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COMMON FIXED POINT AND INVARIANT APPROXIMATION RESULTS

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ABSTRACT. Necessary conditions for the existence of common fixed points for noncommuting mappings satisfying generalized contractive conditions in the setup of certain metrizable topological vector spaces are obtained. As applications, related results on best approximation are derived. Our results extend, generalize and unify various known results in the literature.

1. Introduction and preliminaries

Let X be a linear space. A p-norm on X is a real valued function $\|.\|_p$ on X with $0 < p \leq 1$, satisfying the following conditions:

(i) $\|x\|_p \geq 0$ and $\|x\|_p = 0$ if and only if $x = 0$,
(ii) $\|\lambda x\|_p = |\lambda|^p \|x\|_p$,
(iii) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$,

for all $x, y \in X$ and all scalars λ. The pair $(X, \|.\|_p)$ is called a p-normed space. It is a metric linear space with $d_p(x, y) = \|x - y\|_p$ for all $x, y \in X$, defining a translation invariant metric d_p on X. If $p = 1$, then we obtain the concept of a normed linear space. It is well known that the topology of every Hausdorff locally bounded topological linear space is given by some p-norm, $0 < p \leq 1$ (see [8], [12] and references mentioned therein). The spaces l_p and $L_p[0, 1]$, $0 < p \leq 1$ are p-normed spaces. A p-normed space is not necessarily a locally convex space. Recall that the dual space X^* (the dual of X) separates points of X if for each non-zero x in X, there exists f in X^* such that $f(x) \neq 0$. In this case weak topology on X is well defined and is Hausdorff. We mention that, if X is not locally convex, then X^* need not separate the point of X. For example, if $X = L_p[0, 1]$, $0 < p < 1$, then $X^* = \{0\}$ ([17]). However there are some non-locally convex spaces (such as the p- normed spaces l_p $0 < p < 1$) whose dual separates the points of X ([13]).

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In the sequel, we will assume that X^* separates points of a p-normed space X whenever weak topology is under consideration.

Definition 1.1. Let $f : X \to X$ be a mapping. A point $x \in X$ is called a *fixed point* of f if $fx = x$. We denote the set of fixed points of f by $F(f)$.

Definition 1.2. Let E be a subset of a p-normed space X. For $u \in E$, the set E is called *u-starshaped or starshaped with respect to u* if $tx + (1-t)u \in E$ for each $x \in E$. Note that E is *convex* if E is starshaped with respect to every $u \in E$.

Definition 1.3. Let E be a q-starshaped subset of a p-normed space X, $q \in F(g)$ and E be both f and g invariant where $f, g : X \to X$. Put,

$$Y_q^{fx} = \{y_\lambda : y_\lambda = (1-\lambda)q + \lambda fx, \ \ \lambda \in (0,1]\}.$$

Now, for each x in X, $d_p(gx, Y_q^{fx}) = \inf\limits_{\lambda \in [0,1]} \|gx - y_\lambda\|_p$.

Definition 1.4. Let $f, g : X \to X$ be mappings. The map f is said to be:

(1) *g-contractive* if there exists $k \in (0,1)$ such that
$$\|fx - fy\|_p \le k \|gx - gy\|_p$$

(2) *asymptotically g-nonexpansive* if there exists a sequence $\{k_n : k_n \ge 1\}$ with $\lim\limits_{n\to\infty} k_n = 1$ such that
$$\|f^n x - f^n y\|_p \le k_n \|gx - gy\|_p$$
for each x, y in E and $n \in N$. If $g = I$ (identity map), then f is asymptotically nonexpansive mapping. If $k_n = 1$, for all $n \in N$, then f is known as a g-nonexpansive mapping;

(3) *R-weakly commuting* if there exists a real number $R > 0$ such that
$$\|fgx - gfx\|_p \le R \|gx - fx\|_p$$
for all x in E. If $R = 1$, then maps are called weakly commuting [20];

(4) *R-subweakly commuting* if there exists a real number $R > 0$ such that
$$\|fgx - gfx\|_p \le R d_p(gx, Y_q^{fx});$$
for all $x \in E$;

(5) *uniformly R-subweakly commuting* if there exists a real number $R > 0$ such that
$$\|f^n gx - gf^n x\|_p \le R d_p(gx, Y_q^{f^n x});$$
for all $x \in E$.

(6) C_q-commuting if $gfx = fgx$ for all $x \in C_q(g,f)$, where $C_q(g,f) = U\{C(g, f_\lambda) : 0 \le \lambda \le 1\}$ and $f_\lambda x = (1-\lambda)q + \lambda fx$.

Clearly C_q-commuting mappings are weakly compatible but not conversely in general. R-subcommuting and R-subweakly commuting maps are C_q-commuting but the converse does not hold in general [2].

Definition 1.5. A self mapping f on a p-normed space X is said to be

(1) *affine* on E in X, if
$$f((1-\lambda)x+\lambda fy)=(1-\lambda)fx+\lambda fy,$$
for all $x, y \in E$ and $\lambda \in (0,1)$;

(2) *uniformly asymptotically regular* on E if for each $\epsilon > 0$ there exists a positive integer N such that $\|f^n x - f^n y\|_p < \epsilon$ for all $n \geq N$ and for all x in E.

Definition 1.6. Let X be a p-normed space, M a closed subset of X. If there exists a $y_0 \in M$ such that $\|x - y_0\|_p = d_p(x, M)$, then y_0 is called a *best approximation to x out of M*. We denote by $P_M(x)$, the set of all best approximation to x out of M.

Jungck [9] generalized the notion of weak commutativity by introducing compatible maps and then weakly compatible maps ([11]). There are examples that show each of these generalizations of commutativity is a proper extension of the previous definition. Also during this time, many authors established fixed point theorems for pairs of maps (see for examples, [1], [4], [5], [10], and references therein). Shahzad [18] introduced the class of noncommuting mappings called R-subweakly commuting mappings and applied it to S-nonexpansive mappings in normed spaces. It is well known that uniformly R-subweakly commuting map is a R-subweakly commuting but converse does not hold in general.

This paper deals with the study of common fixed point for C_q-commuting and uniformly R-subweakly commuting mappings in the setting of locally bounded topological vector spaces and locally convex topological vector spaces, and hence extends several results in ([3], [14], [7] and [18]). We also establish results on invariant approximation for these mappings.

2. Common fixed point results

The following result extends and improves Theorem 2.2 of [1], Theorem 2.1 of [7] and Theorem 2.2 of [19].

Theorem 2.1. *Let E be a nonempty q-starshaped complete subset of a p-normed space X, and T, f and g be self-mappings on X. Suppose $q \in F(f) \cap F(g)$, f and g are continuous and affine on E, and $T(E) \subset f(E) \cap g(E)$. If the pairs $\{T, f\}$ and $\{T, g\}$ are C_q-commuting and satisfy for all $x, y \in E$,*

$$\|Tx - Ty\|_p \leq \max\{\|fx - gy\|_p, d_p(fx, Y_q^{T(x)}), d_p(gy, Y_q^{T(y)}), \frac{1}{2}[d_p(fx, Y_q^{T(y)}) + d_p(gy, Y_q^{T(x)})]\}, \tag{2.1}$$

then T, f and g have a common fixed point in E provided one of the following conditions holds;

(i) *E is complete, T is continuous and $cl(T(E))$ is compact;*

(ii) *E is weakly compact, $(f - T)$ is demiclosed at 0 and X is complete.*

Proof. Define $T_n : E \to E$ by

$$T_n x = (1-\lambda_n)q + \lambda_n Tx,$$

where $\lambda_n \in (0,1)$ with $\lim_{n\to\infty} \lambda_n = 1$. Since E is q-starshaped, T_n is self mapping on E for each $n \geq 1$. As f and T are C_q-commuting and f is affine on E with $fq = q$. For each $x \in C_q(f,T)$

$$\begin{aligned} fT_n x &= f((1-\lambda_n)q + \lambda_n Tx) = (1-\lambda_n)q + \lambda_n fTx \\ &= (1-\lambda_n)q + \lambda_n Tfx = T_n fx. \end{aligned}$$

Thus f and T_n commute for each $x \in C(f,T_n) \subset C_q(f,T)$. Hence f and T_n are weakly compatible for all n. Also since g and T are C_q- commuting and g is affine on E with $gq = q$, therefore g and T_n are weakly compatible for all n. Also,

$$\begin{aligned} \|T_n x - T_n y\|_p &\leq (\lambda_n)^p \|Tx - Ty\|_p \\ &\leq (\lambda_n)^p \max\{\|fx - gy\|_p, d_p(fx, Y_q^{T(x)}), \\ &\quad d_p(gy, Y_q^{T(y)}), \frac{1}{2}[d_p(fx, Y_q^{T(y)}) + d_p(gy, Y_q^{T(x)})]\} \\ &\leq (\lambda_n)^p \max\{\|fx - gy\|_p, \|fx - T_n x\|_p \\ &\quad \|gy - T_n y\|_p, \frac{1}{2}[\|fx - T_n y\|_p + \|gy - T_n x\|_p]\}. \end{aligned}$$

By Corollary 3.1 of [6], for each $n \geq 1$ there exists x_n in E such that x_n is a common fixed point of f, g, and T_n. Hence we have the result for each case (i) and (ii).

(i) The compactness of $cl(T(E))$ implies that there exists a subsequence $\{Tx_k\}$ of $\{Tx_n\}$ such that $Tx_k \to y$ as $k \to \infty$. Now, the definition of T_k implies $x_k = (1-\lambda_k)q + \lambda_k Tx_k$, $\lambda_k \longrightarrow 1$ as $k \longrightarrow \infty$ further implies $x_k \to y$, so by continuity of T, f and g, we have $y \in F(T) \cap F(f) \cap F(g)$.

(ii) Since E is weakly compact, there is a subsequence $\{x_k\}$ of $\{x_n\}$ converging weakly to some y in E. But f and g being affine and continuous are weakly continuous, also weak topology on X is Hausdorff so $fy = y = gy$. Since E is bounded, so $(f-T)x_k = (1-(\lambda_k)^{-1})(x_k - q) \to 0$ as $k \to \infty$. Now demiclosed of $f - T$ implies $(f-T)y = 0$ and hence the result is obtained.

This completes the proof. □

Corollary 2.2. *Let E be a nonempty q-starshaped subset of a p-normed space X, and T, f and g be self-mappings on X. Suppose $q \in F(f) \cap F(g)$, f and g are continuous and affine on E, and $T(E) \subset f(E) \cap g(E)$. If the pairs $\{T, f\}$ and $\{T, g\}$ are R-subweakly commuting mappings satisfying (2.1), then T, f and g have a common fixed point in E provided one of the following conditions holds;*

(i) *E is complete, T is continuous and $cl(T(E))$ is compact;*

(ii) *E is weakly compact, $(f-T)$ is demiclosed at 0 and X is complete.*

Corollary 2.3. *Let E be a nonempty closed q-starshaped subset of a p-normed space X, and T, f be two R-subweakly commuting mappings on E such that $T(E) \subset f(E)$, $q \in F(f)$ and $cl(T(E))$ is compact. If T is continuous f-nonexpansive and f is affine on E, then $F(T) \cap F(f)$ is nonempty.*

Theorem 2.4. *Let E be a nonempty closed subset of a complete p-normed space X and f, g be two mappings on E such that $f(E - \{u\}) \subset g(E - \{u\})$, where $u \in F(g)$. Suppose that f is g-contractive and continuous. If f and g are R-weakly commuting mappings on $E - \{u\}$, then $F(f) \cap F(g)$ is a singleton in E.*

Proof. It can be proved following the similar arguments of those given in the proof of Theorem 1 of [16]. □

Theorem 2.5. *Let E be a nonempty closed q-starshaped subset of a complete p-normed space X, and f, g be two uniformly R-subweakly commuting mappings on $E - \{q\}$ such that $g(E) = E$ and $f(E - \{q\}) \subset g(E - \{q\})$, where $q \in F(g)$. Suppose that f is continuous asymptotically g-nonexpansive with sequence $\{k_n\}$ and g is affine on E. For each $n \geq 1$, define a mapping f_n on E by $f_n x = (1 - \alpha_n)q + \alpha_n f^n x$, where $\alpha_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\lim_{n\to\infty} \lambda_n = 1$. Then for each $n \in N$, $F(f_n) \cap F(g)$ is a singleton.*

Proof. For all $x, y \in E$, we have

$$\begin{aligned} \|f_n x - f_n y\|_p &= (\alpha_n)^p \|f^n x - f^n y\|_p \\ &\leq (\lambda_n)^p \|f^n x - f^n y\|_p . \end{aligned}$$

Moreover,

$$\begin{aligned} \|f_n g x - g f_n x\|_p &= (\alpha_n)^p \|f^n g x - g f^n x\|_p \\ &\leq (\alpha_n)^p R d_p(gx, Y_q^{f^n(x)}) \\ &\leq (\alpha_n)^p R \|gx - f_n x\| , \end{aligned}$$

which show that f_n and g are $(\alpha_n)^p R$-weakly commuting for each $n \in N$. Hence the result immediately follows from Theorem 2.4. □

Theorem 2.6. *Let E be a nonempty closed q-starshaped subset of a p-normed space X and f, g be continuous self-mappings on E such that $g(E) = E$ and $f(E - \{q\}) \subset g(E - \{q\})$, $q \in F(g)$. Suppose f is uniformly asymptotically regular, asymptotically g-nonexpansive and g is affine on E. If $cl(E - \{q\})$ is compact and f and g are uniformly R-subweakly commuting mappings on $E - \{q\}$, then $F(f) \cap F(g)$ is a singleton in E.*

Proof. From Theorem 2.5, for each $n \in N$, $F(f_n) \cap F(g)$ is a singleton in E. Thus,

$$g x_n = x_n = (1 - \alpha_n)q + \alpha_n f^n x_n.$$

Also,

$$\|x_n - f^n x_n\|_p = (1 - \alpha_n)^p \|q - f^n x_n\|_p .$$

Since $f(E-\{q\})$ is bounded, $\|x_n - f^n x_n\|_p \to 0$ as $n \to \infty$. Now,

$$\begin{aligned}\|x_n - fx_n\|_p &\leq \|x_n - f^n x_n\|_p + \left\|f^n x_n - f^{n+1}x_n\right\|_p + \left\|f^{n+1}x_n - fx_n\right\|_p \\ &\leq \|x_n - f^n x_n\|_p + \left\|f^n x_n - f^{n+1}x_n\right\|_p + k_1 \|gf^n x_n - gx_n\|_p \\ &\leq \|x_n - f^n x_n\|_p + \left\|f^n x_n - f^{n+1}x_n\right\|_p + k_1 \|g(f^n x_n - x_n)\|_p ,\end{aligned}$$

which implies that, $\|x_n - fx_n\|_p \to 0$, when $n \to \infty$. As $cl(E-\{q\})$ is compact and E is closed, there exists a subsequence $\{x_k\}$ of $\{x_n\}$ such that $x_k \to x_0 \in E$ as $k \to \infty$, continuity of f implies $f(x_0) = x_0$. Since $f(E-\{q\}) \subset g(E-\{q\})$, it follows that $x_0 = f(x_0) = gy$, for some $y \in E$. Moreover

$$\|fx_k - fy\|_p \leq k_1 \|gx_k - gy\|_p = k_1 \|x_k - x_0\|_p .$$

Taking the limit as $k \to \infty$, we obtain $fx_0 = fy$. Thus, $fx_0 = gy = fy = x_0$. Since f and g are uniformly R-subweakly commuting on $E-\{q\}$, therefore

$$\|fx_0 - gx_0\|_p = \|fgy - gfy\|_p \leq R \|gy - fy\|_p = 0.$$

Hence result follows. □

3. Invariant approximation results

Meinardus [15] was the first to employ fixed point theorem to prove the existence of an invariant approximation in Banach spaces. Subsequently, several interesting and valuable results appeared in the literature of approximation theory ([1], [18] and [21]). In this section, we obtain some results on best approximation as a fixed point of uniformly R-subweakly commuting mappings and C_q-commuting mappings in the setting of a p-normed space.

Theorem 3.1. *Let M be a nonempty subset of a p-normed space X, and f, g be two continuous self-mappings on X such that and $f(\partial M \cap M) \subset M$, $u \in F(f) \cap F(g)$ for some u in X. Suppose f is uniformly asymptotically regular, asymptotically g-nonexpansive and g is affine on $P_M(u)$ with $g(P_M(u)) = P_M(u)$, $q \in F(g)$ and $P_M(u)$ is q-starshaped. If $cl(P_M(u))$ is compact and $P_M(u)$ is complete, f and g are uniformly subweakly commuting mappings on $P_M(u) \cup \{u\}$ satisfying $\|fx - fu\|_p \leq \|gx - gu\|_p$, then $P_M(u) \cap F(f) \cap F(g) \neq \phi$.*

Proof. Let $x \in P_M(u)$. Then $\|x-u\|_p = d_p(x, M)$. Note that for any $\lambda \in (0,1)$

$$\|\lambda u + (1-\lambda)x - u\|_p \leq (1-\lambda)^p \|x-u\|_p < \|x-u\|_p = d_p(x, M),$$

which shows that a line segment $\{\lambda u + (1-\lambda)x : 0 < \lambda < 1\}$ and M are disjoint. Thus x is in interior of M so $x \in \partial M \cap M$ which further implies $fx \in M$, as $f(\partial M \cap M) \subset M$. Since $gx \in P_M(u)$, u is common fixed point of f and g, therefore by given contractive condition, we obtain

$$\begin{aligned}\|fx - u\|_p &= \|fx - fu\|_p \\ &\leq \|gx - gu\|_p = \|gx - u\|_p = d_p(u, M).\end{aligned}$$

Thus $P_M(u)$ is f-invariant. Hence,

$$f(P_M(u)) \subset P_M(u) = g(P_M(u)).$$

□

Now the result follows from Theorem 2.6.

Theorem 3.2. *Let M be a nonempty subset of a p-normed space X and T, f and g be self-mappings on X such that u is common fixed point of f, g and T and $T(\partial M \cap M) \subset M$. Suppose f and g are affine and continuous on $P_M(u)$, where $P_M(u)$ is q-starshaped with $f(P_M(u)) = P_M(u) = g(P_M(u))$ and $q \in F(f) \cap F(g)$. If the pairs $\{T, f\}$ and $\{T, g\}$ are C_q-commuting and satisfy for all $x \in P_M(u) \cup \{u\}$*

(3.1)

$$\|Tx - Ty\|_p \leq \begin{cases} \|fx - gu\|_p, & \text{if } y = u \\ \max\{\|fx - gy\|_p, d_p(fx, Y_q^{T(x)}), d_p(gy, Y_q^{T(y)}), \\ \frac{1}{2}[d_p(fx, Y_q^{T(y)}) + d_p(gy, Y_q^{T(x)})]\}, & \text{if } y \in P_M(u), \end{cases}$$

then $P_M(u) \cap F(T) \cap F(f) \cap F(g)$ is nonempty provided one of the following conditions holds:

(i) *$cl(P_M(u))$ is compact and $P_M(u)$ is complete;*
(ii) *$P_M(u)$ is weakly compact, $(f - T)$ is demiclosed at 0 and X is complete.*

Proof. Let $x \in P_M(u)$. Then $\|x - u\|_p = d_p(x, M)$. Note that for any $\lambda \in (0, 1)$

$$\begin{aligned} \|\lambda u + (1-\lambda)x - u\|_p &= (1-\lambda)^p \|x - u\|_p < \|x - u\|_p \\ &= d_p(x, M), \end{aligned}$$

which shows that a line segment $\{\lambda u + (1-\lambda)x : 0 < \lambda < 1\}$ and M is disjoint. Thus x is in interior of M so $x \in \partial M \cap M$ which further implies $Tx \in M$, as $T(\partial M \cap M) \subset M$. Since $fx \in P_M(u)$, u is common fixed point of f, g and T, therefore by given contractive condition we obtain

$$\begin{aligned} \|Tx - u\|_p &= \|Tx - Tu\|_p \\ &\leq \|fx - gu\|_p = \|fx - u\|_p = d_p(u, M). \end{aligned}$$

Thus $P_M(u)$ is T-invariant. Hence,

$$T(P_M(u)) \subset P_M(u) = f(P_M(u)) = g(P_M(u)).$$

The result follows from Theorem 2.1. □

Corollary 3.3. *Let M be a nonempty subset of a p-normed space X and T, f and g be self-mappings on X such that u is common fixed point of f, g and T and $T(\partial M \cap M) \subset M$. Suppose f and g are affine and continuous on $P_M(u)$, where $P_M(u)$ is q-starshaped with $f(P_M(u)) = P_M(u) = g(P_M(u))$ and $q \in F(f) \cap F(g)$. If the pairs $\{T, f\}$ and $\{T, g\}$ are R-subweakly commuting and satisfy (3.1) for all $x \in P_M(u) \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g)$ is nonempty provided one of the following conditions holds:*

(i) *$cl(P_M(u))$ is compact and $P_M(u)$ is complete;*
(ii) *$P_M(u)$ is weakly compact, $(f-T)$ is demiclosed at* 0 *and X is complete.*

Remark 3.4. Theorem 2.6 extends Theorem 3.4 of [3] to p-normed space. Moreover all results of the paper remain valid in the setup of metrizable locally convex topological vector space (X, d), where d is translation invariant and $d(\lambda x, \lambda y) \leq \lambda d(x, y)$, for each $\lambda \in (0, 1)$ and $x, y \in X$ (recall that d_p is translation invariant and satisfies $d_p(\lambda x, \lambda y) \leq (\lambda)^p d(x, y)$ for any scalar $\lambda \geq 0$).

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