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WEAK AND STRONG CONVERGENCE CRITERIA OF MODIFIED NOOR ITERATIONS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE

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Abstract. In this paper weak and strong convergence theorems of modified Noor iterations to fixed points for asymptotically nonexpansive mappings in the intermediate sense in Banach spaces are established. In one theorem where we establish strong convergence we assume an additional property of the operator whereas in another theorem where we establish weak convergence assume an additional property of the space.

1. Introduction

In recent years, one-step and two-step iterative schemes (including Mann and Ishikawa iteration processes as the most important cases) have been studied extensively by many authors to solve the nonlinear operator equations as well as variational inequalities in Hilbert spaces and Banach spaces, see [6]-[10], [17]-[19], and [24]. Fixed point iteration process for asymptotically nonexpansive mappings in Banach spaces including Mann and Ishikawa iteration process have been studied extensively by many authors to solve the nonlinear operator equations as well as variational inequalities in Hilbert spaces and Banach spaces. Noor [13], [14] introduced and analyzed three-step iterative methods to study the approximate solutions of variational inclusions (inequalities) in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. Glowinski and Le Tallec [3] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory and eigenvalue computation. It has been shown in [3] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. In 1998, Haubruge, Nguyen and Strodiot [5] studied the convergence analysis of three-step schemes of Glowinski and Le Tallec [3] and applied these schemes to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization

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of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus we conclude that three-step iterative schemes play an important and significant part in solving various problems which arise in pure and applied sciences. Recently Xu and Noor [25] introduced and studied a three-step iteration scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. In 2004, Cho et. al. [2] extended the work of Xu and Noor to the three-step iteration scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in Banach space. Moreover, Suantai [21] gave weak and strong convergence theorems for a new three-step iterative scheme which can be viewed as an extension for three-step and two-step iterative schemes of Glowinski and Le Tallec [3], Noor [13], Xu and Noor [25], Ishikawa [7]. Inspired and motivated by these works we address the problem of Suantai [21] to more general class of operators that is to asymptotically nonexpansive mappings in the intermediate sense.

Let X be a normed space, C be a nonempty convex subset of X and $T : C \to C$ be a given mapping. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative scheme of asymptotically nonexpansive mappings.

$$\begin{aligned} x_{n+1} &= (1-\alpha_n-\beta_n)x_n + \alpha_n T^n y_n + \beta_n T^n z_n \\ y_n &= (1-b_n-c_n)x_n + b_n T^n z_n + c_n T^n x_n \\ z_n &= (1-a_n)x_n + a_n T^n x_n, \qquad n \geq 1, \end{aligned} \tag{1.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ are appropriate sequences in $[0,1]$. The iterative scheme (1.1) is called the modified Noor iterations [21]. Noor iterations include the Mann-Ishikawa iterations as special cases. If $c_n = \beta_n \equiv 0$, then (1.1) reduces to Noor iterations defined by Xu and Noor [25] :

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n \\ y_n &= (1-b_n)x_n + b_n T^n z_n \\ z_n &= (1-a_n)x_n + a_n T^n x_n, \qquad n \geq 1, \end{aligned} \tag{1.2}$$

where $\{a_n\}, \{b_n\}, \{\alpha_n\}$ are appropriate sequences in $[0,1]$. For $a_n = c_n = \beta_n \equiv 0$, then (1.1) reduces to the usual Ishikawa iterative scheme

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n \\ y_n &= (1-b_n)x_n + b_n T^n x_n, \qquad n \geq 1, \end{aligned} \tag{1.3}$$

where $\{b_n\}, \{\alpha_n\}$ are appropriate sequences in $[0,1]$. For $a_n = b_n = c_n = \beta_n \equiv 0$, then (1.1) reduces to the usual Mann iterative scheme

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \qquad n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}$ are appropriate sequences in $[0,1]$.

Now we recall some well known results and definitions:

Let X be a normed space, C be a nonempty subset of X and let $T : C \to C$ be a given mapping. Then T is said to be asymptotically nonexpansive [4] if there exist a sequence $\{k_n\}, k_n \geq 1$, with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y||$$

for all $x, y \in C$ and each $n \geq 1$. The weaker definition [9] requires that

$$\limsup_{n\to\infty} \sup_{x,y\in C} (||T^n x - T^n y|| - ||x - y||) \leq 0$$

for every $x \in C$ and that T^N be continuous for some $N \geq 1$.

Bruck et. al. [1] gave a definition which is somewhere between these two : T is called asymptotically nonexpansive mapping in the intermediate sense [1] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (||T^n x - T^n y|| - ||x - y||) \leq 0.$$

T is said to be uniformly L-Lipschitzian if there exist a constant $L > 0$ such that

$$||T^n x - T^n y|| \leq L||x - y||$$

for all $x, y \in C$ and all $n \geq 1$. From the above definitions it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive mapping in the intermediate sense and uniformly L-Lipschitzian mapping. But the converse does not hold:

Example. ([8]) Let $X = R, C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in C$, define

$$T(x) = \begin{cases} kx \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then T is asymptotically nonexpansive mapping in the intermediate sense, but is not a Lipschitzian mapping but $T^n x \to 0$ uniformly so that it is not asymptotically nonexpansive mapping.

A Banach space X is said to satisfy Opial's condition [15] if $x_n \rightharpoonup x$ and $x \neq y$ imply

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||.$$

A Banach space X is said to satisfy τ-Opial condition [1] if for every bounded $\{x_n\} \in X$ that τ-converges to $x \in X$ then

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||$$

for every $x \neq y$, where τ is a Housdorff linear topology on X. A Banach space X has the uniform τ-Opial property [1] if for each $c > 0$ there exists $r > 0$ with the property that for each $x \in X$ and each sequence $\{x_n\}$ such that $\{x_n\}$ is τ-convergent to 0 and

$$1 \leq \limsup_{n\to\infty} ||x_n|| < \infty, ||x|| \geq c$$

imply that $\limsup_{n\to\infty} ||x_n - x|| \geq 1 + r$. Clearly uniform τ-Opial condition implies τ-Opial condition. Note that a uniformly convex space which has the τ-Opial property necessarily has the uniform τ-Opial property, where τ is a Housdorff linear topology on X. Let $\{x_n\}$ be a given sequence in C. A mapping $T : C \to C$ with nonempty fixed point set $F(T)$ in C satisfying Condition (A) with respect to the sequence $\{x_n\}$ [20] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$f(d(x_n, F(T))) \leq ||x_n - Tx_n|| \text{ for all } n \geq 1.$$

Lemma 1.1 ([22, Lemma 1]). *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of non-negative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \forall n = 1, 2, \ldots.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(i) $\lim_{n\to\infty} a_n$ *exists*

(ii) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 1.2 ([23, Lemma 2]). *Let $p > 1, r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$||\lambda x + (1-\lambda)y||^p \leq \lambda||x||^p + (1-\lambda)||y||^p - \omega_p(\lambda)g(||x - y||)$$

for all $x, y \in B_r = \{x \in X : ||x|| \leq r\}$ and $\lambda \in [0,1]$, where $\omega_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

Lemma 1.3 ([2, Lemma 1.4]). *Let X be a uniformly convex Banach space and $B_r = \{x \in X : ||x|| \leq r\}$, $r > 0$. Then there exist a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$||\lambda x + \beta y + \gamma z||^2 \leq \lambda||x||^2 + \beta||y||^2 + \gamma||z||^2 - \lambda\beta g(||x - y||)$$

for all $x, y, z \in B_r$ and all $\lambda, \beta, \gamma \in [0,1]$ with $\lambda + \beta + \gamma = 1$.

From Lemma 1.3 we easily get

Lemma 1.4. *Let X be a uniformly convex Banach space and $B_r = \{x \in X : ||x|| \leq r\}$, $r > 0$. Then there exist a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$||\lambda x + \beta y + \gamma z||^2 \leq \lambda||x||^2 + \beta||y||^2 + \gamma||z||^2 - \frac{1}{2}\gamma(\lambda g(||x - z||) + \beta g(||y - z||))$$

for all $x, y, z \in B_r$ and all $\lambda, \beta, \gamma \in [0,1]$ with $\lambda + \beta + \gamma = 1$.

Lemma 1.5 ([1]). *Suppose a Banach space X has the uniform τ-opial property, C is a norm bounded, sequentially τ-compact subset of X and $T : C \to C$ is asymptotically nonexpansive in the weak sense. If $\{y_n\}$ is a sequence in C such that $\lim_{n\to\infty} ||y_n - z||$ exist for each fixed point z of T and if $\{y_n - T^k y_n\}$is τ-convergent to 0 for each $k \in N$, then $\{y_n\}$ is τ-convergent to a fixed point of T.*

2. Main results

In this section we begin with the following lemmas.

Lemma 2.1. *Let X be a uniformly convex Banach space, and let C be a nonempty closed, bounded and convex subset of X. Let T be an asymptotically nonexpansive mapping in the intermediate sense. Put $d_n = \sup_{x,y\in C}(||T^n x - T^n y|| - ||x-y||) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ and $b_n + c_n$ are in $[0,1]$ for all $n \geq 1$. For a given $x_1 \in C$, let the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.1).*

(i) *if q is a fixed point of T, then $\lim_{n\to\infty} ||x_n - q||$ exists.*

(ii) *if $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then*
$$\lim_{n\to\infty} ||T^n y_n - x_n|| = 0.$$

(iii) *if $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then*
$$\lim_{n\to\infty} ||T^n z_n - x_n|| = 0.$$

(iv) *if $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$ and $\limsup_{n\to\infty}(b_n + c_n) < 1$ then*
$$\lim_{n\to\infty} ||T^n x_n - x_n|| = 0.$$

Proof. Existence of fixed point of T follows from [9]. So $F(T) \neq \emptyset$. Let $x^* \in F(T)$. Choose a real number $r > 0$ such that $C \subseteq B_r$ and $C - C \subseteq B_r$. By Lemma 1.2 there exists a continuous strictly increasing and convex function $g_1 : [0,\infty) \to [0,\infty)$, $g_1(0) = 0$ such that

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - \omega_2(\lambda)g_1(||x-y||) \tag{2.1}$$

for all $x, y \in B_r = \{x \in X : ||x|| \leq r\}$ and $\lambda \in [0,1]$, where $\omega_2(\lambda) = \lambda(1-\lambda)^2 + \lambda^2(1-\lambda)$. From (2.1) and (1.1) we get

$$\begin{aligned}
&||z_n - x^*||^2 \\
&= ||(1-a_n)(x_n - x^*) + a_n(T^n x_n - x^*)||^2 \\
&\leq a_n||T^n x_n - x^*||^2 + (1-a_n)||x_n - x^*||^2 - \omega_2(a_n)g_1(||T^n x_n - x_n||) \\
&\leq a_n(||x_n - x^*|| + d_n)^2 + (1-a_n)||x_n - x^*||^2 - \omega_2(a_n)g_1(||T^n x_n - x_n||) \\
&\leq ||x_n - x^*||^2 + 2d_n a_n||x_n - x^*|| + a_n d_n^2.
\end{aligned} \tag{2.2}$$

By Lemma 1.4 there exists a continuous strictly increasing and convex function $g_2 : [0,\infty) \to [0,\infty)$, $g_2(0) = 0$ such that

$$\begin{aligned}
&||\lambda x + \beta y + \gamma z||^2 \\
\leq\ & \lambda||x||^2 + \beta||y||^2 + \gamma||z||^2 - \frac{1}{2}\gamma(\lambda g_2(||x-z||) + \beta g_2(||y-z||))
\end{aligned} \tag{2.3}$$

for all $x, y, z \in B_r$ and all $\lambda, \beta, \gamma \in [0,1]$ with $\lambda + \beta + \gamma = 1$. It follows from (2.2) and (1.1) that

(2.4)
$$\begin{aligned}
&||y_n - x^*||^2\\
&= ||b_n(T^n z_n - x^*) + (1 - b_n - c_n)(x_n - x^*) + c_n(T^n x_n - x^*)||^2\\
&\le b_n||T^n z_n - x^*||^2 + (1 - b_n - c_n)||x_n - x^*||^2 + c_n||T^n x_n - x^*||^2\\
&\quad - \frac{1}{2}(1 - b_n - c_n)(b_n g_2(||T^n z_n - x_n||) + c_n g_2(||T^n x_n - x_n||))\\
&\le b_n(||z_n - x^*|| + d_n)^2 + (1 - b_n - c_n)||x_n - x^*||^2 + c_n(||x_n - x^*|| + d_n)^2\\
&\quad - \frac{1}{2}b_n(1 - b_n - c_n)g_2(||T^n z_n - x_n||)\\
&= b_n||z_n - x^*||^2 + 2b_n d_n||z_n - x^*|| + b_n d_n^2 + (1 - b_n - c_n)||x_n - x^*||^2\\
&\quad + c_n||x_n - x^*||^2 + 2c_n d_n||x_n - x^*|| + c_n d_n^2\\
&\quad - \frac{1}{2}b_n(1 - b_n - c_n)g_2(||T^n z_n - x_n||).
\end{aligned}$$

From (1.1) and using (2.2), (2.3), (2.4) we get,

(2.5)
$$\begin{aligned}
&||x_{n+1} - x^*||^2\\
&= ||\alpha_n(T^n y_n - x^*) + (1 - \alpha_n - \beta_n)(x_n - x^*) + \beta_n(T^n z_n - x^*)||^2\\
&\le \alpha_n||T^n y_n - x^*||^2 + \beta_n||T^n z_n - x^*||^2 + (1 - \alpha_n - \beta_n)||x_n - x^*||^2\\
&\quad - \frac{1}{2}(1 - \alpha_n - \beta_n)(\alpha_n g_2(||T^n y_n - x_n||) + \beta_n g_2(||T^n z_n - x_n||))\\
&\le ||x_n - x^*||^2 + [(2a_n b_n \alpha_n + 2a_n\beta_n + 2b_n\alpha_n + 2c_n\alpha_n + 2\alpha_n + 2\beta_n)||x_n - x^*||\\
&\quad + (\alpha_n b_n + \beta_n)a_n + 4\alpha_n b_n + 3\alpha_n + 3\beta_n + \alpha_n(b_n + c_n)d_n]d_n\\
&\quad - \frac{1}{2}\alpha_n b_n(1 - b_n - c_n)g_2(||T^n z_n - x_n||)\\
&\quad - \frac{1}{2}(1 - \alpha_n - \beta_n)(\alpha_n g_2(||T^n y_n - x_n||) + \beta_n g_2(||T^n z_n - x_n||)).
\end{aligned}$$

Since $\{d_n\}$ and K are bounded so $\exists$ a constant $M > 0$ such that

$$\begin{aligned}
&(2a_n b_n \alpha_n + 2a_n\beta_n + 2b_n\alpha_n + 2c_n\alpha_n + 2\alpha_n + 2\beta_n)||x_n - x^*||\\
&+ (\alpha_n b_n + \beta_n)a_n + 4\alpha_n b_n + 3\alpha_n + 3\beta_n + \alpha_n(b_n + c_n)d_n \le M.
\end{aligned} \tag{2.6}$$

From (2.5) and (2.6) we get,

$$\begin{aligned}
&||x_{n+1} - x^*||^2\\
&\le ||x_n - x^*||^2 + Md_n - \frac{1}{2}\alpha_n b_n(1 - b_n - c_n)g_2(||T^n z_n - x_n||)\\
&\quad - \frac{1}{2}(1 - \alpha_n - \beta_n)(\alpha_n g_2(||T^n y_n - x_n||) + \beta_n g_2(||T^n z_n - x_n||)).
\end{aligned} \tag{2.7}$$

Thus it follows from (2.7) that

(2.8) $\alpha_n(1-\alpha_n-\beta_n)g_2(||T^n y_n - x_n||) \leq 2(||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + Md_n)$,

(2.9) $\beta_n(1-\alpha_n-\beta_n)g_2(||T^n z_n - x_n||) \leq 2(||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + Md_n)$

and

(2.10) $\alpha_n b_n(1-b_n-c_n)g_2(||T^n z_n - x_n||) \leq 2(||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + Md_n)$.

(i) If $q \in F(T)$, by taking $x^* = q$ in (2.7) we have

$$||x_{n+1} - q||^2 \leq ||x_n - q||^2 + Md_n$$

since $\sum_{n=0}^{\infty} d_n < \infty$, so from Lemma 1.1 we have $\lim_{n\to\infty} ||x_n - q||$ exists.

(ii) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then $\exists n_0 \in N$ and $\eta, \eta\prime \in (0,1)$ such that $0 < \eta < \alpha_n$ and $\alpha_n + \beta_n < \eta\prime < 1$, $\forall n \geq n_0$. From (2.8) we get,

$$\eta(1-\eta\prime)g_2(||T^n y_n - x_n||) \leq 2(||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + Md_n), \forall n \geq n_0.$$

Thus from above we have

$$\eta(1-\eta\prime)\sum_{n=n_0}^{\infty} g_2(||T^n y_n - x_n||) \leq 2(||x_{n_0} - x^*||^2 + M\sum_{n=n_0}^{\infty} d_n) < \infty.$$

This implies that

$$\sum_{n=n_0}^{\infty} g_2(||T^n y_n - x_n||) < \infty.$$

which implies that $\lim_{n\to\infty} g_2(||T^n y_n - x_n||) = 0$. Since g_2 is strictly increasing and continuous at 0 with $g(0) = 0$ so we have $\lim_{n\to\infty} ||T^n y_n - x_n|| = 0$.

(iii) Similarly if $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then from (2.9) we have

$$\lim_{n\to\infty} ||T^n z_n - x_n|| = 0.$$

(iv) Now

$$\begin{aligned} ||T^n x_n - x_n|| &\leq ||T^n x_n - T^n y_n|| + ||T^n y_n - x_n|| \\ &\leq ||x_n - y_n|| + d_n + ||T^n y_n - x_n||. \end{aligned}$$

Again

$$||y_n - x_n|| \leq b_n||T^n z_n - x_n|| + c_n||T^n x_n - x_n||.$$

Thus

(2.11)

$$\begin{aligned} ||T^n x_n - x_n|| &\leq ||x_n - y_n|| + d_n + ||T^n y_n - x_n|| \\ &\leq b_n||T^n z_n - x_n|| + c_n||T^n x_n - x_n|| + d_n + ||T^n y_n - x_n||. \end{aligned}$$

Let $\{m_j\}$ be a subsequence of $\{n\}$. If $\liminf_{j\to\infty} b_{m_j} > 0$, then from (2.10) we get $\lim_{j\to\infty} ||T^{m_j} z_{m_j} - x_{m_j}|| = 0$. Again since $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then from ($ii$) we get $\lim_{j\to\infty} ||T^{m_j} y_{m_j} - x_{m_j}|| = 0$. So from (2.11) we get $\lim_{j\to\infty}(1 - c_{m_j})||T^{m_j} x_{m_j} - x_{m_j}|| = 0$. Thus we have

$\lim_{j\to\infty} ||T^{m_j} x_{m_j} - x_{m_j}|| = 0$. Again if $\liminf_{j\to\infty} b_{m_j} = 0$, then there exists a subsequence $\{b_{n_k}\}$ of $\{b_{m_j}\}$ such that $\lim_{k\to\infty} b_{n_k} = 0$. From *(ii)* and (2.11) we get, $\lim_{k\to\infty}(1 - c_{n_k})||T^{n_k} x_{n_k} - x_{n_k}|| = 0$. Thus we have $\lim_{k\to\infty} ||T^{n_k} x_{n_k} - x_{n_k}|| = 0$. Thus we have $\lim_{n\to\infty} ||T^n x_n - x_n|| = 0$. □

Theorem 2.1. *Let X be a uniformly convex Banach space and C be a nonempty closed bounded and convex subset of X. Let T be asymptotically nonexpansive self map of C in the intermediate sense. Put $d_n = \sup_{x,y\in C}(||T^n x - T^n y|| - ||x - y||) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (1.1) with $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n$ and $b_n + c_n$ are in $[0, 1]$ for all $n \geq 1$ and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$ *and*
(ii) $\limsup_{n\to\infty}(b_n + c_n) < 1$.

If T satisfies Condition (A) *with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.1(iv) we have $\lim_{n\to\infty} ||T^n x_n - x_n|| = 0$. Now

$$\begin{aligned} ||T^n z_n - x_n|| &\leq ||T^n z_n - T^n x_n|| + ||T^n x_n - x_n|| \\ &\leq ||z_n - x_n|| + d_n + ||T^n x_n - x_n|| \\ &= a_n||T^n x_n - x_n|| + d_n + ||T^n x_n - x_n|| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Again

$$\begin{aligned} & ||T^n y_n - x_n|| \\ \leq\ & ||T^n y_n - T^n x_n|| + ||T^n x_n - x_n|| \\ \leq\ & ||y_n - x_n|| + d_n + ||T^n x_n - x_n|| \\ \leq\ & b_n||T^n z_n - x_n|| + c_n||T^n x_n - x_n|| + d_n + ||T^n x_n - x_n|| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Now

$$||x_{n+1} - x_n|| \leq \alpha_n||T^n y_n - x_n|| + \beta_n||T^n z_n - x_n|| \to 0 \text{ as } n \to \infty.$$

$$\begin{aligned} ||x_n - Tx_n|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - T^{n+1}x_{n+1}|| \\ &\quad + ||T^{n+1}x_{n+1} - T^{n+1}x_n|| + ||T^{n+1}x_n - Tx_n|| \\ &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - T^{n+1}x_{n+1}|| + ||x_{n+1} - x_n|| + d_{n+1} \\ &\quad + ||T^{n+1}x_n - Tx_n|| \\ &= 2||x_n - x_{n+1}|| + ||x_{n+1} - T^{n+1}x_{n+1}|| + d_{n+1} \\ &\quad + ||T^{n+1}x_n - Tx_n|| \end{aligned}$$

as T is uniformly continuous so

$$\lim_{n\to\infty} ||x_n - Tx_n|| = 0. \tag{2.12}$$

Since T satisfies Condition (A) with respect to the sequence $\{x_n\}$ so there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all

$r \in (0, \infty)$ such that

$$f(d(x_n, F(T))) \leq ||x_n - Tx_n|| \to 0, \text{ as } n \to \infty.$$

So $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Now by (1.1) we get

$$\begin{aligned}
&||x_{n+1} - x^*|| \\
&\leq \alpha_n ||T^n y_n - x^*|| + (1 - \alpha_n - \beta_n)||x_n - x^*|| + \beta_n ||T^n z_n - x^*|| \\
&\leq \alpha_n (||y_n - x^*|| + d_n) + (1 - \alpha_n - \beta_n)||x_n - x^*|| + \beta_n (||z_n - x^*|| + d_n) \\
&\leq ||x_n - x^*|| + (\alpha_n b_n + 2\alpha_n + 2\beta_n) d_n \\
&\leq ||x_n - x^*|| + 5d_n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
||x_{n+m} - x^*|| &\leq ||x_{n+m-1} - x^*|| + 5d_{n+m-1} \\
&\leq ||x_{n+m-2} - x^*|| + 5d_{n+m-2} + 5d_{n+m-1} \\
&\leq \cdots\cdots \\
&\leq ||x_n - x^*|| + 5 \sum_{j=n}^{n+m-1} d_j.
\end{aligned} \tag{2.13}$$

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} d_n < \infty$, so for given $\epsilon > 0$, there exist $N_0 \in N$ such that $d(x_n, F(T)) < \frac{\epsilon}{4}, \sum_{j=n}^{\infty} d_j < \frac{\epsilon}{20}, \forall n \geq N_0$. In particular $d(x_{N_0}, F(T)) < \frac{\epsilon}{4}$. So there exists $q \in F(T)$ such that $||x_{N_0} - q|| = d(x_{N_0}, q) < \frac{\epsilon}{4}$. From (2.13) we get

$$\begin{aligned}
||x_{n+m} - x_n|| &\leq ||x_{n+m} - q|| + ||x_n - q|| \\
&\leq 2||x_n - q|| + 5 \sum_{j=n}^{n+m-1} d_j \\
&\leq 2||x_{N_0} - q|| + 10 \sum_{j=N_0}^{n-1} d_j + 5 \sum_{j=n}^{n+m-1} d_j \\
&< 2||x_{N_0} - q|| + 10 \sum_{j=N_0}^{\infty} d_j \\
&< 2\frac{\epsilon}{4} + 10\frac{\epsilon}{20} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in C. So by completeness of C we get there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$. By (2.12) and continuity of T we get $Tp = p$ that is $p \in F(T)$. This completes the proof of the theorem. □

For $c_n = \beta_n \equiv 0$ in Theorem 2.1 we obtain the following result.

Corollary 2.1. *Let X be a uniformly convex Banach space and C be a non-empty closed, bounded and convex subset of X. Let T be asymptotically non-expansive self map of C in the intermediate sense. Put $d_n = \sup_{x,y\in C}(||T^n x -$*

$T^n y|| - ||x - y||) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (1.2) with $\{a_n\}, \{b_n\}, \{\alpha_n\}$ be real sequences in $[0,1]$ such that

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*
(*ii*) $\limsup_{n\to\infty} b_n < 1$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

For $a_n = c_n = \beta_n \equiv 0$ in Theorem 2.1 we obtain the following result.

Corollary 2.2. *Let X be a uniformly convex Banach space and C be a nonempty closed, bounded and convex subset of X. Let T be asymptotically nonexpansive self map of C in the intermediate sense. Put $d_n = \sup_{x,y\in C}(||T^n x - T^n y|| - ||x - y||) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (1.3) with $\{b_n\}, \{\alpha_n\}$ be real sequences in $[0,1]$ such that*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*
(*ii*) $\limsup_{n\to\infty} b_n < 1$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

For $a_n = b_n = c_n = \beta_n \equiv 0$ in Theorem 2.1 we obtain the following result.

Corollary 2.3. *Let X be a uniformly convex Banach space and C be a nonempty closed, bounded and convex subset of X. Let T be asymptotically nonexpansive self map of C in the intermediate sense. Put $d_n = \sup_{x,y\in C}(||T^n x - T^n y|| - ||x - y||) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (1.4) with $\{\alpha_n\}$ be real sequences in $[0,1]$ such that*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Since every asymptotically nonexpansive mapping is uniformly continuous, we immediately get

Corollary 2.4. *Let X be a uniformly convex Banach space and C be a nonempty closed, bounded and convex subset of X. Let T be asymptotically nonexpansive self map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence defined as in (1.1) with $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ and $b_n + c_n$ are in $[0,1]$ for all $n \geq 1$ and*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$ *and*
(*ii*) $\limsup_{n\to\infty}(b_n + c_n) < 1$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since $\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty}(k_n - 1)diam(C) < \infty$, where $diam(C) = \sup_{x,y \in C} \|x - y\| < \infty$, so the conclusion follows immediately from Theorem 2.1. □

The Corollary 2.4 is obtained without the restriction $\liminf_{n\to\infty} b_n > 0$ as was in [21]. Furthermore if T is completely continuous, then T satisfies Condition (A) with respect to the sequence $\{x_n\}$ ([2], Corollary 2.5). Also Corollary 2.4 includes Theorem 2.2 and Theorem 2.3 of [25].

For $c_n = \beta_n \equiv 0$ in Theorem 2.1 we obtain the following result which improves Theorem 2.1 of [25].

Corollary 2.5. *Let X be a uniformly convex Banach space and C be a non-empty closed, bounded and convex subset of X. Let T be asymptotically nonexpansive self map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence defined as in* (1.2) *with $\{a_n\}, \{b_n\}, \{\alpha_n\}$ be real sequences in* $[0, 1]$ *such that*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*
(*ii*) $\limsup_{n\to\infty} b_n < 1$.

If T satisfies Condition (A) *with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Since $\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty}(k_n - 1)diam(C) < \infty$, where $diam(C) = \sup_{x,y \in C} \|x - y\| < \infty$, so the conclusion follows immediately from Corollary 2.1. □

For $a_n = c_n = \beta_n \equiv 0$ in Theorem 2.1 we obtain the following result.

Corollary 2.6. *Let X be a uniformly convex Banach space and C be a non-empty closed, bounded and convex subset of X. Let T be asymptotically non-expansive self map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence defined as in* (1.3) *with $\{b_n\}, \{\alpha_n\}$ be real sequences in* $[0, 1]$ *such that*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*
(*ii*) $\limsup_{n\to\infty} b_n < 1$.

If T satisfies Condition (A) *with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Since $\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty}(k_n - 1)diam(C) < \infty$, where $diam(C) = \sup_{x,y \in C} \|x - y\| < \infty$, so the conclusion follows immediately from Corollary 2.2. □

For $a_n = b_n = c_n = \beta_n \equiv 0$ in Theorem 2.1 we obtain the following result.

Corollary 2.7. *Let X be a uniformly convex Banach space and C be a non-empty closed, bounded and convex subset of X. Let T be asymptotically nonexpansive self map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let*

$\{x_n\}$ be the sequence defined as in (1.4) with $\{\alpha_n\}$ be real sequences in $[0,1]$ such that

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since $\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty}(k_n - 1)diam(C) < \infty$, where $diam(C) = \sup_{x,y\in C} ||x - y|| < \infty$, so the conclusion follows immediately from Corollary 2.3. □

Corollary 2.7 extends Theorem 2.6 of [21].

Theorem 2.2. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed, bounded and convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense. Put $d_n = \sup_{x,y\in C}(||T^n x - T^n y|| - ||x - y||) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (1.1) with $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ and $b_n + c_n$ are in $[0,1]$ for all $n \geq 1$ and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$ *and*
(ii) $\limsup_{n\to\infty}(b_n + c_n) < 1$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. From (2.12) we have $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ and so $\lim_{n\to\infty} ||x_n - T^m x_n|| = 0$ for all $m \in N$ by the uniform continuity of T. Then we can apply Lemma 1.2 with the τ-topology taken as weak topology and get the conclusion as follows: $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} ||x_n - z||$ exist for each fixed point $z \in F(T)$. Since $\lim_{n\to\infty} ||x_n - T^m x_n|| = 0$ for all $m \in N$ so $\{x_n - T^m x_n\}$ is weakly convergent to 0 for each $m \in N$. So by Lemma 1.2 we get that $\{x_n\}$ converges weakly to a fixed point of T. □

For $c_n = \beta_n \equiv 0$ in Theorem 2.2 we obtain the following result.

Corollary 2.8. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed, bounded and convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense. Put $d_n = \sup_{x,y\in C}(||T^n x - T^n y|| - ||x - y||) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (1.2) with $\{a_n\}, \{b_n\}, \{\alpha_n\}$ be real sequences in $[0,1]$ for all $n \geq 1$ such that*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*
(ii) $\limsup_{n\to\infty} b_n < 1$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

For $a_n = c_n = \beta_n \equiv 0$ in Theorem 2.2 we obtain the following result:

Corollary 2.9. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed, bounded and convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense. Put $d_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (1.3) with $\{b_n\}, \{\alpha_n\}$ be real sequences in $[0,1]$ for all $n \geq 1$ such that*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*
(*ii*) $\limsup_{n\to\infty} b_n < 1$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

For $a_n = b_n = c_n = \beta_n \equiv 0$ in Theorem 2.2 we obtain the following result.

Corollary 2.10. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed, bounded and convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense. Put $d_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (1.4) with $\{\alpha_n\}$ be real sequence in $[0,1]$ for all $n \geq 1$ such that*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

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