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PROJECTIONS OF PSEUDOSPHERE IN THE LORENTZ 3-SPACE

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Abstract. In this paper, we study the map projections from pseudosphere S_1^2 onto the non-lightlike surfaces in the 3-dimensional Lorentzian space, L^3, with curvature zero. We show geometrical means and properties of $\mathbb{R} \times S_1^1$−cylindrical, $S^1 \times L$−cylindrical and $\mathbb{R} \times H_0^1$−cylindrical projections defined on S_1^2 to cylinders $\mathbb{R} \times S_1^1$, $S^1 \times L$ and $\mathbb{R} \times H_0^1$, respectively, and orthographic and stereographic projections on S_1^2 to Lorentzian plane, L^2.

1. Introduction

In Euclidean geometry, the most map projection can be at least partially visualized geometrically as being projected onto one of three surfaces: the cylinder, the cone, and the plane.

The aim of this paper is to study four map projections defined on S_1^2 to non-lightlike surfaces in the 3-dimensional Lorentzian space, L^3, with curvature zero. These surfaces are the cylinders $\mathbb{R} \times S_1^1$, $S^1 \times L$ and $\mathbb{R} \times H_0^1$, and the Lorentzian plane, L^2. Hence, we study the maps: $\mathbb{R} \times S_1^1$−cylindrical, $S^1 \times L$−cylindrical and $\mathbb{R} \times H_0^1$−cylindrical projections of S_1^2 onto $\mathbb{R} \times S_1^1$, $S^1 \times L$ and $\mathbb{R} \times H_0^1$, respectively, and orthographic and stereographic projections onto L^2.

We find that $\mathbb{R} \times S_1^1$−cylindrical projection and $\mathbb{R} \times H_0^1$−cylindrical projection on $S_1^2 - \{x \in S_1^2; |x_1| = 1\}$ are diffeomorphisms. Also, we define the extended $\mathbb{R} \times S_1^1$−cylindrical projection and $\mathbb{R} \times H_0^1$−cylindrical projection on S_1^2 which are one-to-one and onto maps.

Projection of Euclidean sphere onto straight cylinder is a geodesical mapping, [4]. The $S^1 \times L$−cylindrical projection preserves non-lightlike geodesics and preserves the causality of the non-lightlike geodesic curves.

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Analogous to the Euclidean case, orthographic projection of S_1^2 onto L^2 is not one-to-one map.

The stereographic projection of $S_1^2 - \{x \in S_1^2; x_1 = 1\}$ onto $L^2 - H_0^1$ is a diffeomorphism. We define the extended stereographic projection of S_1^2 onto $L^2 - \{(0,0,-1)\}$ which is one-to-one and onto map. Thus, we find that, in the 3-dimensional space of Lorentz, it is necessary to add two points to the pseudosphere to compact it. In addition, the plane of Lorentz is simple connected but the pseudosphere is not it, for that reason to preserve this property it is necessary to remove one point of the plane.

First, in section two, we will remind well known definitions and formulas about our concerns.

2. Preliminaries

Let x and y be two vectors in the $n-$dimensional vector space $\mathbb{R}^n$, $n \in \{2;3\}$. As it is well known, [1], [3], the Lorentzian inner product of x and y is defined by

$$\langle x, y \rangle = \sum_{i=1}^{n-1} x_i y_i - x_n y_n.$$

Thus the square ds^2 of an element of arc-length is given by

$$ds^2 = \sum_{i=1}^{n-1} dx_i^2 - dx_n^2.$$

The space $\mathbb{R}^n$ furnished with this metric is called a $n-$dimensional Lorentzian space, or Lorentz $n-$space. We write L^n instead of $(\mathbb{R}^n,\ ds)$.

We say that a vector x in L^n is timelike if $\langle x, x \rangle < 0$, spacelike if $\langle x, x \rangle > 0$, and null if $\langle x, x \rangle = 0$. The nulls vectors are also said to be lightlike.

We say that x is orthogonal to y if $\langle x, y \rangle = 0$, $x \neq y$.

A curve with spacelike, timelike or lightlike tangent vectors is said to be spacelike, timelike or lightlike, respectively.

We shall give a surface M in L^3 by expressing its coordinates x_i as functions of two parameters in a certain interval. We consider the functions x_i to be real functions of real variables.

We say that M is a non-lightlike surface if at every $p \in M$ its tangent plane T_pM is furnished with positive definite or Lorentzian metric.

The pseudosphere S_1^2 in L^3 is the surface

$$S_1^2 = \left\{(x_1, x_2, x_3) \in L^3 : x_1^2 + x_2^2 - x_3^2 = 1\right\}.$$

Let us recall that S_1^2 can be parametrized by:

$$\text{(1)} \qquad \begin{cases} x_1 = & \cos\theta \cosh\omega \\ x_2 = & \sin\theta \cosh\omega \\ x_3 = & \sinh\omega, \end{cases}$$

where $\omega \in \mathbb{R}$ and $0 \leq \theta < 2\pi$.

3. Map projections

3.1. The cylindrical projection

In classical way, a $C-$cylindrical projection of points on a unit pseudosphere S centered at O, consists of extending the line OP for each point $P \in S$ until it intersects a cylinder C tangent to S .

In what follows, we will denote the warped product by $\times$, the Lorentzian line by L, and

$$\begin{aligned} S_1^1 &= \left\{x \in L^2 : \langle x, x\rangle = 1\right\}, \\ H_0^1 &= \left\{x \in L^2 : \langle x, x\rangle = -1\right\}, \\ S^1 &= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\right\}. \end{aligned}$$

Let us recall that $\mathbb{R} \times S_1^1$, $S^1 \times L$ and $\mathbb{R} \times H_0^1$ are the cylinders in L^3, and these are non-lightlike surfaces with zero curvature (cf. [1]).

We now study the cylindrical projection when $S = S_1^2$. We distinguish three cases.

3.1.1. $C = \mathbb{R} \times S_1^1$. We consider the cylinder in L^3 defined by:

$$\mathbb{R} \times S_1^1 = \left\{(x_1, x_2, x_3) \in L^3 : x_2^2 - x_3^2 = 1\right\}.$$

The two sheets of $\mathbb{R} \times S_1^1$ are characterized by $x_2 > 0$ and $x_2 < 0$. It follows that $\mathbb{R} \times S_1^1$ admits the parametrizations

$$\text{(2)} \qquad \begin{cases} x_1 = t \\ x_2 = \pm \cosh \omega \\ x_3 = \sinh \omega, \end{cases}$$

where $\omega, t \in \mathbb{R}$.

Let us note that cylinder $\mathbb{R} \times S_1^1$ is tangent to S_1^2 at points $(0, \pm \cosh \omega, \sinh \omega)$. We can not apply the geometric process above mentioned on the lightlike straight lines determined by the intersection of S_1^2 with the planes $x_1 = -1$ and $x_1 = 1$, that is on the curves $(-1, \pm u, u)$ and $(1, \pm u, u)$, respectively, with $u \in \mathbb{R}$.

Definition 1. Let $\Lambda_1 = \left\{x \in S_1^2; |x_1| = 1\right\}$ and $\Lambda_2 = \left\{x \in \mathbb{R} \times S_1^1; |x_1| = 1\right\}$. The $\mathbb{R} \times S_1^1-$cylindrical projection is defined:

$$\pi_1 : S_1^2 - \Lambda_1 \to \mathbb{R} \times S_1^1 - \Lambda_2$$

such that

$$\pi_1(x_1, x_2, x_3) = \begin{cases} \frac{1}{\sqrt{x_2^2 - x_3^2}} (x_1, x_2, x_3) & \text{if} \quad |x_2| > |x_3|, \\ \frac{1}{\sqrt{x_3^2 - x_2^2}} (x_1, x_3, x_2) & \text{if} \quad |x_2| < |x_3|, \end{cases}$$

for all $(x_1, x_2, x_3) \in S_1^2 - \Lambda_1$.

Proposition 2. *The map π_1 is a diffeomorphism.*

Proof. Let f_1 and f_2 be the parametrizations (1) and (2), respectively. We have

$$(\theta,\omega) \overset{f_1}{\to} (\cos\theta\cosh\omega, \sin\theta\cosh\omega, \sinh\omega) \overset{\pi_1}{\to} (t, \pm\cosh\varpi, \sinh\varpi) \overset{f_2^{-1}}{\to} (t,\varpi)$$

where

$$t = t(\theta,\omega) = \begin{cases} \frac{\cos\theta}{\sqrt{\sin^2\theta - \tanh^2\omega}} & \text{if} \quad |\sin\theta| > |\tanh\omega| \\ \frac{\cos\theta}{\sqrt{\tanh^2\omega - \sin^2\theta}} & \text{if} \quad |\sin\theta| < |\tanh\omega| \end{cases}$$

and

$$\varpi = \varpi(\theta,\omega) = \begin{cases} \sinh^{-1}\left(\frac{\tanh\omega}{\sqrt{\sin^2\theta - \tanh^2\omega}}\right) & \text{if} \quad |\sin\theta| > |\tanh\omega| \\ \sinh^{-1}\left(\frac{\sin\theta}{\sqrt{\tanh^2\omega - \sin^2\theta}}\right) & \text{if} \quad |\sin\theta| < |\tanh\omega| \end{cases}$$

are diffeomorphisms. □

Definition 3. An extended $\mathbb{R}\times S_1^1$−cylindrical projection is defined:

$$\overline{\pi_1} : S_1^2 \to \mathbb{R}\times S_1^1 - \{(1,-1,0),(-1,-1,0)\}$$

such that

$$\overline{\pi_1}(x_1,x_2,x_3) = \begin{cases} \frac{1}{\sqrt{x_2^2 - x_3^2}}(x_1,x_2,x_3) & \text{if} \quad |x_1| \neq 1 \text{ and } |x_2| > |x_3|, \\ \frac{1}{\sqrt{x_3^2 - x_2^2}}(x_1,x_3,x_2) & \text{if} \quad |x_1| \neq 1 \text{ and } |x_2| < |x_3|, \\ \left(x_1, \sqrt{1+x_3^2}, x_3\right) & \text{if} \quad |x_1| = 1 \text{ and } x_2 = x_3, \\ \left(x_1, -\sqrt{1+x_3^2}, x_3\right) & \text{if} \quad |x_1| = 1 \text{ and } x_2 = -x_3 \neq 0, \end{cases}$$

for all $(x_1,x_2,x_3) \in S_1^2$.

Proposition 4. *The map $\overline{\pi_1}$ is one to one and onto. Also, its inverse map is given by*

$$\overline{\pi_1}^{-1}(x_1,x_2,x_3) = \begin{cases} \frac{1}{\sqrt{x_1^2+1}}(x_1,x_2,x_3) & \text{if} \quad |x_1| \neq 1 \text{ and } |x_2| > |x_3|, \\ \frac{1}{\sqrt{x_1^2-1}}(x_1,x_3,x_2) & \text{if} \quad |x_1| \neq 1 \text{ and } |x_2| < |x_3|, \\ (x_1,x_3,x_3) & \text{if} \quad |x_1| = 1 \text{ and } x_2 = \sqrt{1+x_3^2}, \\ (x_1,-x_3,x_3) & \text{if} \quad |x_1| = 1 \text{ and } x_2 = -\sqrt{1+x_3^2}, \end{cases}$$

for all $(x_1,x_2,x_3) \in \mathbb{R}\times S_1^1 - \{(1,-1,0),(-1,-1,0)\}$.

Proof. Let $x, y \in S_1^2$. If $\overline{\pi_1}(x) = \overline{\pi_1}(y)$, then we have:

$$\frac{1}{\sqrt{x_1^2+1}}(x_1,x_2,x_3) = \frac{1}{\sqrt{y_1^2+1}}(y_1,y_2,y_3),$$

or

$$\frac{1}{\sqrt{x_1^2-1}}(x_1,x_3,x_2) = \frac{1}{\sqrt{y_1^2-1}}(y_1,y_3,y_2),$$

or

$$\left(x_1, \sqrt{1+x_3^2}, x_3\right) = \left(y_1, \sqrt{1+y_3^2}, y_3\right),$$

or

$$\left(x_1, \sqrt{1+x_3^2}, x_3\right) = \left(y_1, -\sqrt{1+y_3^2}, y_3\right).$$

By computing, $x = y$ and $\overline{\pi_1}$ is one to one.

Let $z \in \mathbb{R} \times S_1^1 - \{(1,-1,0), (-1,-1,0)\}$. If $z_1 \neq \pm 1$, since $z \in S_1^1$ then $|z_1| > 1$ and $|z_2| < |z_3|$, or $|z_1| < 1$ and $|z_2| > |z_3|$. Hence, if $|z_1| > 1$, there exists $x = \frac{1}{\sqrt{z_1^2-1}}(z_1, z_3, z_2) \in S_1^2$ such that $z = \overline{\pi_1}(x)$. If $|z_1| < 1$, there exists $x = \frac{1}{\sqrt{z_1^2+1}}(z_1, z_2, z_3) \in S_1^2$ such that $z = \overline{\pi_1}(x)$.

If $z_1 = \pm 1$, since $z \in S_1^1$ then $z_2 = \pm\sqrt{1+z_3^2}$. Hence, if $z_2 > 0$, there exists $x = (z_1, z_3, z_3) \in S_1^2$ such that $z = \overline{\pi_1}(x)$. If $z_2 < 0$, there exists $x = (z_1, -z_3, z_3) \in S_1^2$ such that $z = \overline{\pi_1}(x)$. Thus, $\overline{\pi_1}$ is onto.

It is easy to check that $\overline{\pi_1}^{-1} \circ \overline{\pi_1} = \overline{\pi_1} \circ \overline{\pi_1}^{-1} = identity$. □

Remark 5. In the classical cylindrical projection in Euclidean space given by

$$\sigma_1 : S^2 - \{(1,0,0), (-1,0,0)\} \to S^1 \times \mathbb{R}$$

we remove two points of S^2, and in the extended $\mathbb{R} \times S_1^1$−cylindrical projection in Lorentzian space given by

$$\overline{\pi_1} : S_1^2 \to \mathbb{R} \times S_1^1 - \{(1,-1,0), (-1,-1,0)\}$$

we remove two points of the cylinder.

3.1.2. $C = S^1 \times L$. In Lorentz 3-space, there exists an unique relative position for the straight cylinder in map projection: touching the pseudosphere along the curve $(\cos\theta, \sin\theta, 0)$ and parallel to x_3−axes.

Let us recall that the cylinder $S^1 \times L$ in L^3 is defined by:

$$S^1 \times L = \left\{(x_1, x_2, x_3) \in L^3 : x_1^2 + x_2^2 = 1\right\},$$

and it is given by:

$$\text{(3)} \qquad \begin{cases} x_1 = \cos\theta \\ x_2 = \sin\theta \\ x_3 = t, \end{cases}$$

where $t \in \mathbb{R}$ and $0 < \theta < 2\pi$.

Let $\Sigma = \left\{x \in S^1 \times L; -1 < x_3 < 1\right\}$. According to the geometric process above mentioned, we define the $S^1 \times L$−cylindrical projection.

Definition 6. The $S^1 \times L$−cylindrical projection

$$\pi_2 : S_1^2 \to \Sigma$$

is defined by

$$\pi_2(x_1, x_2, x_3) = \frac{1}{\sqrt{1+x_3^2}}(x_1, x_2, x_3)$$

for all $(x_1, x_2, x_3) \in S_1^2$.

The $S^1 \times L-$cylindrical projection can be visualized geometrically as having a pseudosphere S_1^2 wrapped around Σ.

Proposition 7. *$S^1 \times L-$cylindrical projection is a diffeomorphism with inverse map:*

$$\pi_2^{-1}(x_1, x_2, x_3) = \frac{1}{\sqrt{1 - x_3^2}}(x_1, x_2, x_3).$$

Proof. Let f_1 and f_3 be the parametrizations (1) and (3), respectively. We have

$$(\theta, \omega) \xrightarrow{f_1} (\cos\theta\cosh\omega, \sin\theta\cosh\omega, \sinh\omega) \xrightarrow{\pi_2} \left(\cos\overline{\theta}, \sin\overline{\theta}, t\right) \xrightarrow{f_3^{-1}} \left(\overline{\theta}, t\right),$$

where $t = t(\theta, \omega) = \tanh\omega$ and $\overline{\theta} = \overline{\theta}(\theta, \omega) = \theta$ are diffeomorphisms.

It is easy to check that $\pi_2^{-1} \circ \pi_2 = \pi_2 \circ \pi_2^{-1} =$ *identity.* □

Let us note that π_2 sends the timelike geodesics of S_1^2 to vertical straight lines in $S^1 \times L$ (that is, timelike geodesics of cylinder), and the spacelike geodesics of S_1^2 to Euclidean circle in $S^1 \times L$ (that is, spacelike geodesics of cylinder).

Thus, we say that $S^1 \times L-$cylindrical projection preserves non-lightlike geodesics and preserves the causality of the non-lightlike geodesic curves.

3.1.3. $\boldsymbol{C = \mathbb{R} \times H_0^1}$. This cylinder is defined by:

$$\mathbb{R} \times H_0^1 = \left\{(x_1, x_2, x_3) \in L^3 : x_2^2 - x_3^2 = -1\right\}.$$

The two sheets of $\mathbb{R} \times H_0^1$ are characterized by $x_3 > 0$ and $x_3 < 0$. It follows that $\mathbb{R} \times H_0^1$ admits the parametrizations

$$\text{(4)} \qquad \begin{cases} x_1 = t \\ x_2 = \sinh\omega \\ x_3 = \pm\cosh\omega, \end{cases}$$

where $\omega, t \in \mathbb{R}$.

We can not extend the line OP if P lies in the lightlike straight lines determined by the intersection of S_1^2 with the planes $x_1 = -1$ and $x_1 = 1$.

Definition 8. Let $\Lambda_1 = \left\{x \in S_1^2; |x_1| = 1\right\}$ and $\Lambda_3 = \left\{x \in \mathbb{R} \times H_0^1; |x_1| = 1\right\}$. The $\mathbb{R} \times H_0^1-$cylindrical projection

$$\pi_3 : S_1^2 - \Lambda_1 \to \mathbb{R} \times H_0^1 - \Lambda_3$$

is defined by

$$\pi_3(x_1, x_2, x_3) = \begin{cases} \frac{1}{\sqrt{x_3^2 - x_2^2}}(x_1, x_2, x_3) & \text{if} \quad |x_2| < |x_3|, \\ \frac{1}{\sqrt{x_2^2 - x_3^2}}(x_1, x_3, x_2) & \text{if} \quad |x_2| > |x_3|, \end{cases}$$

for all $(x_1, x_2, x_3) \in S_1^2 - \Lambda_1$.

Proposition 9. *The map π_3 is a diffeomorphism.*

Proof. Let f_1 and f_4 be the parametrizations (1) and (4), respectively. We have

$$(\theta,\omega) \overset{f_1}{\rightarrow} (\cos\theta\cosh\omega, \sin\theta\cosh\omega, \sinh\omega) \overset{\pi_3}{\rightarrow} (t, \sinh\varpi, \pm\cosh\varpi) \overset{f_4^{-1}}{\rightarrow} (t,\varpi),$$

where

$$t = t(\theta,\omega) = \begin{cases} \frac{\cos\theta}{\sqrt{\sin^2\theta - \tanh^2\omega}} & \text{if} \quad |\sin\theta| > |\tanh\omega| \\ \frac{\cos\theta}{\sqrt{\tanh^2\omega - \sin^2\theta}} & \text{if} \quad |\sin\theta| < |\tanh\omega| \end{cases}$$

and

$$\varpi = \varpi(\theta,\omega) = \begin{cases} \sinh^{-1}\left(\frac{\tanh\omega}{\sqrt{\sin^2\theta - \tanh^2\omega}}\right) & \text{if} \quad |\sin\theta| > |\tanh\omega| \\ \sinh^{-1}\left(\frac{\sin\theta}{\sqrt{\tanh^2\omega - \sin^2\theta}}\right) & \text{if} \quad |\sin\theta| < |\tanh\omega| \end{cases}$$

are diffeomorphisms. □

Definition 10. An extended $\mathbb{R}\times H_0^1$−cylindrical projection

$$\overline{\pi_3} : S_1^2 \rightarrow \mathbb{R}\times H_0^1 - \{(1,0-1),(-1,0,-1)\}$$

is defined by

$$\overline{\pi_3}(x_1,x_2,x_3) = \begin{cases} \frac{1}{\sqrt{x_3^2-x_2^2}}(x_1,x_2,x_3) & \text{if} \quad |x_1| \neq 1 \text{ and } |x_2| < |x_3|, \\ \frac{1}{\sqrt{x_2^2-x_3^2}}(x_1,x_3,x_2) & \text{if} \quad |x_1| \neq 1 \text{ and } |x_2| > |x_3|, \\ \left(x_1,x_2,\sqrt{1+x_2^2}\right) & \text{if} \quad |x_1| = 1 \text{ and } x_2 = x_3, \\ \left(x_1,x_2,-\sqrt{1+x_2^2}\right) & \text{if} \quad |x_1| = 1 \text{ and } x_2 = -x_3 \neq 0, \end{cases}$$

for all $(x_1,x_2,x_3) \in S_1^2$.

Proposition 11. *The map $\overline{\pi_3}$ is one to one and onto. Also, its inverse map is given by*

$$\overline{\pi_3}^{-1}(x_1,x_2,x_3) = \begin{cases} \frac{1}{\sqrt{1-x_1^2}}(x_1,x_2,x_3) & \textit{if} \quad |x_1| \neq 1 \textit{ and } |x_2| < |x_3|, \\ \frac{1}{\sqrt{1-x_1^2}}(x_1,x_3,x_2) & \textit{if} \quad |x_1| \neq 1 \textit{ and } |x_2| > |x_3|, \\ (x_1,x_2,x_2) & \textit{if} \quad |x_1| = 1 \textit{ and } x_3 = \sqrt{1+x_2^2}, \\ (x_1,x_2,-x_2) & \textit{if} \quad |x_1| = 1 \textit{ and } x_3 = -\sqrt{1+x_2^2}, \end{cases}$$

for all $(x_1,x_2,x_3) \in \mathbb{R}\times H_0^1 - \{(1,0-1),(-1,0,-1)\}$.

Proof. This proof is analogous to the proof of Proposition 4. □

Remark 12. Similarly to the extended $\mathbb{R}\times S_1^1$−cylindrical projection, in the extended $\mathbb{R}\times H_0^1$−cylindrical projection we remove two points of the cylinder.

3.2. Orthographic projection

In Lorentzian geometry, the orthographic projection is a parallel projection which the parallel projection lines are orthogonal to the projection plane.

According to the signature $(+,+,-)$, the planes $x_1 = 0$ and $x_2 = 0$ are congruent planes to plane L^2. Also let us note that these planes differ from a Euclidean rigid movement. Without loss of generality, we choose the plane $x_1 = 0$ to develop our study.

Definition 13. Let Π be the plane $x_1 = 0$. Then, the orthographic projection is defined

$$\pi_O : S_1^2 \to \Pi,$$

such that $\pi_O(x_1, x_2, x_3) = (0, x_2, x_3)$ for all $(x_1, x_2, x_3) \in S_1^2$.

Let us note that

$$\pi_O(x_1, x_2, x_3) = \pi_O(-x_1, x_2, x_3),$$

that is, π_O is not a one-to-one map.

The lightlike straight lines determined by the intersection of S_1^2 with the planes $x_1 = -1$ and $x_1 = 1$ are send to lightlike straight lines in Π. In particular, $\pi_O(1,0,0) = \pi_O(-1,0,0) = (0,0,0)$.

Also, the orthographic projection does not preserve geodesics. Thus, π_O sends the spacelike geodesics of S_1^2 to horizontal straight segments (that is, parth spacelike geodesics of Π) but π_O sends the most timelike geodesics of S_1^2 to branch of Lorentzian circles in Π, which are not timelike geodesics of Lorentizian plane.

3.3. Stereographic projection

Instead of having parallel projection lines, the stereographic projection is projected from a fixed point of S to projection plane. That fixed point is called pole of projection.

In what follows, we will consider $S = S_1^2$ and the point $N = (1,0,0)$ as the pole.

A variety of different transformation formulas are possible depending on the relative positions of the projection plane and x_3−axis. Without loss of generality, we choose the plane $\Pi : x_1 = 0$ to be projection plane.

Let $P = (x_1, x_2, x_3) \in S_1^2$. We extend the line NP until it intersects the plane Π if $x_1 \neq 1$.

Definition 14. Let $\Lambda_4 = \left\{x \in S_1^2; x_1 = 1\right\}$. We define the stereographic projection π_N by:

$$\pi_N : S_1^2 - \Lambda_4 \to \Pi - H_0^1$$

such that

$$\pi_N(x_1, x_2, x_3) = \left(0, \frac{x_2}{1-x_1}, \frac{x_3}{1-x_1}\right)$$

for all $(x_1, x_2, x_3) \in S_1^2 - \Lambda_4$.

Let $\Pi_a : x_1 = a$ be a plane in L^3. These planes are congruent planes to plane L^2 for all $a \in \mathbb{R}$.

We remark that if $-1 < a < 1$ and $P \in S_1^2 \cap \Pi_a$, then $\pi_N(P)$ is spacelike in Π.

- If $a < -1$ and $P \in S_1^2 \cap \Pi_a$, then $\pi_N(P)$ is timelike in Π and its sign is preserved.
- If $a > 1$ and $P \in S_1^2 \cap \Pi_a$, then $\pi_N(P)$ is timelike in Π and its sign is reserved.
- If $a = -1$ and $P = (-1, x_2, \pm x_2)$, then $\pi_N(P)$ are lightlike in Π.

Proposition 15. *The map π_N is a diffeomorphism.*

Proof. Let $P = (x_1, x_2, x_3), N = (1,0,0)$ and $Q_P = P - \langle P, N\rangle N$. Then, $\pi_N(P) = \frac{1}{1-\langle P,N\rangle} Q_P$ and

$$\pi_N^{-1}(P) = \frac{1}{\langle Q_P, Q_P\rangle + 1}\left(\langle Q_P, Q_P\rangle - 1, 2\langle P,(0,1,0)\rangle, 2\langle P,(0,0,1)\rangle\right).$$

□

Definition 16. An extended stereographic projection $\overline{\pi_N}$ is defined by:

$$\overline{\pi_N} : S_1^2 \to \Pi - \{(0,0,-1)\}$$

such that

$$\overline{\pi_N}(x_1,x_2,x_3) = \begin{cases} \left(0, \frac{x_2}{1-x_1}, \frac{x_3}{1-x_1}\right) & \text{if} \quad x_1 \neq 1 \text{ and } |x_2| \neq |x_3|, \\ \left(0, x_2, \sqrt{1+x_2^2}\right) & \text{if} \quad x_1 = 1 \text{ and } x_2 = x_3, \\ \left(0, x_2, -\sqrt{1+x_2^2}\right) & \text{if} \quad x_1 = 1 \text{ and } x_2 = -x_3 \neq 0, \end{cases}$$

for all $(x_1,x_2,x_3) \in S_1^2$.

Proposition 17. *The map $\overline{\pi_N}$ is one to one and onto. Also, its inverse map is given by*

$$\overline{\pi_N}^{-1}(x_1,x_2,x_3) = \begin{cases} \frac{1}{x_2^2 - x_3^2 + 1}\left(x_2^2 - x_3^2 - 1, 2x_2, 2x_3\right) & \textit{if} \quad |x_2| \neq |x_3|, \\ (1, x_2, x_2) & \textit{if} \quad x_3 = \sqrt{1+x_2^2}, \\ (1, x_2, -x_2) & \textit{if} \quad x_3 = -\sqrt{1+x_2^2}, \end{cases}$$

for all $(x_1,x_2,x_3) \in \Pi - \{(0,0,-1)\}$.

Proof. This proof is analogous to the proof of Proposition 4. □

Remark 18. In the classical stereographic projection in Euclidean space given by

$$\sigma_N : S^2 - \{(1,0,0)\} \to \mathbb{R}^2$$

we remove one point of S^2, and in the extended stereographic projection in Lorentzian space given by

$$\overline{\pi_N} : S_1^2 \to L^2 - \{(0,0,-1)\}$$

we remove one point of the plane. Then, it is necessary to add two points to the pseudosphere to compact it.

4. Remarks on map projections of S_1^2 in Lorentz 3-space

In the development of this work we noticed that points $(x_1, x_2, x_3) \in S_1^2$ such that $|x_1| = 1$ are of particular interest when we want to project S_1^2 onto cylinders or Lorentzian plane. The $\mathbb{R} \times S_1^1$−cylindrical projection and $\mathbb{R} \times H_0^1$−cylindrical projection on $S_1^2 - \{x \in S_1^2; |x_1| = 1\}$ are diffeomorphisms.

On the other hand, the pseudosphere "wraps" the surface Σ which is contained in the straight cylinder. Also, $S^1 \times L$−cylindrical projection preserves non-lightlike geodesics and preserves the causality of the non-lightlike geodesic curves.

The orthographic projection is not one-to-one map and it does not preserve geodesics either.

The stereographic projection of the Euclidean sphere on the Euclidean plane shows that to compact the plane it is necessary to add one point it. In the 3-dimensional space of Lorentz, the extended stereographic projection shows that it is necessary to add two points to the pseudosphere to compact it. In addition, the plane of Lorentz is simple connected but the pseudosphere is not it, for that reason to preserve this property it is necessary to remove one point of the plane.

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