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BOUNDED SOLUTIONS FOR THE SCHRÖDINGER OPERATOR ON RIEMANNIAN MANIFOLDS

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Abstract. Let M be a complete Riemannian manifold and $\mathcal{L}$ be a Schrödinger operator on M. We prove that if M has finitely many $\mathcal{L}$-nonparabolic ends, then the space of bounded $\mathcal{L}$-harmonic functions on M has the same dimension as the sum of dimensions of the spaces of bounded $\mathcal{L}$-harmonic functions on each $\mathcal{L}$-nonparabolic end, which vanish at the boundary of the end.

1. Introduction

Let M be a complete Riemannian manifold and $\mathcal{L} = \Delta - V$ be a Schrödinger operator on M, where Δ is the Laplacian on M and the potential V is a nonnegative function on M. A function u on an open subset Ω of M is called an $\mathcal{L}$-solution (-supersolution, -subsolution, respectively,) on Ω if $\mathcal{L}u = 0$ (≤ 0, ≥ 0, respectively,) on Ω. This equation is understood in the sense of distribution. We say that a function u is $\mathcal{L}$-harmonic on Ω if u is a continuous $\mathcal{L}$-solution on Ω. In the case when the potential term V of the Schrödinger operator $\mathcal{L}$ is continuous, one can achieve the continuity of $\mathcal{L}$-solutions. More generally, such a result can be extended to potentials in the local Kato class. (See [4].)

This paper is motivated by the previous works of present authors [5] and [6]. By the result of [5], the dimension of the space of bounded $\mathcal{L}$-harmonic functions on a complete Riemannian manifold is equal to the number of $\mathcal{L}$-nonparabolic ends in the case when each $\mathcal{L}$-nonparabolic end is regular. On the other hand, the present authors in [6] proved that the dimension of the space of bounded energy finite $\mathcal{L}$-harmonic functions on a complete Riemannian manifold is equal to the maximal number of $\mathcal{L}$-massive subsets of the manifold. In this paper, we propose the space of bounded $\mathcal{L}$-harmonic functions on ends of a complete Riemannian manifold and give the relation between the space of bounded $\mathcal{L}$-harmonic functions on the whole manifold and those of its ends. In particular, we prove that the dimension of the space of bounded $\mathcal{L}$-harmonic functions on

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the whole manifold is equal to the sum of dimension of the spaces of bounded $\mathcal{L}$-harmonic functions on its ends as follows:

Theorem 1.1. *Let M be a complete Riemannian manifold and $\mathcal{L} = \Delta - V$, where Δ denotes the Laplacian on M and V is a nonnegative continuous function on M. Let $E_1, E_2, \ldots, E_l$, $l \geq 1$, be $\mathcal{L}$-nonparabolic ends of M. Then $\mathcal{HB}_{\mathcal{L}}(M)$ has the same dimension as the dimension of $\prod_{i=1}^{l} \mathcal{HB}_{\mathcal{L}}(E_i, \partial E_i)$, where $\mathcal{HB}_{\mathcal{L}}(X)$ and $\mathcal{HB}_{\mathcal{L}}(X, \partial X)$ denote the space of bounded $\mathcal{L}$-harmonic functions on X and the subspace of elements of $\mathcal{HB}_{\mathcal{L}}(X)$ vanishing at ∂X, respectively.*

In particular, in the case when $\mathcal{HB}_{\mathcal{L}}(M)$ is finite dimensional, there exists an isomorphism

$$\Phi : \mathcal{HB}_{\mathcal{L}}(M) \to \prod_{i=1}^{l} \mathcal{HB}_{\mathcal{L}}(E_i, \partial E_i).$$

In the case when the potential term of the Schrödinger operator is identically zero, $\mathcal{L}$-harmonic functions become harmonic functions. Therefore, this result partially generalizes those of Yau [10], of Grigor'yan [1], [2], [3], of Li-Tam [7], [8] and of Sung- Tam-Wang [9].

2. $\mathcal{L}$-massivity and bounded $\mathcal{L}$-harmonic functions on manifolds

Let M be a complete Riemannian manifold and o be a fixed point in M. Throughout this paper, Δ always denotes the Laplacian on M and V is a nonnegative continuous function on M. Also $\mathcal{L} = \Delta - V$ denotes a Schrödinger operator on M.

An open proper subset $\Omega \subset M$ is said to be $\mathcal{L}$-massive if there exists a continuous function w on M such that $0 \leq w \leq 1$ on M,

$$\begin{cases} \mathcal{L}\, w = 0 & \text{on } \Omega; \\ \quad w = 0 & \text{on } M \setminus \Omega; \\ \sup_{\Omega} w = 1. \end{cases}$$

Such a function w is called an inner potential of Ω.

Arguing similarly as in [3], we get the following useful properties of $\mathcal{L}$-massive sets:

Proposition 2.1. *Suppose $\Omega' \subset \Omega$ are open proper subsets of a complete Riemannian manifold and $\mathcal{L} = \Delta - V$. Then*

(i) *if Ω' is $\mathcal{L}$-massive, then Ω is also $\mathcal{L}$-massive;*
(ii) *if Ω is $\mathcal{L}$-massive and $\overline{\Omega} \setminus \Omega'$ is compact, then Ω' is also $\mathcal{L}$-massive.*

We denote by $\mathcal{B}(M)$ the space of all bounded continuous functions on M. Let $\mathcal{HB}_{\mathcal{L}}(M)$ denote the subspace of all $\mathcal{L}$-harmonic functions in $\mathcal{B}(M)$. Then we can prove that the dimension of $\mathcal{HB}_{\mathcal{L}}(M)$ is equal to the supremum of the number of mutually disjoint $\mathcal{L}$-massive subsets of M as follows:

Theorem 2.2. *Let M be a complete Riemannian manifold and $\mathcal{L} = \Delta - V$. Then for each $m \in \mathbf{N}$, $\dim \mathcal{HB}_{\mathcal{L}}(M) \geq m$ if and only if there exist mutually disjoint $\mathcal{L}$-massive subsets $\Omega_1, \Omega_2, \ldots, \Omega_m$ of M.*

Proof. Let $\Omega_1, \Omega_2, \ldots, \Omega_m$ be the mutually disjoint $\mathcal{L}$-massive subsets of M and w_i be an inner potential of Ω_i for each $i = 1, 2, \ldots, m$. Then for each $i = 1, 2, \ldots, m$ and $r > 0$, define a continuous function $f_{i,r}$ on $B_r(o)$ such that

$$\begin{cases} \mathcal{L} f_{i,r} = 0 & \text{on } B_r(o); \\ f_{i,r} = w_i & \text{on } \partial B_r(o) \cap \Omega_i; \\ f_{i,r} = 0 & \text{on } \partial B_r(o) \setminus \Omega_i, \end{cases}$$

where $B_r(o)$ denotes the metric r-ball centered at o. By the comparison principle, $w_i \leq f_{i,r} \leq 1$ on $B_r(o)$. Since $f_{i,r'} \geq w_i = f_{i,r}$ on $\partial B_r(o)$ for $r' > r$, we have $f_{i,r'} \geq f_{i,r}$ on $B_r(o)$. Thus $\{f_{i,r}\}$ is increasing in r, hence has a limit function f_i. In particular, f_i is an $\mathcal{L}$-harmonic function on M satisfying $0 \leq w_i \leq f_i \leq 1$. Since $\sup_{\Omega_i} w_i = 1$, we have $\sup_{\Omega_i} f_i = 1$.

On the other hand, since $\Omega_1, \Omega_2, \ldots, \Omega_m$ are mutually disjoint, $\sum_{i=1}^m w_i = \max_{i=1,2,\ldots,m} w_i$, hence $\sup_M \sum_{i=1}^m w_i = 1$ and $\sup_M \sum_{i=1}^m f_i = 1$. Since $\sup_{\Omega_i} w_i = 1$, there is a sequence $\{x_{i,n}\}_{n \in \mathbf{N}}$ in Ω_i such that $\lim_{n\to\infty} w_i(x_{i,n}) = 1$ for each $i = 1, 2, \ldots, m$. From the fact that $0 \leq w_i \leq f_i \leq 1$ and $\sum_{i=1}^m f_i \leq 1$, the sequence $\{x_{i,n}\}$ satisfies

$$\lim_{n\to\infty} f_j(x_{i,n}) = \delta_{ij} \tag{2.1}$$

for each $i = 1, 2, \ldots, m$, where δ_{ij} is Kronecker's delta.

Suppose that

$$a_1 f_1 + a_2 f_2 + \cdots + a_m f_m = 0$$

for some $a_1, a_2, \ldots, a_m \in \mathbf{R}$. Then (2.1) implies that $a_i = 0$ for each $i = 1, 2, \ldots, m$, hence $f_1, f_2, \ldots, f_m$ are linearly independent. Consequently,

$$\dim \mathcal{HB}_{\mathcal{L}}(M) \geq m.$$

Conversely, suppose that $\dim \mathcal{HB}_{\mathcal{L}}(M) \geq m$. Then there exist linearly independent $\mathcal{L}$-harmonic functions $u_1, u_2, \ldots, u_m$ in $\mathcal{HB}_{\mathcal{L}}(M)$. Let $\hat{M}$ be the Stone-Cech compactification of M and $\partial \hat{M} = \hat{M} \setminus M$. Then every function $u \in \mathcal{B}(M)$ can be extended to a continuous function $\overline{u}$ on $\hat{M}$.

We can extend u_i to $\overline{u}_i$ on $\hat{M}$ in such a way that $\overline{u}_i|_{\partial \hat{M}}$, denoted by f_i, is continuous on $\partial \hat{M}$. By using the linear independence of $u_1, u_2, \ldots, u_m$ and the comparison principle, $f_1, f_2, \ldots, f_m$ are also linearly independent. Then there exist continuous functions $F_1, F_2, \ldots, F_m$, each of which is a linear combination of $f_1, f_2, \ldots, f_m$ and is not identically zero, such that $\{x \in \partial \hat{M} : F_i(x) = \max_{\partial \hat{M}} F_i\}$'s are mutually disjoint. (See [3].) Since each F_i is a linear combination of $f_1, f_2, \ldots, f_m$, there exists a linear combination v_i of $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_m$ such that $v_i = F_i$ on $\partial \hat{M}$. We may assume that $\max_{\partial \hat{M}} F_i > 0$ for each $i = 1, 2, \ldots, m$. For given $\epsilon > 0$, put $\Omega_i^\epsilon = \{x \in M : v_i(x) > \max_{\partial \hat{M}} F_i - \epsilon\}$. Then Ω_i^ϵ is an $\mathcal{L}$-massive subset of M.

We claim that Ω_i^ϵ's are mutually disjoint for sufficiently small $\epsilon > 0$. If this is not the case, then for some $i \neq j$, there exists a sequence $\{\epsilon_n\}_{n\in\mathbf{N}}$ such that $\lim_{n\to\infty} \epsilon_n = 0$ and $\Omega_i^{\epsilon_n} \cap \Omega_j^{\epsilon_n} \neq \emptyset$ for all $n \in \mathbf{N}$. Let $x_n \in \Omega_i^{\epsilon_n} \cap \Omega_j^{\epsilon_n}$ for each $n \in \mathbf{N}$. Since $\hat{M}$ is compact, there exists a convergent subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ with a limit point, say $x_0 \in \hat{M}$, as $k \to \infty$. Clearly, we have $v_i(x_0) = \max_{\partial\hat{M}} F_i = \sup v_i$ and $v_j(x_0) = \max_{\partial\hat{M}} F_j = \sup v_j$. If $x_0 \in M$, then by the maximum principle, we have a contradiction. If $x_0 \in \partial\hat{M}$, then $v_i(x_0) = \max_{\partial\hat{M}} F_i$ and $v_j(x_0) = \max_{\partial\hat{M}} F_j$, i.e., x_0 is a common maximum point of F_i and F_j, which is a contradiction. This proves the claim. □

By constructing a basis from inner potentials of $\mathcal{L}$-massive subsets of a complete Riemannian manifold, we can explicitly describe the space of bounded $\mathcal{L}$-harmonic functions on the manifold as follows:

Theorem 2.3. *Let M be a complete Riemannian manifold whose maximal number of mutually disjoint $\mathcal{L}$-massive subsets is $m \in \mathbf{N}$, where $\mathcal{L} = \Delta - V$. Suppose $\Omega_1, \Omega_2, \ldots, \Omega_m$ are mutually disjoint $\mathcal{L}$-massive subsets of M. Let w_i be an inner potential of Ω_i for each $i = 1, 2, \ldots, m$. Then we can construct a basis $\{f_1, f_2, \ldots, f_m\}$ for $\mathcal{HB}_{\mathcal{L}}(M)$ such that*

(i) $0 \le w_i \le f_i \le 1$ *on* Ω_i *for each* $i = 1, 2, \ldots, m$;
(ii) $\sup_M \sum_{i=1}^m f_i = 1$.

In particular, for given real numbers $a_1, a_2, \ldots, a_m \in \mathbf{R}$, there exists an $\mathcal{L}$-harmonic function $h \in \mathcal{HB}_{\mathcal{L}}(M)$ such that for each $i = 1, 2, \ldots, m$,

$$\lim_{n\to\infty} h(x_{i,n}) = a_i, \tag{2.2}$$

where $\{x_{i,n}\}_{n\in\mathbf{N}}$ is a sequence in Ω_i satisfying (2.1).

Conversely, each $\mathcal{L}$-harmonic function $h \in \mathcal{HB}_{\mathcal{L}}(M)$ is uniquely determined by the values in (2.2).

Proof. Since the maximal number of mutually disjoint $\mathcal{L}$-massive subsets contained in M is m, by Theorem 2.2, $\dim \mathcal{HB}_{\mathcal{L}}(M) = m$. Let $\Omega_1, \Omega_2, \ldots, \Omega_m$ be the mutually disjoint $\mathcal{L}$-massive subsets of M and w_i be an inner potential of Ω_i for each $i = 1, 2, \ldots, m$. Then one can check that the bounded $\mathcal{L}$-harmonic functions $f_i, f_2, \ldots, f_m$ constructed in the proof of Theorem 2.2 form a basis for $\mathcal{HB}_{\mathcal{L}}(M)$ satisfying

(i) $0 \le w_i \le f_i \le 1$ on Ω_i for each $i = 1, 2, \ldots, m$;
(ii) $\sup_M \sum_{i=1}^m f_i = 1$.

For given real numbers $a_1, a_2, \ldots, a_m \in \mathbf{R}$, define $h = \sum_{j=1}^m a_j f_j$. Then since the sequence $\{x_{i,n}\}$ satisfies (1), we have

$$\lim_{n\to\infty} h(x_{i,n}) = \sum_{j=1}^m a_j \lim_{n\to\infty} f_j(x_{i,n}) = \sum_{j=1}^m a_j \delta_{ij} = a_i$$

for each $i = 1, 2, \ldots, m$.

Conversely, let h be a function in $\mathcal{HB}_{\mathcal{L}}(M)$ satisfying (2.2). Clearly, a bounded $\mathcal{L}$-harmonic function $\sum_{j=1}^{m} a_j f_j$ also satisfies (2.2). Putting $g = h - \sum_{j=1}^{m} a_j f_j$, there exist $c_1, c_2, \ldots, c_m \in \mathbf{R}$ such that $g = \sum_{j=1}^{m} c_j f_j$. Then from the definition of $\{x_{i,n}\}$, we have

$$c_i = \lim_{n\to\infty} g(x_{i,n}) = \lim_{n\to\infty} h(x_{i,n}) - \sum_{j=1}^{m} a_j \lim_{n\to\infty} f_j(x_{i,n}) = a_i - \sum_{j=1}^{m} a_j \delta_{ij} = 0$$

for each $i = 1, 2, \ldots, m$. This implies that $g \equiv 0$ on M, i.e., $h \equiv \sum_{j=1}^{m} a_j f_j$ on M. □

3. $\mathcal{L}$-massivity and bounded $\mathcal{L}$-harmonic functions on ends

Let M be a complete Riemannian manifold and o be a fixed point in M. We denote by $\sharp(r)$ the number of unbounded components of $M \setminus B_r(o)$. It is easy to prove that $\sharp(r)$ is nondecreasing in $r > 0$. Let $\lim_{r\to\infty}\sharp(r) = l$, where l may be infinity, then we say that the number of ends of M is l. If l is finite, then we can choose $r_0 > 0$ in such a way that $\sharp(r) = l$ for all $r \geq r_0$. In this case, there exist mutually disjoint unbounded components $E_1, E_2, \ldots, E_l$ of $M \setminus \overline{B}_{r_0}(o)$ and we call each E_i an end of M for $i = 1, 2, \ldots, l$. We say that an end E of M is $\mathcal{L}$-nonparabolic if there exists a continuous function u_E, called an $\mathcal{L}$-harmonic measure, on $E \setminus B_{r_1}(o)$ for some $r_1 \geq r_0$ such that

$$\begin{cases} \mathcal{L}\, u_E = 0 & \text{on } E \setminus \bar{B}_{r_1}(o); \\ \quad u_E = 0 & \text{on } \partial B_{r_1}(o) \cap E; \\ \sup_{E\setminus \overline{B}_{r_1}(o)} u_E = 1. & \end{cases}$$

Otherwise, E is called an $\mathcal{L}$-parabolic end.

For an end E of M, $\mathcal{HB}_{\mathcal{L}}(E, \partial E)$ denotes the space of all $\mathcal{L}$-harmonic functions on E vanishing at ∂E. Let $\Omega_1, \Omega_2, \ldots, \Omega_s$ be the mutually disjoint $\mathcal{L}$-massive subsets of E and w_i be an inner potential of Ω_i for each $i = 1, 2, \ldots, s$. For each $i = 1, 2, \ldots, s$ and sufficiently large $r > r_1$, define a continuous function $g_{i,r}$ on $B_r(o) \cap E$ such that

$$\begin{cases} \mathcal{L}\, g_{i,r} = 0 & \text{on } B_r(o) \cap E; \\ \quad g_{i,r} = w_i & \text{on } (\partial B_r(o) \cap E) \cap \Omega_i; \\ \quad g_{i,r} = 0 & \text{on } \partial E; \\ \quad g_{i,r} = 0 & \text{on } (\partial B_r(o) \cap E) \setminus \Omega_i. \end{cases}$$

By the comparison principle, $\{g_{i,r}\}$ is increasing in r, hence has a limit function g_i. In particular, $g_1, g_2, \ldots, g_s$ are linearly independent bounded $\mathcal{L}$-harmonic functions on E, each of which satisfies

(i) $0 \leq w_i \leq g_i \leq 1$;
(ii) $\sup_{\Omega_i} g_i = 1$;
(iii) $\sup_E \sum_{i=1}^{s} g_i = 1$.

These together with the assumption that $\Omega_1, \Omega_2, \ldots, \Omega_s$ are the mutually disjoint $\mathcal{L}$-massive sets imply that for each $i = 1, 2, \ldots, s$, there exists a sequence $\{x_{i,n}\}_{n\in\mathbf{N}}$ in Ω_i such that

$$\lim_{n\to\infty} g_j(x_{i,n}) = \delta_{ij}. \tag{3.1}$$

Arguing similarly as in the proof of Theorem 2.2, we have the following theorem:

Theorem 3.1. *Let E be an end of a complete Riemannian manifold and $\mathcal{L} = \Delta - V$. Then for each $s \in \mathbf{N}$, $\dim \mathcal{HB}_{\mathcal{L}}(E, \partial E) \geq s$ if and only if there exist mutually disjoint $\mathcal{L}$-massive subsets $\Omega_1, \Omega_2, \ldots, \Omega_s$ of E.*

Suppose that the maximal number of mutually disjoint $\mathcal{L}$-massive subsets contained in E is $s \in \mathbf{N}$. Then, by Theorem 3.1, $\dim \mathcal{HB}_{\mathcal{L}}(E, \partial E) = s$. Arguing similarly as in the proof of Theorem 2.3, we have the following theorem:

Theorem 3.2. *Let E be an end of a complete Riemannian manifold, whose maximal number of mutually disjoint $\mathcal{L}$-massive subsets in E is $s \in \mathbf{N}$, where $\mathcal{L} = \Delta - V$. Suppose $\Omega_1, \Omega_2, \ldots, \Omega_s$ are mutually disjoint $\mathcal{L}$-massive subsets of E. Let w_i be an inner potential of Ω_i for each $i = 1, 2, \ldots, s$. Then we can construct a basis $\{g_1, g_2, \ldots, g_s\}$ for $\mathcal{HB}_{\mathcal{L}}(E, \partial E)$ such that*

(i) $0 \leq w_i \leq g_i \leq 1$ *on* Ω_i *for each* $i = 1, 2, \ldots, s$;
(ii) $\sup_E \sum_{i=1}^{s} g_i = 1$.

In particular, for given real numbers $a_1, a_2, \ldots, a_s \in \mathbf{R}$, there exists an $\mathcal{L}$-harmonic function $h \in \mathcal{HB}_{\mathcal{L}}(E, \partial E)$ such that for each $i = 1, 2, \ldots, s$,

$$\lim_{n\to\infty} h(x_{i,n}) = a_i, \tag{3.2}$$

where $\{x_{i,n}\}_{n\in\mathbf{N}}$ is a sequence in Ω_i satisfying (3.1).

Conversely, each $\mathcal{L}$-harmonic function $h \in \mathcal{HB}_{\mathcal{L}}(E, \partial E)$ is uniquely determined by the values in (3.2).

4. Proof of main results

In this section, we give the relation between the dimension of various spaces of $\mathcal{L}$-harmonic functions on the whole manifold and those on its ends. To begin with, we give a characterization of $\mathcal{L}$-parabolicity of ends in terms of $\mathcal{L}$-massivity as follows:

Lemma 4.1. *Suppose that the maximal number of mutually disjoint $\mathcal{L}$-massive subsets contained in M is m. Then we can choose mutually disjoint $\mathcal{L}$-massive subsets $\Omega_1, \Omega_2, \ldots, \Omega_m$ in such a way that for each Ω_i, there exists an $\mathcal{L}$-nonparabolic end E such that $\Omega_i \subset E$.*

Proof. Let $\Omega_1, \Omega_2, \ldots, \Omega_m$ be mutually disjoint $\mathcal{L}$-massive subsets of M. We claim that for each $i = 1, 2, \ldots, m$, there exist an $\mathcal{L}$-massive subset $\Omega_i' \subset \Omega_i$ and an $\mathcal{L}$-nonparabolic end E such that $\Omega_i' \subset E$.

By Proposition 2.1, $\Omega_i \setminus \overline{B}_{r_0}(o)$, $i = 1, 2, \ldots, m$, is also $\mathcal{L}$-massive. Let w_1 be an inner potential of $\Omega_1 \setminus \overline{B}_{r_0}(o)$. If an end E of M satisfies

$$\Omega_1 \cap E \neq \emptyset \quad \text{and} \quad \sup_{x \in \Omega_1 \cap E} w_1(x) > 0, \tag{4.1}$$

then $\Omega_1 \cap E$ is an $\mathcal{L}$-massive subset of Ω_1. In this case, other ends cannot satisfy the property (4.1). Otherwise, there is a contradiction to the maximality of the number of mutually disjoint $\mathcal{L}$-massive subsets of M. This implies that even if there is another end $\tilde{E}$ of M with $\Omega_1 \cap \tilde{E} \neq \emptyset$, w_1 must be identically zero on $\Omega_1 \cap \tilde{E}$. Therefore,

$$\Omega_1' = \{x \in \Omega_1 \setminus B_{r_0}(o) : w_1(x) > 0\}$$

is an $\mathcal{L}$-massive subset and E becomes an $\mathcal{L}$-nonparabolic end, hence Ω_1' and E are the desired ones.

Applying the above argument to other $\mathcal{L}$-massive subsets Ω_i, $i = 2, 3, \ldots, m$, we have the claim. □

We are now ready to prove our main result.

Proof of Theorem 1.1. In the case that $\mathcal{HB}_{\mathcal{L}}(M)$ is infinite dimensional, by Theorem 3.1, M can have infinitely many mutually disjoint $\mathcal{L}$-massive subsets. Then by Lemma 4.1, at least one end E of M must contain infinitely many mutually disjoint $\mathcal{L}$-massive subsets, since the number of ends of M is finite. Thus for any $m \in \mathbf{N}$, there are mutually disjoint $\mathcal{L}$-massive subsets $\Omega_1, \Omega_2, \ldots, \Omega_m$ of the end E. Then by Theorem 3.1, the dimension of the space of bounded $\mathcal{L}$-harmonic functions on the end E, which vanish at its boundary ∂E, is greater than or equal to m. Since $m \in \mathbf{N}$ is arbitrarily chosen, the function space $\mathcal{H}B_{\mathcal{L}}(E, \partial E)$ is infinite dimensional.

Conversely, in the case that the function space $\mathcal{H}B_{\mathcal{L}}(E, \partial E)$ on an end E is infinite dimensional, by Theorem 3.1, the end E has infinitely many mutually disjoint $\mathcal{L}$-massive subsets, hence so does M. By Theorem 2.2, this implies that $\mathcal{HB}_{\mathcal{L}}(M)$ is infinite dimensional.

Suppose that the dimension of $\mathcal{HB}_{\mathcal{L}}(M)$ is $m \in \mathbf{N}$. Then by Theorem 3.1 and Lemma 4.1, we can choose mutually disjoint $\mathcal{L}$-massive subsets

$$\Omega_1^1, \Omega_2^1, \ldots, \Omega_{s(1)}^1, \Omega_1^2, \Omega_2^2, \ldots, \Omega_{s(2)}^2, \ldots, \Omega_1^l, \Omega_2^l, \ldots, \Omega_{s(l)}^l,$$

where $\Omega_1^i, \Omega_2^i, \ldots, \Omega_{s(i)}^i$ denote the mutually disjoint $\mathcal{L}$-massive subsets contained in E_i for each $i = 1, 2, \ldots, l$ and $s(1) + s(2) + \cdots + s(l) = m$. This implies that the maximal number of mutually disjoint $\mathcal{L}$-massive subsets contained in E_i is $s(i)$ for each $i = 1, 2, \ldots, l$. Now let w_j^i be an inner potential of Ω_j^i for each $j = 1, 2, \ldots, s(i)$ and $i = 1, 2, \ldots, l$. By Theorem 2.3, we can find a basis

$$\{f_1^1, f_2^1, \ldots, f_{s(1)}^1, f_1^2, f_2^2, \ldots, f_{s(2)}^2, \ldots, f_1^l, f_2^l, \ldots, f_{s(l)}^l\}$$

for $\mathcal{HB}_{\mathcal{L}}(M)$ such that for $j = 1, 2, \ldots, s(i)$ and $i = 1, 2, \ldots, l$,

(i) $0 \leq w_j^i \leq f_j^i \leq 1$;

(ii) $\sup_M \sum_{i=1}^{l} \sum_{j=1}^{s(i)} f_j^i = 1$.

Since $\sup_{\Omega_j^i} w_j^i = 1$, there exists a sequence $\{x_{j,n}^i\}_{n\in\mathbf{N}}$ in Ω_j^i such that for each $j = 1, 2, \ldots, s(i)$ and $i = 1, 2, \ldots, l$, $\lim_{n\to\infty} w_j^i(x_{j,n}^i) = 1$, hence

$$\lim_{n\to\infty} f_r^k(x_{j,n}^i) = \delta_{ik}\delta_{rj}.$$

By Theorem 3.2, we can find a basis $\{g_1^i, g_2^i, \ldots, g_{s(i)}^i\}$ for $\mathcal{HB}_{\mathcal{L}}(E_i, \partial E_i)$ such that for $j = 1, 2, \ldots, s(i)$ and $i = 1, 2, \ldots, l$,

(i) $0 \le w_j^i \le g_j^i \le 1$;

(ii) $\sup_{E_i} \sum_{j=1}^{s(i)} g_j^i = 1$.

Since $\sup_{\Omega_j^i} w_j^i = 1$,

$$\lim_{n\to\infty} g_r^i(x_{j,n}^i) = \delta_{rj}$$

for each $j = 1, 2, \ldots, s(i)$ and $i = 1, 2, \ldots, l$.

Let h be a function in $\mathcal{HB}_{\mathcal{L}}(M)$. Combining Theorem 2.3, Lemma 4.1 and Theorem 3.2, we can construct a unique function h_i in $\mathcal{HB}_{\mathcal{L}}(E_i, \partial E_i)$ in such a way that

$$\lim_{n\to\infty} h_i(x_{j,n}^i) = \lim_{n\to\infty} h(x_{j,n}^i)$$

for each $j = 1, 2, \ldots, s(i)$. In fact, if $h = \sum_{i=1}^{l} \sum_{j=1}^{s(i)} a_j^i f_j^i$, then $h_i = \sum_{j=1}^{s(i)} a_j^i g_j^i$. Let us define $\Phi : \mathcal{HB}_{\mathcal{L}}(M) \to \prod_{i=1}^{l} \mathcal{HB}_{\mathcal{L}}(E_i, \partial E_i)$ by

$$\Phi(h) = (h_1, h_2, \ldots, h_l).$$

Then by the uniqueness of the $\mathcal{L}$-harmonic functions $h_1, h_2, \ldots, h_l$, the map Φ is well defined.

Clearly, the map Φ is linear.

If $h = \sum_{i=1}^{l} \sum_{j=1}^{s(i)} a_j^i f_j^i \in \ker\Phi$, i.e., $\Phi(h) = (h_1, h_2, \ldots, h_l) = (0, 0, \ldots, 0)$, then

$$a_j^i = \lim_{n\to\infty} h(x_{j,n}^i) = \lim_{n\to\infty} h_i(x_{j,n}^i) = 0$$

for each $j = 1, 2, \ldots, s(i)$ and $i = 1, 2, \ldots, l$. Hence $h \equiv 0$ on M. Therefore, the map Φ is injective.

Let $(h_1, h_2, \ldots, h_l) \in \prod_{i=1}^{l} \mathcal{HB}_{\mathcal{L}}(E_i, \partial E_i)$. Then we may write

$$(h_1, h_2, \ldots, h_l) = \Big(\sum_{j=1}^{s(1)} a_j^1 g_j^1, \sum_{j=1}^{s(2)} a_j^2 g_j^2, \ldots, \sum_{j=1}^{s(l)} a_j^l g_j^l\Big)$$

Let $h = \sum_{i=1}^{l} \sum_{j=1}^{s(i)} a_j^i f_j^i$. Then $h \in \mathcal{HB}_{\mathcal{L}}(M)$ and $\Phi(h) = (h_1, h_2, \ldots, h_l)$, i.e., the map Φ is surjective. □

Arguing similarly as in the proof of Theorem 1.1, we get the same result in the case of bounded energy finite $\mathcal{L}$-harmonic functions as follows:

Corollary 4.2. *Let M be a complete Riemannian manifold and $\mathcal{L} = \Delta - V$. Let $E_1, E_2, \ldots, E_l$, $l \geq 1$, be $\mathcal{L}$-nonparabolic ends of M. Then $\mathcal{HBD}_{\mathcal{L}}(M)$ has the same dimension as the dimension of $\prod_{i=1}^{l} \mathcal{HBD}_{\mathcal{L}}(E_i, \partial E_i)$, where $\mathcal{HBD}_{\mathcal{L}}(X)$ and $\mathcal{HBD}_{\mathcal{L}}(X, \partial X)$ denote the space of bounded energy finite $\mathcal{L}$-harmonic functions on X and the subspace of elements of $\mathcal{HBD}_{\mathcal{L}}(X)$ vanishing at ∂X, respectively.*

In particular, in the case when $\mathcal{HBD}_{\mathcal{L}}(M)$ is finite dimensional, there exists an isomorphism

$$\Phi : \mathcal{HBD}_{\mathcal{L}}(M) \to \prod_{i=1}^{l} \mathcal{HBD}_{\mathcal{L}}(E_i, \partial E_i).$$

Applying our argument to the case of harmonic functions, we have the following isomorphism between the space of bounded harmonic functions (with finite Dirichlet integral, respectively) on a complete Riemannian manifold and the Cartesian product of those on its ends:

Corollary 4.3. *Let M be a complete Riemannian manifold with nonparabolic ends $E_1, E_2, \ldots, E_l$, $l \geq 1$. Then $\mathcal{HB}(M)$ has the same dimension as the dimension of $\prod_{i=1}^{l} \mathcal{HB}(E_i, \partial E_i)$, where $\mathcal{HB}(X)$ and $\mathcal{HB}(X, \partial X)$ denote the space of bounded harmonic functions on X and the subspace of elements of $\mathcal{HB}(X)$ vanishing at ∂X, respectively.*

In particular, in the case when $\mathcal{HB}(M)$ is finite dimensional, there exists an isomorphism

$$\Phi : \mathcal{HB}(M) \to \prod_{i=1}^{l} \mathcal{HB}(E_i, \partial E_i).$$

Also, $\mathcal{HBD}(M)$ has the same dimension as that of $\prod_{i=1}^{l} \mathcal{HBD}(E_i, \partial E_i)$, where $\mathcal{HBD}(X)$ and $\mathcal{HBD}(X, \partial X)$ denote the space of bounded harmonic functions with finite Dirichlet integral on X and the subspace of elements of $\mathcal{HBD}(X)$ vanishing at ∂X, respectively.

In particular, in the case when $\mathcal{HBD}(M)$ is finite dimensional, there exists an isomorphism

$$\Phi : \mathcal{HBD}(M) \to \prod_{i=1}^{l} \mathcal{HBD}(E_i, \partial E_i).$$

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