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THE ZEROS OF CERTAIN FAMILY OF SELF-RECIPROCAL POLYNOMIALS

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ABSTRACT. For integral self-reciprocal polynomials $P(z)$ and $Q(z)$ with all zeros lying on the unit circle, does there exist integral self-reciprocal polynomial $G_r(z)$ depending on r such that for any r, $0 \le r \le 1$, all zeros of $G_r(z)$ lie on the unit circle and $G_0(z) = P(z)$, $G_1(z) = Q(z)$? We study this question by providing examples. An example answers some interesting questions. Another example relates to the study of convex combination of two polynomials. From this example, we deduce the study of the sum of certain two products of finite geometric series.

1. Introduction and examples

Throughout this paper, U denotes the unit circle and n is a positive integer. A polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \tag{1.1}$$

of degree n is said to be a self-reciprocal polynomial of degree n if it satisfies $P(z) = z^n P(1/z)$, or equivalently, $a_j = a_{n-j}$ for $0 \le j \le n$. In what follows, let $P(z)$ be a real self-reciprocal polynomial of the form (1.1). By Cohn's theorem [1], a polynomial $Q(z)$ of degree n with all its zeros on U must be of the form $Q(z) = \mu z^n Q(1/z)$ for some μ, $|\mu| = 1$. Hence it is interesting to mention the condition for $P(z)$ with all zeros on U.

Cohn (see [1] or p.206 of [5]) proved that all zeros of $P(z)$ lie on U if and only if all zeros of its derivative lie on or inside U. Recently, by Lakatos [4], it has been known that if $|a_n| \ge \sum_{k=1}^{n-1} |a_k - a_n|$, all zeros of $P(z)$ lie on U. Using Lakatos's theorem [4], we can easily see that, for t real and

$$|t| \ge |a_n| + \sum_{k=1}^{n-1} |a_k - a_n|,$$

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we have that

$$P(z) + t(z^n + z^{n-1} + \cdots + 1) \tag{1.2}$$

has all its zeros on U since $|a_n + t| \geq |t| - |a_n|$. The second term (with deleting t) of (1.2) is also self-reciprocal and has all its zeros on U. This suggests to study polynomials having a form something like $sP(z) + tQ(z)$ where $P(z)$ and $Q(z)$ are integral self-reciprocal polynomials with same degree having all their zeros on U. As a special case, one may ask whether all zeros of convex combination of $P(z)$ and $Q(z)$ are on U. The answer is negative. For example, let

$$P(z) = (z^2 - z + 1)(z^4 + 1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1),$$
$$Q(z) = (z^4 - z^3 + z^2 - z + 1)(z^8 - z^6 + z^4 - z^2 + 1).$$

Each of $P(z)$ and $Q(z)$ has all its zeros on U, but $(P(z) + Q(z))/2$ has zeros $0.86603\cdots \pm i\,0.77115\cdots$ of modulus $1.15963\cdots$. Thus $rP(z) + (1-r)Q(z)$ $(0 \leq r \leq 1)$ "fails" in a sense: it does not keep all zeros on U. Hence, it is natural to ask a generalized question: for integral self-reciprocal polynomials $P(z)$, $Q(z)$ with all zeros lying on U, does there exist integral self-reciprocal polynomial $G_r(z)$ depending on r such that for any r, $0 \leq r \leq 1$, all zeros of $G_r(z)$ lie on U and $G_0(z) = P(z)$, $G_1(z) = Q(z)$? For convenience, if this question is true, we write

$$P \sim Q,$$

and

$$P \hookrightarrow Q$$

if $G_r(z) = (1-r)P(z) + rQ(z)$. In case of $P \not\hookrightarrow Q$, it seems to be difficult to find a suitable $G_r(z)$. In Section 2, we get $G_r(z)$ for the case

$$P(z) = (z^{2n} + 1)^2, \quad Q(z) = (z^2 - 1)(z^{4n-2} - 1),$$

where $P \not\hookrightarrow Q$. It is of interest that this $G_r(z)$ solves the following question. If

$$z^8 - z^6 - z^2 + 1 = (z^2 + z + 1)(z^2 - z + 1)(z + 1)^2(z - 1)^2 = 0,$$

then $|z| = 1$ and $(z^4 + 1)^2 = (z^3 + z)^2$, so $|z^4 + 1| = |z^3 + z|$. For $0 < r < 1$, it is not clear that

$$|z^4 + 1| = |(rz)^3 + rz|$$

has a solution on U. The $G_r(z)$ in our example will give an answer of this question for general case

$$\left|z^{2n} + 1\right| = \left|(rz)^{2n-1} + rz\right|$$

in Section 2.

We now turn to the case $P \hookrightarrow Q$. In Section 3, using Fell's lemma [2], we will show that

$$\frac{z^\alpha - 1}{z - 1}\frac{z^\beta - 1}{z - 1} \hookrightarrow \frac{z^\gamma - 1}{z - 1}\frac{z^\eta - 1}{z - 1} \tag{1.3}$$

provided

$$(\alpha, \beta; \gamma, \eta) = (n, n+3; n+1, n+2).$$

In fact, it seems that (1.3) is true for all $\alpha, \beta, \gamma, \eta$ with $\alpha + \beta = \gamma + \eta$. But we have not resolved this question. However, for $\alpha + \beta = \gamma + \eta$, we will prove in Section 3 that the sum of two products of finite geometric series

$$\frac{z^\alpha - 1}{z-1}\frac{z^\beta - 1}{z-1} + \frac{z^\gamma - 1}{z-1}\frac{z^\eta - 1}{z-1}$$

has all its zeros on U. In general, if we define, for positive integers α_k and β_k, $1 \le k \le n$, with $\sum_{k=1}^n \alpha_k = \sum_{k=1}^n \beta_k$,

$$\Phi_n(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; z) := \prod_{k=1}^n \frac{z^{\alpha_k} - 1}{z-1} + \prod_{k=1}^n \frac{z^{\beta_k} - 1}{z-1},$$

then Φ_n is an integral monic self-reciprocal polynomial in z of degree $\sum_{k=1}^n \alpha_k - n$, and each summand of Φ_n obviously has all its zeros on U. The zeros of Φ_1 clearly lie on U. All zeros of Φ_2 lying on U will be proved in Section 3. However, for $n = 3, 4$, we may find some $\alpha_k' s$ and $\beta_k' s$ with $\sum_{k=1}^n \alpha_k = \sum_{k=1}^n \beta_k$ such that not all zeros of Φ_n lie on U. For example, the polynomial equation

$$(z-1)(z^3-1)(z^{23}-1) + (z^2-1)(z^{11}-1)(z^{14}-1) = 0$$

has four nonreal zeros with modulus $\neq 1$. Hence, for $n \ge 3$, it seems that, for some $\alpha_k' s$ and $\beta_k' s$ with $\sum_{k=1}^n \alpha_k = \sum_{k=1}^n \beta_k$, not all zeros of Φ_n lie on U.

2. The case $P \sim Q$ but $P \not\leftrightarrow Q$

Using Cohn's theorem (see [1] or p.206 of [5]) we have

Theorem 1. *Let* $0 \le r \le 1$. *Then we have*

$$(z^{2n}+1)^2 \sim (z^2-1)(z^{4n-2}-1),$$

where

$$G_r(z) = z^{4n} - r^{2n}z^{4n-2} - (r^{4n-2} + r^2 - 2)z^{2n} - r^{2n}z^2 + 1.$$

Proof. Note that $G_0(z) = (z^{2n}+1)^2$ and $G_1(z) = (z^2-1)(z^{4n-2}-1)$. Suppose $0 < r < 1$. Then we have

$$\frac{G_r'(z)}{2z} = 2nz^{4n-2} - (2n-1)r^{2n}z^{4n-4} + n(2 - r^2 - r^{4n-2})z^{2n-2} - r^{2n}.$$

Let

$$f(z) = 2nz^{4n-2},$$
$$g(z) = -(2n-1)r^{2n}z^{4n-4} + n(2 - r^2 - r^{4n-2})z^{2n-2} - r^{2n}.$$

On $|z|=1$,

$$\begin{aligned}|g(z)| &\le (2n-1)r^{2n} + n(2-r^2-r^{4n-2}) + r^{2n}\\ &= n(2r^{2n}+2-r^2-r^{4n-2})\\ &= n\left(2-\frac{(-r+r^n)^2(r+r^n)^2}{r^2}\right)\\ &< 2n = |f(z)|.\end{aligned}$$

Hence, by Rouché (see p.2 of [5]), $G'_r(z)$ has all its zeros inside U. It follows from Cohn's theorem that the proof is complete. □

Remark 2. We observe that

$$(z^{2n}+1)^2 \not\leftrightarrow (z^2-1)(z^{4n-2}-1).$$

In fact, for $n=13$, $r=1/5$,

$$(1-r)(z^{2n}+1)^2 + r(z^2-1)(z^{4n-2}-1)$$

has four zeros with modulus $\neq 1$.

If $z^8-z^6-z^2+1=(z^2+z+1)(z^2-z+1)(z+1)^2(z-1)^2=0$, then $|z|=1$ and

$$(z^4+1)^2=(z^3+z)^2,$$

so

$$|z^4+1|=|z^3+z|.$$

For $0<r<1$, it is not clear that

$$|z^4+1|=|(rz)^3+rz|$$

has a solution on U. The polynomial $G_r(z)$ in Theorem 1 gives the answer of this.

Proposition 3. *Let* $0\le r\le 1$. *A zero of*

$$G_r(z)=z^{4n}-r^{2n}z^{4n-2}-(r^{4n-2}+r^2-2)z^{2n}-r^{2n}z^2+1$$

satisfies

$$\left|z^{2n}+1\right|=\left|(rz)^{2n-1}+rz\right|.$$

Proof. We can compute that $G_r(z)=0$ is equivalent to

$$(z^{2n}+1)^2=r^2z^2(z^{2n-2}+r^{2n-2})((rz)^{2n-2}+1).$$

Thus

$$|z^{2n}+1|^2=r^2|z^2||z^{2n-2}+r^{2n-2}||(rz)^{2n-2}+1|$$

and we can compute that, for $|z|=1$,

$$|z^{2n-2}+r^{2n-2}|=|(rz)^{2n-2}+1|,$$

which implies that, for $|z|=1$,

$$|z^{2n}+1|^2=r^2|z^2||(rz)^{2n-2}+1|^2$$

and

$$|z^{2n}+1| = |(rz)^{2n-1}+rz|.$$

□

3. The case $P \hookrightarrow Q$

Before studying the case $P \hookrightarrow Q$, we prove

Theorem 4. *Let α, β, γ, and η be positive integers with $\alpha+\beta=\gamma+\eta$. Then the integral self-reciprocal polynomial*

$$\Phi_2(\alpha,\beta,;\gamma,\eta;z) := \frac{z^\alpha-1}{z-1}\frac{z^\beta-1}{z-1} + \frac{z^\gamma-1}{z-1}\frac{z^\eta-1}{z-1}$$

has all its zeros on U.

Proof. Let n, α and γ be positive integers with $n \geq 2$ and $\alpha < \gamma < \lfloor n/2 \rfloor$. Then it suffices to show that

$$\Psi(z) := (z^\alpha-1)(z^{n-\alpha}-1)+(z^\gamma-1)(z^{n-\gamma}-1)$$

has all its zeros on U. Expanding $\Psi(z)$ in z derives that

$$\Psi(z) = 2z^n - z^{n-\alpha} - z^{n-\gamma} - z^\gamma - z^\alpha + 2,$$

which is self-reciprocal. On the other hand, we can compute that

$$\begin{aligned}
&\Psi'(z)\\
&= z^{\alpha-1}\left(2nz^{n-\alpha}-(n-\alpha)z^{n-2\alpha}-(n-\gamma)z^{n-\gamma-\alpha}-\gamma z^{\gamma-\alpha}-\alpha\right)\\
&= z^{\alpha-1}(z-1)\left(\alpha(1+z+\cdots+z^{\gamma-\alpha-1})+(\alpha+\gamma)z^{\gamma-\alpha}(1+z+\cdots+z^{n-2\gamma-1})\right.\\
&\quad \left.+(n+\alpha)z^{n-\gamma-\alpha}(1+z+\cdots+z^{\gamma-\alpha-1})+(2n)z^{n-2\alpha}(1+z+\cdots+z^{\alpha-1})\right).
\end{aligned}$$

Hence, by Eneström-Kakeya theorem (see p.136 of [5]), $\Psi'(z)$ has all its zeros lie inside or on U. The result follows from Cohn's theorem [1]. □

Perhaps one can use Fell's lemma [2] when we expect that $P \hookrightarrow Q$ though proof by using this lemma seems to be very long and complicated. In this section, we will see an example of using Fell's lemma. Fell [2] gave necessary and sufficient conditions for the zeros of $(1-r)P+rQ$ $0 \leq r \leq 1$ to all lie on U provided the zeros of monic polynomials P and Q with the same degree all lie on U.

Lemma 5. *Let $P_0(z)$ and $P_1(z)$ be real monic polynomials of degree n with their zeros contained in U except for -1 and 1. Denote the zeros of $P_0(z)$ by $w_1, w_2, \ldots, w_n$ and of $P_1(z)$ by $z_1, z_2, \ldots, z_n$. Assume that*

$$w_i \neq z_j \quad (1 \leq i, j \leq n)$$

and

$$\begin{aligned}
&0 < \arg(w_i) \leq \arg(w_j) < 2\pi,\\
&0 < \arg(z_i) \leq \arg(z_j) < 2\pi \quad (1 \leq i < j \leq n).
\end{aligned}$$

Let α_i be the smaller open arc of U bounded by w_i and z_i $(i = 1, \ldots, n)$. Then the locus of zeros of $(1-r)P_0(z) + rP_1(z)$ $(0 \le r \le 1)$ is contained in U if and only if the arcs α_i are disjoint.

In the rest of this section, for n positive integer, we again consider the integral self-reciprocal polynomials

$$P(z) = \frac{z^\alpha - 1}{z-1}\frac{z^\beta - 1}{z-1} \quad \text{and} \quad Q(z) = \frac{z^\gamma - 1}{z-1}\frac{z^\eta - 1}{z-1}.$$

We conjecture that $P(z) \hookrightarrow Q(z)$ for all $\alpha, \beta, \gamma, \eta$ with $\alpha + \beta = \gamma + \eta$ which is a generalization of Theorem 4. We have not resolved this question. However we show that the assertion is true for suitably chosen $(\alpha, \beta; \gamma, \eta)$ in terms of n. In fact we have

Theorem 6. *Let n be a positive integer. Then*

$$\frac{z^\alpha - 1}{z-1}\frac{z^\beta - 1}{z-1} \hookrightarrow \frac{z^\gamma - 1}{z-1}\frac{z^\eta - 1}{z-1}$$

provided

$$(\alpha, \beta; \gamma, \eta) = (n, n+3; n+1, n+2).$$

For the proof of this, we rely on Lemma 5. For convenience, we denote $\frac{z^\alpha - 1}{z-1}$ by $[\alpha]$ for a positive integer α. Then the arguments of the zeros of $[\alpha]$ between 0 and 2π are $2k\pi/\alpha$, $(1 \le k \le \alpha - 1)$. So, by removing the constant 2π, the zeros of $[\alpha]$ can be identified with the ascending chain of rational numbers $1/\alpha, 2/\alpha, \ldots, (\alpha-1)/\alpha$. When applying Lemma 5 in the proof, we can use an ascending chain of rational numbers instead of angle arguments. In order to show Theorem 6, one can consider the following three cases separately, namely

(a) for $n \ge 2$, $(\alpha, \beta; \gamma, \eta) = (2n, 2n+2; 2n-2, 2n+4)$,
(b) for $n \ge 1$, $(\alpha, \beta; \gamma, \eta) = (4n+1, 4n+3; 4n-1, 4n+5)$,
(c) for $n \ge 0$, $(\alpha, \beta; \gamma, \eta) = (4n+3, 4n+5; 4n+1, 4n+7)$.

In Proposition 7 below, we give the zero distribution of each case whose proof is complicated and straightforward. Theorem 6 is immediately obtained from Lemma 5 and Proposition 7. In Proposition 7, we denote $(c)^k$ by k c's.

Proposition 7. (a) *Let $n \ge 2$ be an integer. If we indicate elements of $[2n][2n+2]$ and $[2n-2][2n+4]$ by a and b, respectively, then the ascending chain for the elements of both $[2n][2n+2]$ and $[2n-2][2n+4]$ is of the form*

$$(baab)^{2n}.$$

(b) *Let $n \ge 1$ be an integer. If we indicate elements of $[4n+1][4n+3]$ and $[4n-1][4n+5]$ by a and b, respectively, then the ascending chain for the elements of both $[4n+1][4n+3]$ and $[4n-1][4n+5]$ is of the form*

$$(baab)^n(abba)^{2n+1}(baab)^n.$$

(c) *Let $n \geq 1$ be an integer. If we indicate elements of $[4n+3][4n+5]$ and $[4n+1][4n+7]$ by a and b, respectively, then the ascending chain for the elements of both $[4n+3][4n+5]$ and $[4n+1][4n+7]$ is of the form*

$$(baab)^n(baba)(abba)^n ab(baab)^n(baba)(abba)^n ab.$$

Proof. (a) We recall that $[\alpha] = \frac{x^\alpha - 1}{x-1}$. Let n be an integer ≥ 2. We observe that

$$[2n] \text{ or } [2n+4] = \begin{cases} (x+1)\cdot \text{integral polynomial}, & n \text{ odd}, \\ (x+1)(x^2+1)\cdot \text{integral polynomial}, & n \text{ even}, \end{cases} \tag{3.1}$$

and

$$[2n+2] \text{ or } [2n-2] = \begin{cases} (x+1)(x^2+1)\cdot \text{integral polynomial}, & n \text{ odd}, \\ (x+1)\cdot \text{integral polynomial}, & n \text{ even}. \end{cases} \tag{3.2}$$

We first find the ascending chain for the zeros of both $[2n]$ and $[2n+2]$. By (3.1) and (3.2), both $[2n][2n+2]$ and $[2n-2][2n+4]$ have the common factor $(x+1)^2(x^2+1)$. So we may assume that, for n even, the ascending chains for the zeros of $[2n]$ and $[2n+2]$ are

$$a_{1e} = \left\{\frac{k}{2n} : 1 \leq k \leq 2n-1, k \neq n, \frac{n}{2}, \frac{3n}{2}\right\},$$
$$a_{2e} = \left\{\frac{k}{2n+2} : 1 \leq k \leq 2n+1, k \neq n+1\right\},$$

respectively, and, for n odd,

$$a_{1o} = \left\{\frac{k}{2n} : 1 \leq k \leq 2n-1, k \neq n\right\},$$
$$a_{2o} = \left\{\frac{k}{2n+2} : 1 \leq k \leq 2n+1, k \neq n+1, \frac{n+1}{2}, \frac{3n+3}{2}\right\},$$

respectively. Also, we may assume that, for n even, the ascending chains for the zeros of $[2n-2]$ and $[2n+4]$ are

$$b_{1e} = \left\{\frac{k}{2n-2} : 1 \leq k \leq 2n-3, k \neq n-1\right\},$$
$$b_{2e} = \left\{\frac{k}{2n+4} : 1 \leq k \leq 2n+3, k \neq n+2, \frac{n+2}{2}, \frac{3n+6}{2}\right\},$$

respectively, and, for n odd,

$$b_{1o} = \left\{\frac{k}{2n-2} : 1 \leq k \leq 2n-3, k \neq n-1, \frac{n-1}{2}, \frac{3n-3}{2}\right\},$$
$$b_{2o} = \left\{\frac{k}{2n+4} : 1 \leq k \leq 2n+3, k \neq n+2\right\}.$$

Write $a_e = a_{1e} \cup a_{2e}$, $a_o = a_{1o} \cup a_{2o}$, $b_e = b_{1e} \cup b_{2e}$ and $b_o = b_{1o} \cup b_{2o}$. Regardless of whether n is even or odd, we have $a_e = a_o$, $b_e = b_o$. So, if we show that the elements of a_e and b_e form good pairs (for the definition of good pair, see Definition 4.2 of [3]), then so do the elements of a_o and b_o. Hence we only consider the case n even. Moreover, since the numbers after adding the elements of a_e, a_o, b_e and b_o in $(0, 1/2)$ by $1/2$ again belong to a_e, a_o, b_e and b_o in $(1/2, 1)$, and the cardinality of a_e, a_o, b_e and b_o in $(0, 1/2)$ is even, respectively, we only need to consider the elements of a_e and b_e in the interval $(0, 1/2)$. Clearly the ascending chain for the zeros of both $[2n]$ and $[2n+2]$ begins with $1/(2n+2)$. Since 2 is the only common divisor of $2n$ and $2n+2$, we have $a_{1e} \cap a_{2e} = \emptyset$. Note that $k < n$ if and only if $(k+1)/(2n+2) > k/(2n)$. So the ascending chain for a_e in $(0, 1/2)$ is the following;

$$\begin{aligned}&\frac{1}{2n+2}, \frac{1}{2n}, \frac{2}{2n+2}, \frac{2}{2n}, \ldots, \frac{n/2}{2n+2},\\ &\frac{n/2+1}{2n+2}, \frac{n/2+1}{2n}, \ldots, \frac{n-1}{2n}, \frac{n}{2n+2}.\end{aligned} \tag{3.3}$$

In (3.3), the only consecutive numbers except for $\{(n/2)/(2n+2), (n/2+1)/(2n+2)\}$ are of the form either

$$\frac{k}{2n+2}, \frac{k}{2n} \quad (1 \le k \le n-1,\ k \ne n/2) \tag{3.4}$$

or

$$\frac{k}{2n}, \frac{k+1}{2n+2} \quad (1 \le k \le n-1,\ k \ne n/2). \tag{3.5}$$

Now we check the existence of elements of b_e between two numbers in (3.4) and (3.5). Let k be an integer with $1 \le k \le n-1$, $k \ne n/2$. First, we consider (3.4). If

$$\frac{k}{2n+2} \le \frac{x}{2n-2} \le \frac{k}{2n} \tag{3.6}$$

for some integer x with $1 \le x \le 2n-3$, $x \ne n-1$, then (3.6) implies that $k - (2k)/(n+1) \le x \le k - k/n$. If $(2k)/(n+1) \ge 1$, i.e., $k \ge (n+1)/2$, then $k - 2 < k - (2k)/(n+1) \le k-1$, and $k - 1 \le k - k/n \le k - (n+1)/(2n) < k$. So $x = k-1$ (except for $k = 1$). Otherwise, $k - 1 < x \le k - k/n$, and so there is no integer x satisfying (3.6). If

$$\frac{k}{2n+2} \le \frac{x}{2n+4} \le \frac{k}{2n} \tag{3.7}$$

for some integer x with $1 \le x \le 2n+3$, $k \ne n+2$ and $(n+2)/2$, then (3.7) implies that $k + k/(n+1) \le x \le k + (2k)/n$. If $(2k)/n > 1$, i.e., $k > n/2$, then $k + (2k)/n > k+1$, and $k < k + k/(n+1) \le k+1$. So $x = k+1$. Otherwise, $k < k + (2k)/n < k+1$, and $k < k + k/(n+1) < k+1$. So there is no integer x satisfying (3.7). Thus, from (3.6) and (3.7), for $(n+1)/2 \le k \le n-1$, there are two numbers $(k-1)/(2n-2)$ and $(k+1)/(2n+4)$ of b_e between

$k/(2n+2)$ and $k/(2n)$. But, $(k-1)/(2n-2) = (k+1)/(2n+4)$ only if $k = (2n+1)/3$ is an integer (here, $n \equiv 1 \pmod 3$). For $k = (2n+1)/3$, $(k-1)/(2n-2) = (k+1)/(2n+4) = 1/3$. Hence the following two cases arise.

Case 1-1 $n \not\equiv 1 \pmod 3$

$$\begin{cases} 1 \le k \le \frac{n}{2} - 1 & \Rightarrow \text{there is no element of } b_e, \\ \frac{n}{2} + 1 \le k \le n & \Rightarrow \text{there are two different elements of } b_e, \end{cases}$$

in $\left[\frac{k}{2n+2}, \frac{k}{2n}\right]$.

Case 1-2 $n \equiv 1 \pmod 3$

$$\begin{cases} 1 \le k \le \frac{n}{2} - 1 & \Rightarrow \text{there is no element of } b_e, \\ \frac{n}{2} + 1 \le k \le n,\ k \ne \frac{2n+1}{3} & \Rightarrow \text{there are two different } b_e's, \\ k = \frac{2n+1}{3} & \Rightarrow \text{there are two same elements } \left(= \frac{1}{3}\right) \text{ of } b_e, \end{cases}$$

in $\left[\frac{k}{2n+2}, \frac{k}{2n}\right]$.

Now we consider the case (3.5). First we suppose that

$$\frac{k}{2n} \le \frac{x}{2n-2} \le \frac{k+1}{2n+2} \tag{3.8}$$

for some integer x with $1 \le x \le 2n-3$ and $x \ne n-1$. Then (3.8) implies that $k - k/n \le x \le k + 1 - (2(k+1))/(n+1)$. If $(2(k+1))/(n+1) \le 1$, i.e $k \le (n-1)/2$, then $k - k/n > k-1$ and $k + 1 - (2(k+1))/(n+1) < k+1$, so $x = k$. Otherwise, there is no x such that (3.8) holds. Suppose that

$$\frac{k}{2n} \le \frac{x}{2n+4} \le k + 12n + 2 \tag{3.9}$$

for some integer x with $1 \le x \le 2n+3$, $k \ne n+2$ and $(n+2)/2$. Then (3.9) implies that $k + (2k)/n \le x \le k + 1 + (k+1)/(n+1)$. If $(2k)/n < 1$, i.e $k < n/2$, then $x = k+1$. Otherwise, there is no x such that (3.9) holds. Thus, from (3.8) and (3.9), for $1 \le k \le (n-1)/2$, there are two numbers $k/(2n-2)$ and $(k+1)/(2n+4)$ of b_e between $k/(2n)$ and $(k+1)/(2n+2)$. But, $k/(2n-2) = (k+1)/(2n+4)$ only if $k = (n-1)/3$ is an integer (here, $n \equiv 1 \pmod 3$). For $k = (n-1)/3$, $k/(2n-2) = (k+1)/(2n+4) = 1/6$.

Case 1-1' $n \not\equiv 1 \pmod 3$

$$\begin{cases} 1 \le k \le \frac{n}{2} - 1 & \Rightarrow \text{there are two different elements of } b_e, \\ \frac{n}{2} + 1 \le k \le n-1 & \Rightarrow \text{there is no element of } b_e, \end{cases}$$

in $\left[\frac{k}{2n}, \frac{k+1}{2n+2}\right]$.

Case 1-2' $n \equiv 1 \pmod 3$

$$\begin{cases} 1 \le k \le \frac{n}{2} - 1,\ k \ne \frac{n-1}{3} & \Rightarrow \text{ there are two different elements of } b_e, \\ k = \frac{n-1}{3} & \Rightarrow \text{ there are two same elements } \left(= \frac{1}{6}\right) \text{ of } b_e, \\ \frac{n}{2} + 1 \le k \le n-1 & \Rightarrow \text{ there is no element of } b_e, \end{cases}$$

in $\left[\frac{k}{2n}, \frac{k+1}{2n+2}\right]$.

If we indicate elements of a_e and b_e by a and b, respectively, then it follows from either (3.3) and Case 1-1' or (3.3) and Case 1-2' that the ascending chain for both a_e and b_e is of the following form (obviously it starts from $b\,a$);

$$\begin{gathered} b\left(= \frac{1}{2n+4}\right), \frac{1}{2n+2}, \frac{1}{2n}, b, b, \frac{2}{2n+2}, \frac{2}{2n}, b, \ldots, \\ b, \frac{n/2-1}{2n+2}, \frac{n/2-1}{2n}, b, b, \frac{n/2}{2n+2}, \frac{n/2+1}{2n+2}, b, b, \\ \frac{n/2+1}{2n}, \frac{n/2+2}{2n+2}, b, \ldots, b, \frac{n-1}{2n}, \frac{n}{2n+2}, b\left(= \frac{n+1}{2n+4}\right), \end{gathered}$$

and so the ascending chain up to $1/2$ is of the form

$$baab\ baab \cdots baab \quad (n \text{ blocks of } (baab)).$$

Since the pattern repeats on $(1/2, 1)$, we have the form $(baab)^{2n}$. (b), (c) Let n be an integer ≥ 1. The ascending chains for the zeros of $[4n+1]$ and $[4n+3]$ are

$$a_1 = \left\{\frac{k}{4n+1} : 1 \le k \le 4n\right\},$$
$$a_2 = \left\{\frac{k}{4n+3} : 1 \le k \le 4n+2\right\},$$

respectively. Also, the ascending chains for the zeros of $[4n-1]$ and $[4n+5]$ are

$$b_1 = \left\{\frac{k}{4n-1} : 1 \le k \le 4n-2\right\},$$
$$b_2 = \left\{\frac{k}{4n+5} : 1 \le k \le 4n+4\right\},$$

respectively. Write $a = a_1 \cup a_2$ and $b = b_1 \cup b_2$. Since the numbers after adding the elements of a and b in $(0, 1/2)$ by $1/2$ again belong to a, b in $(1/2, 1)$ and the cardinality of a_1, a_2, b_1 and b_2 in $(0, 1/2)$ is even, respectively, we only need to consider the elements of a and b in the interval $(0, 1/2)$. We first find the ascending chain for a in $(0, 1/2)$. Since $\gcd(4n+1, 4n+3) = 1$, we have $a_1 \cap a_2 = \emptyset$. Clearly the ascending chain for the zeros of both $[4n+1]$ and $[4n+3]$ begins with $1/(4n+3)$. We note that $k > (4n+1)/2$ if and only if

$(k+1)/(4n+3) < k/(4n+1)$. So the ascending chain for either a in $(0, 1/2)$ is the following;

$$\frac{1}{4n+3}, \frac{1}{4n+1}, \frac{2}{4n+3}, \frac{2}{4n+1}, \ldots, \frac{2n}{4n+3}, \frac{2n}{4n+1}, \frac{2n+1}{4n+3}. \tag{3.10}$$

In (3.10), the only consecutive numbers are of the form either

$$\frac{k}{4n+3}, \frac{k}{4n+1} \quad (1 \leq k \leq 2n) \tag{3.11}$$

or

$$\frac{k}{4n+1}, \frac{k+1}{4n+2} \quad (1 \leq k \leq 2n). \tag{3.12}$$

Now we check the existence of elements of b between two numbers in (3.11) and (3.12). First, we consider (3.11). If

$$\frac{k}{4n+3} \leq \frac{x}{4n-1} \leq \frac{k}{4n+1} \tag{3.13}$$

for some integer x with $1 \leq x \leq 4n-2$, then (3.13) implies that $k-(4k)/(4n+3) \leq x \leq k-(2k)/(4n+1)$. If $(4k)/(4n+3) \geq 1$, i.e., $k \geq (4n+3)/4$, then $k-2 < k-(4k)/(4n+3) \leq k-1$, and $k-1 \leq k-(2k)/(4n+1) < k$. So $x = k-1$. Otherwise, $k-1 < x \leq k-(2k)/(4n+1)$, and so there is no integer x satisfying (3.13). If

$$\frac{k}{4n+3} \leq \frac{x}{4n+5} \leq \frac{k}{4n+1} \tag{3.14}$$

for some integer x with $1 \leq x \leq 4n+4$, then (3.14) implies that $k+(2k)/(4n+3) \leq x \leq k+(4k)/(4n+1)$. If $(4k)/(4n+1) \geq 1$, i.e. $k \geq n+1$, then $x = k+1$. Otherwise, there is no integer x satisfying (3.14). Thus, from (3.13) and (3.14), for $n+1 \leq k \leq 2n$, there are two numbers $(k-1)/(4n-1)$ and $(k+1)/(4n+5)$ of b between $k/(4n+3)$ and $k/(4n+1)$. But, $(k-1)/(4n-1) = (k+1)/(4n+5)$ only if $k = (4n+2)/3$ is an integer (here, $n \equiv 1 \pmod 3$). For $k = (4n+2)/3$, $(k-1)/(4n-1) = (k+1)/(4n+5) = 1/3$.

<u>Case 2-1 $n \not\equiv 1 \pmod 3$</u>

$$\begin{cases} 1 \leq k \leq n & \Rightarrow \text{ there is no } b, \\ n+1 \leq k \leq 2n, & \Rightarrow \text{ there are two different } b's, \end{cases}$$

in $\left[\frac{k}{4n+3}, \frac{k}{4n+1}\right]$.

<u>Case 2-2 $n \equiv 1 \pmod 3$</u>

$$\begin{cases} 1 \leq k \leq n & \Rightarrow \text{ there is no } b, \\ n+1 \leq k \leq 2n, k \neq \frac{4n+2}{3} & \Rightarrow \text{ there are two different } b's, \\ k = \frac{4n+2}{3} & \Rightarrow \text{ there are two same } b's \left(= \frac{1}{3}\right), \end{cases}$$

in $\left[\frac{k}{4n+3}, \frac{k}{4n+1}\right]$.

We consider the case for (3.12). First we suppose that

$$\frac{k}{4n+1} \leq \frac{x}{4n-1} \leq \frac{k+1}{4n+3} \tag{3.15}$$

for some integer x with $1 \leq x \leq 4n-2$. Then (3.15) implies that $k-(2k)/(4n+1) \leq x \leq k+1-(4k+4))/(4n+3)$. If $(4k+4))/(4n+3) < 1$, i.e., $k \leq n-1$, then $k-(2k)/(4n+1) > k-1$ and $k+1-(4k+4))/(4n+3) > k$, so $x = k$. Otherwise, there is no x such that (3.15) holds. Suppose that

$$\frac{k}{4n+1} \leq \frac{x}{4n+5} \leq \frac{k+1}{4n+3} \tag{3.16}$$

for some integer x with $1 \leq x \leq 4n+4$. Then (3.16) implies that $k+(4k)/(4n+1) \leq x \leq k+1+(2k+2)/(4n+3)$. If $(4k)/(4n+1) < 1$, i.e., $k \leq n$, then $x = k+1$. Otherwise, there is no x such that (3.16)holds. Thus, from (3.15) and (3.16), for $1 \leq k \leq (n-1)/2$, there are two numbers $k/(4n-1)$ and $(k+1)/(4n+5)$ of b between $k/(4n-1)$ and $(k+1)/(4n+3)$. But, $k/(4n+1) = (k+1)/(4n+3)$ only if $k = (4n-1)/6$ is an integer. But $6 \nmid (4n-1)$ for all n. Hence we have

$$\begin{cases} 1 \leq k \leq n-1 & \Rightarrow \text{ there are two different } b's, \\ k = n & \Rightarrow \text{ there is only one } b, \\ n+1 \leq k \leq 2n & \Rightarrow \text{ there is no } b, \end{cases} \tag{3.17}$$

in $\left[\frac{k}{4n+1}, \frac{k+1}{4n+3}\right]$. If we indicate elements of a and b by a and b, respectively, then it follows from (3.10), Case 2-1 and (3.17) or (3.10), Case 2-2 and (3.17) that the ascending chain is of the following form (obviously it starts from $b\,a$);

$$\begin{aligned} & b\left(=\frac{1}{4n+5}\right), \frac{1}{4n+3}, \frac{1}{4n+1}, b, b, \frac{2}{4n+3}, \frac{2}{4n+1}, b, \ldots, \\ & \frac{n}{4n+3}, \frac{n}{4n+1}, b, \frac{n+1}{4n+3}, b, b, \frac{n+1}{4n+1}, \frac{n+2}{4n+3}, b, \ldots, \\ & \frac{2n}{4n+1}, \frac{2n+1}{4n+3}, b, b, \frac{2n}{4n+1}, \frac{2n+1}{4n+3}, b\left(=\frac{2n+2}{4n+5}\right), \end{aligned}$$

and so the ascending chain up to 1/2 is of the form

$(baab)^n(abba)^n ab$ (i.e., n blocks of $(baab)$, n blocks of $(abba)$ and ab).

Since the pattern repeats on $(1/2, 1)$, we have the form

$$(baab)^n(abba)^{2n+1}(baab)^n.$$

The case $(\alpha, \beta; \gamma, \eta) = (4n+3, 4n+5; 4n+1, 4n+7)$, where $n \geq 0$, is very similar to the previous case $(\alpha, \beta; \gamma, \eta) = (4n+1, 4n+3; 4n-1, 4n+5)$, where $n \geq 2$. In fact, the ascending chains for the zeros of $[4n+3]$ and $[4n+5]$ are

$$\begin{aligned} a_1 &= \left\{\frac{k}{4n+3} : 1 \leq k \leq 4n+2\right\}, \\ a_2 &= \left\{\frac{k}{4n+5} : 1 \leq k \leq 4n+4\right\}, \end{aligned}$$

respectively. Also, the ascending chains for the zeros of $[4n+1]$ and $[4n+7]$ are

$$b_1 = \left\{ \frac{k}{4n+1} : 1 \leq k \leq 4n \right\},$$

$$b_2 = \left\{ \frac{k}{4n+7} : 1 \leq k \leq 4n+6 \right\},$$

respectively. Write $a = a_1 \cup a_2$ and $b = b_1 \cup b_2$. Then we can show that the ascending chain up to $1/2$ is of the form

$$(baab)^n(baba)(abba)^n ab$$

(i.e., n blocks of $(baab), baba, n$ blocks of $(abba)$ and ab).

Since the pattern repeats on $(1/2, 1)$, we have the form

$$(baab)^n(baba)(abba)^n ab(baab)^n(baba)(abba)^n ab.$$

□

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