ge 6$

By Lemma 3.1 and (3.11), we obtain that for $3 < p < 6$,

$$\|\mathbf{u}\phi\|_p \le c_\delta t^{-\frac{3}{4}+\frac{3}{2p}}(1+t)^{\frac{7}{4}-\frac{3}{2r}+\delta} \quad \text{for } t > 0. \tag{3.12}$$

Now we derive decay rate estimate of $\|\nabla \times \mathbf{v}\|_p$ for $p \ge 6$.

Observe that the estimates for $I_2 + I_4$, $I_5, \dots, I_8$ in the proof of Proposition 3.2 do hold for any $p \ge 2$, and the estimate for the term $I_1 + I_3$ holds only for $p < 6$. Hence we have only to obtain the estimate for $I_1 + I_3$ for $p \ge 6$,

Making use of (3.12) and the temporal estimate (1.4), by Minkowski's and Hölder's inequalities, we have

$$\begin{aligned} \|I_1 + I_3\|_p &\le c \int_0^t \|\mathbf{u}\phi\|_{\frac{3r}{3-2r}} \|\mathbf{u}\|_{\frac{2pr}{2pr-2p+r}} \|\nabla \Gamma_{t-\tau}\|_{\frac{6p}{4p+3}} \\ &\le c_\delta \int_0^t \tau^{-\frac{3}{4}+\frac{3}{4p}}(1+\tau)^{\frac{9}{4}-\frac{3}{r}+\delta}(t-\tau)^{-1+\frac{3}{4p}}\,d\tau \\ &\le c_\delta t^{\max\{\frac{3}{2}-\frac{3}{r}+\frac{3}{2p}+\delta,-1+\frac{3}{4p}\}},\ t \ge 2. \end{aligned} \tag{3.13}$$

If we follow the same reasoning in the last part of the previous lemma, we complete the proof.

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