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APPROXIMATING THE STIELTJES INTEGRAL OF BOUNDED FUNCTIONS AND APPLICATIONS FOR THREE POINT QUADRATURE RULES

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Abstract. Sharp error estimates in approximating the Stieltjes integral with bounded integrands and bounded integrators respectively, are given. Applications for three point quadrature rules of $n-$time differentiable functions are also provided.

1. Introduction

In order to approximate the *Stieltjes integral* $\int_a^b f(t)\,du(t)$ with the simpler expression

$$\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt, \tag{1.1}$$

S. S. Dragomir and I. Fedotov [8] introduced in 1998 the following *error functional*

$$D(f,u;a,b) := \int_a^b f(t)\,du(t) - \frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt, \tag{1.2}$$

provided that both the Stieltjes integral $\int_a^b f(t)\,du(t)$ and the *Riemann integral* $\int_a^b f(t)\,dt$ exist.

If the *integrand* f is *Riemann integrable* on $[a,b]$ and the *integrator* $u:[a,b]\to\mathbb{R}$ is $L-$*Lipschitzian*, i.e.,

$$|u(t)-u(s)| \le L|t-s| \quad \text{for each} \;\; t,s\in[a,b], \tag{1.3}$$

then the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and, as pointed out in [8],

$$|D(f,u;a,b)| \le L\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{1.4}$$

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The inequality (1.4) is sharp in the sense that the multiplicative constant $C = 1$ in front of L cannot be replaced by a smaller quantity. Moreover, if there exist the constants $m, M \in \mathbb{R}$ such that

$$m \leq f(t) \leq M \quad \text{for a.e.} \quad t \in [a,b], \tag{1.5}$$

then [8]

$$|D(f,u;a,b)| \leq \frac{1}{2} L(M-m)(b-a). \tag{1.6}$$

The constant $\frac{1}{2}$ is best possible in (1.6).

A different approach in the case of integrands of bounded variation were considered by the same authors in 2001, see [9], where they proved that

$$|D(f,u;a,b)| \leq \max_{t\in[a,b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| \bigvee_a^b (u), \tag{1.7}$$

provided that f is continuous and u is of bounded variation. Here $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. The inequality (1.7) is also sharp.

If we assume that f is $K-$Lipschitzian, then [9]

$$|D(f,u;a,b)| \leq \frac{1}{2} K(b-a) \bigvee_a^b (u), \tag{1.8}$$

with $\frac{1}{2}$ the best possible constant in (1.8).

For various bounds on the *error functional* $D(f,u;a,b)$ where f and u belong to different classes of functions for which the Stieltjes integral exists, see [2], [5], [6] and [7] and the references therein.

The main aim of the present paper is to estimate the error of approximating the Stieltjes integral $\int_a^b f(t)\,du(t)$ with the simpler expression

$$\frac{m+M}{2} \cdot [u(b) - u(a)] \tag{1.9}$$

provided the integrand f is bounded below by m and above by M.

In the dual case, i.e., when $n \leq u(t) \leq M$ on $[a,b]$, the problem under consideration consists of approximating the same Stieltjes integral $\int_a^b f(t)\,du(t)$ with the quantity

$$\left[u(b) - \frac{n+N}{2}\right] f(b) + \left[\frac{n+N}{2} - u(a)\right] f(a). \tag{1.10}$$

Applications for the three point quadrature rule of $n-$differentiable functions are also given.

2. Inequalities for the Stieltjes integral

The following result may be stated.

Theorem 1. *Let $u : [a,b] \to \mathbb{R}$ be a function of bounded variation and $f : [a,b] \to \mathbb{R}$ a function such that there exists the constants $m, M \in \mathbb{R}$ with*

$$m \leq f(t) \leq M \quad \text{for each} \quad t \in [a,b], \tag{2.1}$$

and the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists. Then, by defining the error functional

$$\Delta(f,u,m,M;a,b) := \int_a^b f(t)\,du(t) - \frac{m+M}{2}[u(b)-u(a)],$$

we have the bound

$$|\Delta(f,u,m,M;a,b)| \leq \frac{1}{2}(M-m)\bigvee_a^b(u). \tag{2.2}$$

The constant $\frac{1}{2}$ is best possible in (2.2) in the sense that it cannot be replaced by a smaller quantity.

Proof. Since, obviously, the function $f - \frac{m+M}{2}$ satisfies the inequality

$$\left|f(t) - \frac{m+M}{2}\right| \leq \frac{1}{2}(M-m) \quad \text{for any} \quad t \in [a,b]$$

and the Stieltjes integral $\int_a^b \left(f(t) - \frac{m+M}{2}\right) du(t)$ exists, then

$$\begin{aligned}\left|\int_a^b \left(f(t) - \frac{m+M}{2}\right) du(t)\right| &\leq \sup_{t\in[a,b]}\left|f(t) - \frac{m+M}{2}\right|\bigvee_a^b(u) \\ &\leq \frac{1}{2}(M-m)\bigvee_a^b(u)\end{aligned}$$

and the inequality (2.2) is proved.

Now, assume that (2.2) holds with a positive constant C, i.e.,

$$|\Delta(f,u,m,M;a,b)| \leq C(M-m)\bigvee_a^b(u), \tag{2.3}$$

provided u is of bounded variation on $[a,b]$ and f satisfies (2.1).

If we consider the function $f_0(t) := \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ and $u_0(t) = \frac{1}{2}\left(t - \frac{a+b}{2}\right)^2$, then we observe that the Stieltjes integral $\int_a^b f_0(t)\,du_0(t)$ exists, f_0 is bounded above by $M_0 = 1$ and below by $m_0 = -1$, u_0 is of bounded variation and

$$\bigvee_a^b(u_0) = \int_a^b |u_0'(t)|\,dt = \int_a^b \left|t - \frac{a+b}{2}\right| dt = \frac{(b-a)^2}{4}.$$

Also

$$\int_a^b f_0(t)\,du_0(t) = \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\left(t-\frac{a+b}{2}\right)dt$$
$$= \int_a^b \left|t-\frac{a+b}{2}\right| dt = \frac{(b-a)^2}{4}$$

and replacing f_0 and u_0 in (2.3) produces the inequality

$$\frac{(b-a)^2}{4} \le 2C\cdot\frac{(b-a)^2}{4}$$

which implies that $C \ge \frac{1}{2}$. □

The following corollary provides a natural example of functions f that can be chosen to fulfill the conditions in the above theorem.

Corollary 1. *Let $u:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$ and f a continuous function on $[a,b]$. Then*

$$\left|\widetilde{\Delta}(f,u;a,b)\right| \le \frac{1}{2}\left[\max_{t\in[a,b]} f(t) - \min_{t\in[a,b]} f(t)\right]\bigvee_a^b(u), \tag{2.4}$$

where

$$\widetilde{\Delta}(f,u;a,b) := \int_a^b f(t)\,du(t) - \frac{\min_{t\in[a,b]} f(t) + \max_{t\in[a,b]} f(t)}{2}[u(b)-u(a)].$$

The constant $\frac{1}{2}$ is best possible.

Proof. For the sharpness of the constant, we cannot use the above example since f_0 was not continuous on $[a,b]$.

Let us now consider $u_0(t) = \operatorname{sgn}\left(t-\frac{a+b}{2}\right)$ and $f_0(t) = \left|t-\frac{a+b}{2}\right|$. The Stieltjes integral $\int_a^b f_0(t)\,du_0(t)$ exists and

$$\int_a^b f_0(t)\,du_0(t)$$
$$= f_0(t)\,u_0(t)\Big|_a^b - \int_a^b u_0(t)\,df_0(t)$$
$$= \frac{b-a}{2}+\frac{b-a}{2} - \left[\int_a^{\frac{a+b}{2}}(-1)\,d\left(\frac{a+b}{2}-t\right) + \int_{\frac{a+b}{2}}^b (1)\,d\left(t-\frac{a+b}{2}\right)\right]$$
$$= 0$$

we have then

$$\left|\widetilde{\Delta}(f_0,u_0;a,b)\right| = \frac{b-a}{2}.$$

Also

$$\frac{1}{2}\left[\max_{t\in[a,b]} f_0(t) - \min_{t\in[a,b]} f_0(t)\right]\bigvee_a^b(u_0) = \frac{b-a}{2},$$

which shows that the equality case holds in (2.4). □

The following result providing bounds for the Lipshitzain integrators may be stated as well:

Theorem 2. *If $u : [a, b] \to \mathbb{R}$ is $L-$Lipschitzian and $f : [a, b] \to \mathbb{R}$ is Riemann integrable and satisfies the condition* (2.1), *then*

$$|\Delta(f, u, m, M; a, b)| \leq \frac{1}{2}(M - m) L (b - a). \tag{2.5}$$

The constant $\frac{1}{2}$ is best possible.

Proof. It is well known that if p is Riemann integrable on $[a, b]$ and v is $L-$Lipschitzian on $[a, b]$, then the Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and

$$\left| \int_a^b p(t)\, dv(t) \right| \leq L \int_a^b |p(t)|\, dt. \tag{2.6}$$

Now, taking into account that $f - \frac{m+M}{2}$ is Riemann integrable, by making use of (2.6) we have

$$\begin{aligned} \left| \int_a^b \left(f(t) - \frac{m+M}{2} \right) du(t) \right| &\leq L \int_a^b \left| f(t) - \frac{m+M}{2} \right| dt \\ &\leq \frac{1}{2}(M - m) L (b - a) \end{aligned}$$

and the desired inequality (2.5) is obtained.

To prove the sharpness of the constant $\frac{1}{2}$, assume that the inequality (2.5) holds with a positive constant D, i.e.,

$$|\Delta(f, u, m, M; a, b)| \leq D (M - m) L (b - a), \tag{2.7}$$

provided f is Riemann integrable and satisfies (2.1) while u is Lipschitz continuous with the constant $L > 0$.

Consider the functions $f_0(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ and $u_0(t) = \left|t - \frac{a+b}{2}\right|$. It is obvious that f_0 is Riemann integrable and $M_0 = 1$, $m_0 = -1$. Since, by the triangle inequality we have

$$|u_0(t) - u_0(s)| = \left| \left| t - \frac{a+b}{2} \right| - \left| s - \frac{a+b}{2} \right| \right| \leq |t - s|,$$

for any $t, s \in [a, b]$, hence u_0 is Lipschitzian with the constant $L = 1$. Now, observe that

$$\begin{aligned} \int_a^b f_0(t)\, du_0(t) &= \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) d\left(\left| t - \frac{a+b}{2} \right| \right) \\ &= \int_a^{\frac{a+b}{2}} (-1)\, d\left(\frac{a+b}{2} - t \right) + \int_{\frac{a+b}{2}}^b (1)\, d\left(t - \frac{a+b}{2} \right) \\ &= b - a, \end{aligned}$$

and introducing the above values in (2.7) we deduce

$$b-a \le 2D(b-a),$$

which implies that $D \ge \frac{1}{2}$. ☐

Corollary 2. *If f is continuous on $[a,b]$ and u is $L-$Lipschitzian, then:*

$$\left|\widetilde{\Delta}(f,u;a,b)\right| \le \frac{1}{2}\left[\max_{t\in[a,b]} f(t) - \min_{t\in[a,b]} f(t)\right] L(b-a). \tag{2.8}$$

The constant $\frac{1}{2}$ is best possible.

Proof. In order to prove the sharpness of the constant, we cannot use the example from Theorem 2 since f_0 was not continuous.

If $u_0(t) = \left|t-\frac{a+b}{2}\right|$ and f_0 is continuous, then

$$\int_a^b f_0(t)\,d\left|t-\frac{a+b}{2}\right| = \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) f_0(t)\,dt.$$

Consider now the sequence of continuous functions

$$f_{0,n}(t) = \begin{cases} -1 & \text{if } t\in\left[a, \frac{a+b}{2}-\frac{1}{n}\right]; \\ -1+n\left(t-\frac{a+b}{2}+\frac{1}{n}\right) & \text{if } t\in\left(\frac{a+b}{2}-\frac{1}{n}, \frac{a+b}{2}+\frac{1}{n}\right); \\ 1 & \text{if } t\in\left[\frac{a+b}{2}+\frac{1}{n}, b\right], \end{cases}$$

which coincides with $u_0(t) = \operatorname{sgn}\left(t-\frac{a+b}{2}\right)$ on $\left[a,\frac{a+b}{2}-\frac{1}{n}\right]\cup\left[\frac{a+b}{2}+\frac{1}{n},b\right]$ and connects the end segments of this function on $\left[\frac{a+b}{2}-\frac{1}{n},\frac{a+b}{2}+\frac{1}{n}\right]$ respectively. Obviously

$$\begin{aligned} &\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) f_{0,n}(t)\,dt \\ &= \int_a^{\frac{a+b}{2}-\frac{1}{n}} dt + \int_{\frac{a+b}{2}-\frac{1}{n}}^{\frac{a+b}{2}+\frac{1}{n}} \operatorname{sgn}\left(t-\frac{a+b}{2}\right) f_{0,n}(t)\,dt + \int_{\frac{a+b}{2}+\frac{1}{n}}^{b} dt \\ &= b-a+x_n, \end{aligned}$$

where

$$|x_n| = \left|\int_{\frac{a+b}{2}-\frac{1}{n}}^{\frac{a+b}{2}+\frac{1}{n}} \operatorname{sgn}\left(t-\frac{a+b}{2}\right) f_{0,n}(t)\,dt\right| \le \frac{2}{n}.$$

Now, if (2.8) holds with a constant $E>0$, i.e.,

$$\left|\widetilde{\Delta}(f,u;a,b)\right| \le E\left[\max_{t\in[a,b]} f(t) - \min_{t\in[a,b]} f(t)\right] L(b-a),$$

then on choosing $f_{0,n}$ and u_0 as above, we get

$$b-a+x_n \le 2E(b-a)$$

for each $n \in \mathbb{N}$. Letting $n \to \infty$ and taking into account that $\lim_{n\to\infty} x_n = 0$, we deduce $E \geq \frac{1}{2}$, and the corollary is proved. $\square$

Corollary 3. *Let $f, h : [a,b] \to \mathbb{R}$ be Riemann integrable functions, f satisfies* (2.1) *and $|h(t)| \leq N$ for a.e. $t \in [a,b]$. Then*

$$\left| \int_a^b f(t)\, h(t)\, dt - \frac{m+M}{2} \int_a^b h(t)\, dt \right| \leq \frac{1}{2}(M-m)\, N\,(b-a). \tag{2.9}$$

The constant $\frac{1}{2}$ is best possible.

The proof follows by (2.5) on choosing $u(t) = \int_a^t h(s)\, ds$. The details are omitted.

Finally, we can state the following result as well.

Theorem 3. *Let $u : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ a bounded function satisfying* (2.1) *and such that $\int_a^b f(t)\, du(t)$ exists. Then*

$$\begin{aligned} |\Delta(f,u,m,M;a,b)| &\leq \int_a^b \left| f(t) - \frac{m+M}{2} \right| du(t) \\ &\leq \frac{1}{2}(M-m)\,[u(b)-u(a)]. \end{aligned} \tag{2.10}$$

The first inequality in (2.10) *is sharp. The constant $\frac{1}{2}$ is best possible.*

Proof. The inequality

$$\left| \int_a^b \left(f(t) - \frac{m+M}{2} \right) du(t) \right| \leq \int_a^b \left| f(t) - \frac{m+M}{2} \right| du(t)$$

follows by the definition of Stieltjes integrals.

Since

$$\left| f(t) - \frac{m+M}{2} \right| \leq \frac{1}{2}(M-m) \quad \text{for each } t \in [a,b],$$

we also have that

$$\begin{aligned} \int_a^b \left| f(t) - \frac{m+M}{2} \right| du(t) &\leq \frac{1}{2}(M-m) \int_a^b du(t) \\ &= \frac{1}{2}(M-m)\,[u(b)-u(a)] \end{aligned}$$

and the inequality (2.10) is thus proved.

Now, assume that $f_0(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a, b]$. Then for any continuous and monotonic nondecreasing function $u_0 : [a, b] \to \mathbb{R}$ we can state that

$$\begin{aligned}
&\Delta(f_0, u_0, m_0, M_0; a, b) \\
&= \int_a^{\frac{a+b}{2}} (-1)\, du_0(t) + \int_{\frac{a+b}{2}}^b (1)\, du_0(t) \\
&= u_0(a) + u_0(b) - 2u_0\left(\frac{a+b}{2}\right).
\end{aligned}$$

Also,

$$\int_a^b \left| f_0(t) - \frac{m_0 + M_0}{2} \right| du_0(t) = u_0(b) - u_0(a)$$

and

$$\frac{1}{2}(M_0 - m_0)\left[u_0(b) - u_0(a)\right] = u_0(b) - u_0(a),$$

which shows that the last inequality holds with equality in (1.9).

Finally, to have equality in the first part of (2.10) it is sufficient selecting u_0 to vanish in $\left[a, \frac{a+b}{2}\right]$ and being continuous and monotonic nondecreasing on $\left[\frac{a+b}{2}, b\right]$. In this situation we get in all terms of (2.10) the same quantity $u_0(b)$. □

Corollary 4. *If f is continuous on $[a, b]$ and u is monotonic nondecreasing, then*

$$\begin{aligned}
(2.11) \qquad \left|\widetilde{\Delta}(f, u; a, b)\right| &\leq \int_a^b \left| f(t) - \frac{\min_{t\in[a,b]} f(t) + \max_{t\in[a,b]} f(t)}{2} \right| du(t) \\
&\leq \frac{1}{2}\left[\max_{t\in[a,b]} f(t) - \min_{t\in[a,b]} f(t)\right]\left[u(b) - u(a)\right].
\end{aligned}$$

To prove the sharpness of the inequality we use the functions $f_0(t) = \left|t - \frac{a+b}{2}\right|$ and $u_0(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ which produce in all terms of (2.11) the quantity $\frac{b-a}{2}$.

Corollary 5. *If f, w are Riemann integrable on $[a, b]$ and f satisfies* (2.1) *while w is nonnegative, then*

$$\begin{aligned}
(2.12) \quad \left| \int_a^b f(t)\, w(t)\, dt - \frac{m + M}{2} \int_a^b w(t)\, dt \right| &\leq \int_a^b \left| f(t) - \frac{m + M}{2} \right| w(t)\, dt \\
&\leq \frac{1}{2}(M - m) \int_a^b w(t)\, dt.
\end{aligned}$$

The dual case, i.e., when the integrator is bounded below and above, is incorporated in the following result.

Theorem 4. *Assume that u is Riemann integrable on $[a,b]$ and*

$$-\infty < n \le u(t) \le N < \infty \quad \text{for a.e.} \quad t \in [a,b]. \tag{2.13}$$

Define the error functional of generalized trapezoid type

$$\nabla(f,u,n,N;a,b) := \left[u(b) - \frac{n+N}{2}\right] f(b) + \left[\frac{n+N}{2} - u(a)\right] f(a) - \int_a^b f(t)\,du(t).$$

(i) *If f is of bounded variation and such that the Stieltjes integral*

$$\int_a^b f(t)\,du(t)$$

exists, then

$$|\nabla(f,u,n,N;a,b)| \le \frac{1}{2}(N-n)\bigvee_a^b(f). \tag{2.14}$$

The constant $\frac{1}{2}$ is best possible in (2.14).

(ii) *If f is K–Lipschitzian on $[a,b]$, then*

$$|\nabla(f,u,n,N;a,b)| \le \frac{1}{2}(N-n)K(b-a). \tag{2.15}$$

The constant $\frac{1}{2}$ is best possible in (2.15).

(iii) *If f is monotonic nondecreasing on $[a,b]$ such that the Stieltjes integrals, $\int_a^b f(t)\,du(t)$, $\int_a^b \left|u(t) - \frac{n+N}{2}\right| df(t)$ exist, then*

$$\begin{aligned} |\nabla(f,u,n,N;a,b)| &\le \int_a^b \left|u(t) - \frac{n+N}{2}\right| df(t) \\ &\le \frac{1}{2}(N-n)\left[f(b) - f(a)\right]. \end{aligned} \tag{2.16}$$

The first inequality is sharp and the constant $\frac{1}{2}$ is best possible in (2.16).

Proof. The proof follows by Theorems 1 – 3 on utilizing the integral identity:

$$\begin{aligned} &\left[u(b) - \frac{n+N}{2}\right] f(b) + \left[\frac{n+N}{2} - u(a)\right] f(a) - \int_a^b f(t)\,du(t) \\ &= \int_a^b \left[u(t) - \frac{n+N}{2}\right] df(t) \end{aligned}$$

and the details are omitted. □

Remark 1. The above inequalities also hold for continuous functions $u : [a,b] \to \mathbb{R}$ when n is replaced by $\min_{t\in[a,b]} u(t)$ and N is replaced by $\max_{t\in[a,b]} u(t)$. The details are left to the interested reader.

3. Applications for three point quadrature rules

In [1] (see also [10, p. 223]) P. Cerone and S. S. Dragomir established the following three point quadrature rule for $n-$times differentiable functions:

$$\begin{aligned}(3.1)\quad \int_a^b f(t)\,dt = \sum_{k=1}^{n} \frac{1}{k!}\Big\{(1-\gamma)^k\left[(b-x)^k+(-1)^{k-1}(x-a)^k\right] f^{(k-1)}(x) \\ + \gamma^k\left[(x-a)^k f^{(k-1)}(a)+(-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right]\Big\} \\ +(-1)^n\int_a^b C_n(x,t)\, f^{(n)}(t)\,dt,\end{aligned}$$

where

$$(3.2)\qquad C_n(x,t)=\begin{cases}\frac{[t-(\gamma x+(1-\gamma)a)]^n}{n!} & \text{if } t\in[a,x];\\ \frac{[t-(\gamma x+(1-\gamma)b)]^n}{n!} & \text{if } t\in(x,b],\end{cases}$$

and $\gamma\in[0,1]$, $x\in(a,b)$.

This representation comprises amongst others the interior point quadrature rule obtained by Cerone et al. [3] in 1999 for $\gamma=0$ and the trapezoid quadrature rule obtained by Cerone et al. [4] in 2000 for $\gamma=1$.

Consider the function:

$$(3.3)\qquad K_n(x,t):=(-1)^n\begin{cases}\frac{[t-(\gamma x+(1-\gamma)a)]^{n+1}}{(n+1)!} & \text{if } t\in[a,x];\\ \frac{[t-(\gamma x+(1-\gamma)b)]^{n+1}}{(n+1)!} & \text{if } t\in(x,b].\end{cases}$$

The function $K_n(x,\cdot):[a,b]\to\mathbb{R}$, for each fixed $x\in[a,b]$, is of bounded variation and

$$\begin{aligned}\bigvee_a^b(K_n(x,\cdot)) &= \int_a^x\left|\frac{dK_n(x,t)}{dt}\right|dt+\int_x^b\left|\frac{dK_n(x,t)}{dt}\right|dt\\ &= \int_a^x\frac{|t-(\gamma x+(1-\gamma)a)|^n}{n!}dt+\int_x^b\frac{|\gamma x+(1-\gamma)b-t|^n}{n!}dt.\end{aligned}$$

We have

$$\begin{aligned}I_1 &= \int_a^x\frac{|t-(\gamma x+(1-\gamma)a)|^n}{n!}dt\\ &= \int_a^{\gamma x+(1-\gamma)a}\frac{[\gamma x+(1-\gamma)a-t]^n}{n!}dt+\int_{\gamma x+(1-\gamma)a}^{x}\frac{[t-(\gamma x+(1-\gamma)a)]^n}{n!}dt\end{aligned}$$

$$
\begin{aligned}
&= -\left[\left.\frac{[\gamma x+(1-\gamma)a-t]^{n+1}}{(n+1)!}\right|_{a}^{\gamma x+(1-\gamma)a}\right]+\left.\frac{[t-(\gamma x+(1-\gamma)a)]^{n+1}}{(n+1)!}\right|_{\gamma x+(1-\gamma)a}^{x} \\
&= \frac{\gamma^{n+1}(x-a)^{n+1}}{(n+1)!}+\frac{(1-\gamma)^{n+1}(x-a)^{n+1}}{(n+1)!} \\
&= \frac{1}{(n+1)!}(x-a)^{n+1}\left[\gamma^{n+1}+(1-\gamma)^{n+1}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_x^b \frac{|\gamma x+(1-\gamma)b-t|^n}{n!}dt \\
&= \int_x^{\gamma x+(1-\gamma)b} \frac{[\gamma x+(1-\gamma)b-t]^n}{n!}dt+\int_{\gamma x+(1-\gamma)b}^{b}\frac{[t-(\gamma x+(1-\gamma)b)]^n}{n!}dt \\
&= -\left[\left.\frac{[\gamma x+(1-\gamma)b-t]^{n+1}}{(n+1)!}\right|_{x}^{\gamma x+(1-\gamma)b}\right]+\left.\frac{[t-(\gamma x+(1-\gamma)b)]^{n+1}}{(n+1)!}\right|_{\gamma x+(1-\gamma)b}^{b} \\
&= \frac{(1-\gamma)^{n+1}(b-x)^{n+1}}{(n+1)!}+\frac{\gamma^{n+1}(b-x)^{n+1}}{(n+1)!} \\
&= \frac{1}{(n+1)!}(b-x)^{n+1}\left[\gamma^{n+1}+(1-\gamma)^{n+1}\right].
\end{aligned}
$$

Therefore

(3.4)

$$
\bigvee_a^b (K_n(x,\cdot)) = \frac{1}{(n+1)!}\left[\gamma^{n+1}+(1-\gamma)^{n+1}\right]\left[(b-x)^{n+1}+(x-a)^{n+1}\right].
$$

We also have

(3.5)

$$
\begin{aligned}
\int_a^b f^{(n)}(t)\,d(K_n(x,t)) &= \int_a^x f^{(n)}(t)\,d\left[(-1)^n\frac{[t-(\gamma x+(1-\gamma)a)]^{n+1}}{(n+1)!}\right] \\
&\quad+\int_x^b f^{(n)}(t)\,d\left[(-1)^n\frac{[t-(\gamma x+(1-\gamma)b)]^{n+1}}{(n+1)!}\right] \\
&= (-1)^n\int_a^b C_n(t,x)\,f^{(n)}(t)\,dt,
\end{aligned}
$$

with $C_n(t,x)$ defined by (3.2).

We can state the following result in approximating the Riemann integral $\int_a^b f(x)\,dx$ of $n-$times differentiable functions f in terms of three point quadrature rules.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a function such that for $n\geq 1$ the derivative $f^{(n-1)}$ is absolutely continuous and there exists the real constants γ_n,Γ_n such* *that*

$$\gamma_n \le f^{(n)}(t) \le \Gamma_n \qquad \text{for a.e. } t \in [a,b]. \tag{3.6}$$

Then

$$\begin{aligned}
&\int_a^b f(t)\,dt \\
&= \sum_{k=1}^{n} \frac{1}{k!}\Big\{(1-\gamma)^k \left[(b-x)^k + (-1)^{k-1}(x-a)^k\right] f^{(k-1)}(x) \\
&\quad + \gamma^k \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right]\Big\} \\
&\quad + \frac{\gamma_n+\Gamma_n}{2}\left[(-1)^n (b-x)^{n+1} + (x-a)^{n+1}\right]\frac{\gamma^{n+1}}{(n+1)!} + R_n,
\end{aligned} \tag{3.7}$$

and the error R_n satisfies the bound

(3.8)

$$|R_n| \le \frac{1}{2}(\Gamma_n - \gamma_n)\frac{1}{(n+1)!}\left[\gamma^{n+1} + (1-\gamma)^{n+1}\right]\left[(b-x)^{n+1} + (x-a)^{n+1}\right]$$

for $\gamma \in [0,1]$ and $x \in [a,b]$.

Proof. We apply Theorem 1 for the functions $f^{(n)}$ and $K(x,\cdot)$ to get:

$$\begin{aligned}
&\left|\int_a^b f^{(n)}(t)\,dK_n(x,t) - \frac{\gamma_n+\Gamma_n}{2}\left[K_n(x,b) - K_n(x,a)\right]\right| \\
&\le \frac{1}{2}(\Gamma_n - \gamma_n)\bigvee_a^b (K_n(x,\cdot))
\end{aligned}$$

for $x \in [a,b]$.

Since

$$\begin{aligned}
K_n(x,b) &= (-1)^n \frac{[b-(\gamma x + (1-\gamma)b)]^{n+1}}{(n+1)!} \\
&= (-1)^n \frac{\gamma^{n+1}(b-x)^{n+1}}{(n+1)!}
\end{aligned}$$

and

$$\begin{aligned}
K_n(x,a) &= (-1)^n \frac{[a-(\gamma x + (1-\gamma)a)]^{n+1}}{(n+1)!} \\
&= (-1)^n \frac{[\gamma(a-x)]^{n+1}}{(n+1)!} \\
&= -\frac{\gamma^{n+1}(x-a)^{n+1}}{(n+1)!},
\end{aligned}$$

hence by (3.4) and (3.5) we deduce:

$$\begin{aligned}(3.9)\quad &\left|(-1)^n \int_a^b C_n(t,x) f^{(n)}(t)\,dt\right.\\ &\left.-\frac{\gamma_n+\Gamma_n}{2}\left[(-1)^n \frac{\gamma^{n+1}(b-x)^{n+1}}{(n+1)!}+\frac{\gamma^{n+1}(x-a)^{n+1}}{(n+1)!}\right]\right|\\ &\le \frac{1}{2}(\Gamma_n-\gamma_n)\frac{1}{(n+1)!}\left[\gamma^{n+1}+(1-\gamma)^{n+1}\right]\left[(b-x)^{n+1}+(x-a)^{n+1}\right].\end{aligned}$$

Finally, on utilizing the identity (3.1) we deduce from (3.9) the representation (3.7) and the estimate (3.8). □

Remark 2. The above approximation of the integral $\int_a^b f(t)\,dt$ contains some particular cases of interest.

If $\lambda = 0$, then we have

$$(3.10)\quad \int_a^b f(t)\,dt = \sum_{k=1}^{n} \frac{1}{k!}\left[(b-x)^k + (-1)^{k-1}(x-a)^k\right] f^{(k-1)}(x) + T_n,$$

with

$$|T_n| \le \frac{1}{2}(\Gamma_n-\gamma_n)\frac{1}{(n+1)!}\left[(b-x)^{n+1}+(x-a)^{n+1}\right].$$

If $\lambda = \frac{1}{2}$, then we have

$$\begin{aligned}(3.11)\quad \int_a^b f(t)\,dt = &\sum_{k=1}^{n} \frac{1}{2^k k!}\left\{\left[(b-x)^k + (-1)^{k-1}(x-a)^k\right] f^{(k-1)}(x)\right.\\ &\left.+\left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right]\right\}\\ &+\frac{\gamma_n+\Gamma_n}{2^{n+2}(n+1)!}\left[(-1)^n(b-x)^{n+1}+(x-a)^{n+1}\right]+M_n,\end{aligned}$$

with

$$|M_n| \le \frac{1}{2^{n+1}(n+1)!}(\Gamma_n-\gamma_n)\left[(b-x)^{n+1}+(x-a)^{n+1}\right].$$

Finally, if $\lambda = 1$, then we have

$$\begin{aligned}(3.12)\quad \int_a^b f(t)\,dt = &\sum_{k=1}^{n} \frac{1}{k!}\left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right]\\ &+\frac{\gamma_n+\Gamma_n}{2(n+1)!}\left[(-1)^n(b-x)^{n+1}+(x-a)^{n+1}\right]+Q_n,\end{aligned}$$

with

$$|Q_n| \le \frac{1}{2(n+1)!}(\Gamma_n-\gamma_n)\left[(b-x)^{n+1}+(x-a)^{n+1}\right].$$

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