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FUZZY n-INNER PRODUCT SPACE

Srinivasan Vijayabalaji and Natesan Thillaigovindan

Abstract. The purpose of this paper is to introduce the notion of fuzzy n-inner product space. Ascending family of quasi α -n-norms corresponding to fuzzy quasi n-norm is introduced and we provide some results on it.

1. Introduction

An interesting theory of 2-inner product space and n-inner product space has been effectively constructed by C. R. Diminnie, S. Gähler and A. White in [6, 7]. It was further investigated and developed by A. Misiak in [20, 21]. Recent results about n-inner product space can be viewed in [4, 5]. In [11, 12, 13, 15, 19] we can study about the origin and development of n-normed linear space. Different authors introduced the definitions of fuzzy inner product space in [1, 16, 17] and fuzzy normed linear space in [2, 3, 8, 9, 10, 14, 18, 23]. Recently in [22] we have introduced the notion of fuzzy n-normed linear space.

In this present paper we introduce the notion of fuzzy n-inner product space as a further generalization of n-inner product space found in [4]. We further generalize our fuzzy n-normed linear space [22] to fuzzy quasi n-normed linear space provide some results on it.

2. Preliminaries

Before proceeding further, in this section let us recall some familiar concepts which will be needed in the sequel.

Definition 2.1 ([4]). Let n be a natural number greater than 1 and X be a real linear space of dimension greater than or equal to n and let $(\bullet, \bullet|\bullet, \dots, \bullet)$ be a real valued function on $\underbrace{X \times \cdots \times X}_{n+1} = X^{n+1}$ satisfying the following conditions:

(1) (i) $(x, x|x_2, \dots, x_n) \geq 0$,

(ii) $(x, x|x_2, \dots, x_n) = 0$ if any only if $x, x_2, \dots, x_n$ are linearly dependent,

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(2) $(x, y|x_2, \ldots, x_n) = (y, x|x_2, \ldots, x_n)$,
(3) $(x, y|x_2, \ldots, x_n)$ is invariant under any permutation of $x_2, \ldots, x_n$,
(4) $(x, x|x_2, \ldots, x_n) = (x_2, x_2|x, x_3, \ldots, x_n)$,
(5) $(ax, x|x_2, \ldots, x_n) = a(x, x|x_2, \ldots, x_n)$ for every $a \in R$(real),
(6) $(x + x', y|x_2, \ldots, x_n) = (x, y|x_2, \ldots, x_n) + (x', y|x_2, \ldots, x_n)$.

Then $(\bullet, \bullet|\bullet, \ldots, \bullet)$ is called an n-inner product on X and $(X, (\bullet, \bullet|\bullet, \ldots, \bullet))$ is called an n-inner product space.

Definition 2.2 ([4]). Let $n \in$N (natural numbers) and X be a real linear space of dimension greater than or equal to n. A real valued function $||\bullet, \ldots, \bullet||$ on $\underbrace{X \times \cdots \times X}_{n} = X^n$ satisfying the following four properties:

(1) $||x_1, x_2, \ldots, x_n|| = 0$ if any only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(2) $||x_1, x_2, \ldots, x_n||$ is invariant under any permutation,
(3) $||x_1, x_2, \ldots, ax_n|| = |a| \; ||x_1, x_2, \ldots, x_n||$, for any $a \in R$ (real),
(4) $||x_1, x_2, , x_{n-1}, y + z|| \leq ||x_1, x_2, \ldots, x_{n-1}, y|| + ||x_1, x_2, \ldots, x_{n-1}, z||$,

is called an n-norm on X and the pair $(X, ||\bullet, \ldots, \bullet||)$ is called an n-normed linear space.

Remark 2.3. In the above definition if we replace (3) by, $(3)'$ $||x_1, x_2, \ldots, ax_n|| = |a|^p \; ||x_1, x_2, \ldots, x_n||$, for any $a \in R$(real) and $0 \leq p < 1$, then $(X, ||\bullet, \ldots, \bullet||)$ is called a quasi n-normed linear space.

Remark 2.4 ([4]). If an n-inner product space $(X, (\bullet, \bullet|\bullet, \ldots, \bullet))$ is given then $||x_1, x_2, \ldots, x_n|| = \sqrt{(x_1, x_1|x_2, \ldots, x_n)}$ defines an n-norm on X. Further the following extension of Cauchy-Buniakowski inequality is also true

$$|(x, y|x_2, \ldots, x_n)| \leq \sqrt{(x, x|x_2, \ldots, x_n)}\sqrt{(y, y|x_2, \ldots, x_n)}.$$

Definition 2.5 ([22]). Let X be a linear space over a real field F. A fuzzy subset N of $X^n \times R$ (R-set of real numbers) is called a fuzzy n-norm on X if and only if:

($N1$) For all $t \in R$ with $t \leq 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$.
($N2$) For all $t \in R$ with $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent.
($N3$) $N(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.
($N4$) For all $t \in R$ with $t > 0$, $N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$, if $c \neq 0$, $c \in F$ (field).
($N5$) For all $s, t \in R$, $N(x_1, x_2, \ldots, x_n + x_n', s + t) \geq \min\{N(x_1, x_2, \ldots, x_n, s), N(x_1, x_2, \ldots, x_n', t)\}$.
($N6$) $N(x_1, x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in R$ and
$$\lim_{t \to \infty} N(x_1, x_2, \ldots, x_n, t) = 1.$$

Then (X, N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Remark 2.6. In the above Definition 2.5, if we replace $(N4)$ by, $(N4)'$ For all $t \in R$ with $t > 0$ $N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|^p})$, if $c \neq 0$, $c \in F$(field),$0 \leq p < 1$. Then (X, N) is called a fuzzy quasi n-normed linear space or in short f-q-n-NLS.

Theorem 2.7 ([22]). *Let (X, N) be a f-n-NLS. Assume the condition that $(N^*) N(x_1, x_2, \ldots, x_n, t) > 0$ for all $t > 0$ implies $x_1, x_2, \ldots, x_n$ are linearly dependent. Define $||x_1, x_2, \ldots, x_n||_\alpha$=inf $\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}$,$\alpha \in (0,1)$. Then $\{||\bullet, \bullet, \ldots, \bullet||_\alpha : \alpha \in (0,1)\}$,is an ascending family of n-norms on X. We call these n-norms as α-n-norms on X corresponding to the fuzzy n-norm on X.*

3. Fuzzy n-inner product space

In this section we introduce the satisfactory notion of fuzzy n-inner product space as a generalization of Definition 2.1 as follows.

Definition 3.1. Let X be a linear space over a field F. A fuzzy subset J : $X^{n+1} \times R$ (R-set of real numbers) is called a fuzzy n-inner product on X if and only if:

(1) For all $t \in R$ with $t \leq 0$, $J(x, x|x_2, \ldots, x_n, t) = 0$.
(2) For all $t \in R$ with $t > 0$, $J(x, x|x_2, \ldots, x_n, t) = 1$ if and only if $x, x_2, \ldots, x_n$ are linearly dependent.
(3) For all $t > 0$, $J(x, y|x_2, \ldots, x_n, t) = J(y, x|x_2, \ldots, x_n, t)$.
(4) $J(x, y|x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_2, \ldots, x_n$.
(5) For all $t > 0$, $J(x, x|x_2, \ldots, x_n, t) = J(x_2, x_2|x, x_3, \ldots, x_n, t)$.
(6) For all $t > 0$, $J(ax, bx|x_2, \ldots, x_n, t) = J(x, x|x_2, \ldots, x_n, \frac{t}{|ab|})$, $a, b \in R$ (real).
(7) For all $s, t \in R$,
$$J(x + x', y|x_2, \ldots, x_n, t + s)$$
$$\geq \min\{J(x, y|x_2, \ldots, x_n, t), J(x', y|x_2, \ldots, x_n, s)\}.$$
(8) For all $s, t \in R$ with $s > 0$, $t > 0$,
$$J(x, y|x_2, \ldots, x_n, \sqrt{ts})$$
$$\geq \min\{J(x, x|x_2, \ldots, x_n, t), J(y, y|x_2, \ldots, x_n, s)\}.$$
(9) $J(x, y|x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in R$ and
$$\lim_{t \to \infty} J(x, y|x_2, \ldots, x_n, t) = 1.$$

Then (X, J) is called a fuzzy n-inner product space or in short f-n-IPS.

Example 3.2. Let $(X, (\bullet, \bullet|\bullet, \ldots, \bullet))$ be an n-inner product space. Define

$$J(x, y|x_2, \ldots, x_n, t) = \begin{cases} \frac{t}{t + |(x, y|x_2, \ldots, x_n)|}, & \text{when } t > 0, t \in R, \\ & (x, y|x_2, \ldots, x_n) \in X^{n+1} \\ 0, & \text{when } t \leq 0. \end{cases}$$

Then (X, J) is a f-n-IPS.

Proof. The nine conditions for f-n-IPS are verified below:

(1) For all $t \in R$ with $t \leq 0$ we have by our definition, $J(x, x|x_2, \dots, x_n, t) = 0$.

(2) For all $t \in R$ with $t > 0$ we have

$$\begin{aligned} &J(x, x|x_2, \dots, x_n, t) = 1 \\ &\Leftrightarrow \frac{t}{t + |(x, x|x_2, \dots, x_n)|} = 1 \Leftrightarrow |(x, x|x_2, \dots, x_n)| = 0 \\ &\Leftrightarrow (x, x|x_2, \dots, x_n) = 0 \Leftrightarrow x, x_2, \dots, x_n \end{aligned}$$

are linearly dependent.

(3) For all $t > 0$,

$$\begin{aligned} J(x, y|x_2, \dots, x_n, t) &= \frac{t}{t + |(x, y|x_2, \dots, x_n)|} \\ &= \frac{t}{t + |(y, x|x_2, \dots, x_n)|} \\ &= J(y, x|x_2, \dots, x_n, t). \end{aligned}$$

(4) As $(x, x|x_2, \dots, x_n)$ is invariant under any permutation of $x_2, \dots, x_n$, we have $J(x, y|x_2, \dots, x_n, t)$ is invariant under any permutation.

(5) For all $t > 0$,

$$\begin{aligned} J(x, x|x_2, \dots, x_n, t) &= \frac{t}{t + |(x, x|x_2, \dots, x_n)|} \\ &= \frac{t}{t + |(x_2, x_2|x, x_3, \dots, x_n)|} \\ &= J(x_2, x_2|x, x_3, \dots, x_n, t). \end{aligned}$$

(6) For all $t > 0$,

$$\begin{aligned} J(x, x|x_2, \dots, x_n, \frac{t}{|ab|}) &= \frac{\frac{t}{|ab|}}{\frac{t}{|ab|} + |(x, x|x_2, \dots, x_n)|} \\ &= \frac{t}{t + |ab||(x_2, x_2|x, x_3, \dots, x_n)|} \\ &= \frac{t}{t + |(ax, bx|x_2, \dots, x_n)|} \\ &= J(ax, bx|x_2, \dots, x_n, t). \end{aligned}$$

(7) If (a) $s + t < 0$ (b) $s = t = 0$ (c) $s + t > 0; s > 0, t < 0; s < 0, t > 0$, then the above relation is obvious.If (d) $s > 0, t > 0, s + t > 0$. Then

without loss of generality assume that

$$
\begin{aligned}
& J(x, y|x_2, \ldots, x_n, t) \\
& \leq J(x', y|x_2, \ldots, x_n, s) \\
\Rightarrow\ & \frac{t}{t + |(x, y|x_2, \ldots, x_n)|} \leq \frac{s}{s + |(x', y|x_2, \ldots, x_n)|} \\
\Rightarrow\ & \frac{t + |(x, y|x_2, \ldots, x_n)|}{t} \geq \frac{s + |(x', y|x_2, \ldots, x_n)|}{s} \\
\Rightarrow\ & 1 + \frac{|(x, y|x_2, \ldots, x_n)|}{t} \geq 1 + \frac{|(x', y|x_2, \ldots, x_n)|}{s} \\
\Rightarrow\ & \frac{|(x, y|x_2, \ldots, x_n)|}{t} \geq \frac{|(x', y|x_2, \ldots, x_n)|}{s} \\
\Rightarrow\ & \frac{s|(x, y|x_2, \ldots, x_n)|}{t} \geq |(x', y|x_2, \ldots, x_n)| \\
\Rightarrow\ & |(x, y|x_2, \ldots, x_n)| + \frac{s|(x, y|x_2, \ldots, x_n)|}{t} \\
& \geq |(x, y|x_2, \ldots, x_n)| + |(x', y|x_2, \ldots, x_n)| \\
\Rightarrow\ & (1 + \frac{s}{t})|(x, y|x_2, \ldots, x_n)| \geq |(x + x', y|x_2, \ldots, x_n)| \\
\Rightarrow\ & (\frac{s + t}{t})|(x, y|x_2, \ldots, x_n)| \geq |(x + x', y|x_2, \ldots, x_n)| \\
\Rightarrow\ & \frac{|(x, y|x_2, \ldots, x_n)|}{t} \geq \frac{|(x + x', y|x_2, \ldots, x_n)|}{s + t} \\
\Rightarrow\ & 1 + \frac{|(x, y|x_2, \ldots, x_n)|}{t} \geq 1 + \frac{|(x + x', y|x_2, \ldots, x_n)|}{s + t} \\
\Rightarrow\ & \frac{t + |(x, y|x_2, \ldots, x_n)|}{t} \geq \frac{s + t + |(x + x', y|x_2, \ldots, x_n)|}{s + t} \\
\Rightarrow\ & \frac{t}{t + |(x, y|x_2, \ldots, x_n)|} \leq \frac{s + t}{s + t + |(x + x', y|x_2, \ldots, x_n)|} \\
\Rightarrow\ & \min\{J(x, y|x_2, \ldots, x_n, t), J(x', y|x_2, \ldots, x_n, t)\} \\
& \leq J(x + x', y|x_2, \ldots, x_n, s + t).
\end{aligned}
$$

(8) Without loss of generality assume that

$$
\begin{aligned}
& J(x, x|x_2, \ldots, x_n, t) \\
& \leq J(y, y|x_2, \ldots, x_n, s), \text{ for all } s, t \in R \text{ with } s > 0,\ t > 0. \\
\Rightarrow\ & \frac{t}{t + |(x, x|x_2, \ldots, x_n)|} \leq \frac{s}{s + |(y, y|x_2, \ldots, x_n)|} \\
\Rightarrow\ & \frac{t + |(x, x|x_2, \ldots, x_n)|}{t} \geq \frac{s + |(y, y|x_2, \ldots, x_n)|}{s}
\end{aligned}
$$

$$\Rightarrow 1+\frac{|(x,x|x_2,\ldots,x_n)|}{t} \geq 1+\frac{|(y,y|x_2,\ldots,x_n)|}{s}$$
$$\Rightarrow \frac{|(x,x|x_2,\ldots,x_n)|}{t} \geq \frac{|(y,y|x_2,\ldots,x_n)|}{s}$$
$$\Rightarrow \frac{s|(x,x|x_2,\ldots,x_n)|}{t} \geq |(y,y|x_2,\ldots,x_n)|$$
$$\Rightarrow \frac{|(x,x|x_2,\ldots,x_n)|s|(x,x|x_2,\ldots,x_n)|}{t}$$
$$\geq |(x,x|x_2,\ldots,x_n)||(y,y|x_2,\ldots,x_n)|.$$

By Remark 2.4,

$$|(x,x|x_2,\ldots,x_n)|^2\frac{s}{t}$$
$$\geq |(x,y|x_2,\ldots,x_n)|^2$$
$$\Rightarrow |(x,x|x_2,\ldots,x_n)|^2\frac{s}{t^2} \geq \frac{|(x,y|x_2,\ldots,x_n)|^2}{t}$$
$$\Rightarrow \frac{|(x,x|x_2,\ldots,x_n)|^2}{t^2} \geq \frac{|(x,y|x_2,\ldots,x_n)|^2}{st}.$$

Taking square root on both sides,

$$\Rightarrow \frac{|(x,x|x_2,\ldots,x_n)|}{t} \geq \frac{|(x,y|x_2,\ldots,x_n)|}{\sqrt{st}}$$
$$\Rightarrow 1+\frac{|(x,x|x_2,\ldots,x_n)|}{t} \geq 1+\frac{|(x,y|x_2,\ldots,x_n)|}{\sqrt{st}}$$
$$\Rightarrow \frac{t+|(x,x|x_2,\ldots,x_n)|}{t} \geq \frac{\sqrt{st}+|(x,y|x_2,\ldots,x_n)|}{\sqrt{st}}$$
$$\Rightarrow \frac{t}{t+|(x,x|x_2,\ldots,x_n)|} \leq \frac{\sqrt{st}}{\sqrt{st}+|(x,y|x_2,\ldots,x_n)|}$$
$$\Rightarrow \min\{J(x,x|x_2,\ldots,x_n,t), J(y,y|x_2,\ldots,x_n,s)\}$$
$$\leq J(x,y|x_2,\ldots,x_n,\sqrt{ts}).$$

(9) For all $t_1,t_2 \in R$ if $t_1 < t_2 \leq 0$ then, by our definition,

$$J(x,y|x_2,\ldots,x_n,t_1) = J(x,y|x_2,\ldots,x_n,t_2) = 0.$$

Suppose $t_2 > t_1 > 0$ then,

$$\frac{t_2}{t_2+|(x,y|x_2,\ldots,x_n)|} - \frac{t_1}{t_1+|(x,y|x_2,\ldots,x_n)|}$$
$$= \frac{|(x,y|x_2,\ldots,x_n)|(t_2-t_1)}{(t_2+|(x,y|x_2,\ldots,x_n)|)(t_1+|(x,y|x_2,\ldots,x_n)|)} \geq 0,$$

for all $(x,y|x_2,\ldots,x_n) \in X^{n+1}$

$$\Rightarrow \frac{t_2}{t_2 + |(x, y|x_2, \ldots, x_n)|} \geq \frac{t_1}{t_1 + |(x, y|x_2, \ldots, x_n)|}$$
$$\Rightarrow J(x, y|x_2, \ldots, x_n, t_2) \geq J(x, y|x_2, \ldots, x_n, t_1).$$

Thus $J(x, y|x_2, \ldots, x_n, t)$ is a non-decreasing function. Also,

$$\begin{aligned} &\lim_{t\to\infty} J(x, y|x_2, \ldots, x_n, t) \\ &= \lim_{t\to\infty} \frac{t}{t + |(x, y|x_2, \ldots, x_n)|} \\ &= \lim_{t\to\infty} \frac{t}{t(1 + \frac{1}{t}|(x, y|x_2, \ldots, x_n)|)} \\ &= 1. \end{aligned}$$

Thus (X, J) is a f-n-IPS. □

4. Quasi α-n-normed linear space

As a consequence of Definition 2.5 we introduce an interesting notion of ascending family of quasi α-n-norms corresponding to the fuzzy quasi n-norm in the following Theorem.

Theorem 4.1. *Let (X, N) be a f-q-n-NLS. Assume the condition that $(N7)$ $N(x_1, x_2, \ldots, x_n, t) > 0$ for all $t > 0$ implies $x_1, x_2, \ldots, x_n$ are linearly dependent. Define $||x_1, x_2, \ldots, x_n||_\alpha = \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}$, $\alpha \in (0, 1)$. Then $\{||\bullet, \bullet, \ldots, \bullet||_\alpha : \alpha \in (0, 1)\}$,is an ascending family of quasi n-norms on X. We call these quasi n-norms as quasi α-n-norms on X corresponding to the fuzzy quasi n-norm on X.*

Proof. (1) $||x_1, x_2, \ldots, x_n||_\alpha = 0 \Rightarrow \inf\{t : \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha\} = 0$, $\Rightarrow$ For all $t \in R$, $t > 0, \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha > 0, \alpha \in (0, 1) \Rightarrow$ By $(N7)$ $x_1, x_2, \ldots, x_n$ are linearly dependent. Conversely assume that $x_1, x_2, \ldots, x_n$ are linearly dependent. $\Rightarrow$ By $(N2)$, $N(x_1, x_2, \ldots, x_n, t) = 1$ for all $t > 0$. $\Rightarrow$ For all $\alpha \in (0, 1)$, $\inf\{t : \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha\} = 0$. $\Rightarrow ||x_1, x_2, \ldots, x_n||_\alpha = 0$.

(2) As $N(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation it follows that $||x_1, x_2, \ldots, x_n||_\alpha$ is invariant under any permutation.

$(3)'$ For all $c \in F$, $0 \leq p < 1$, then,

$$\begin{aligned} &||x_1, x_2, \ldots, cx_n||_\alpha \\ &= \inf\{s : \mathrm{N}(x_1, x_2, \ldots, cx_n, s) \geq \alpha\} \\ &= \inf\left\{s : \mathrm{N}(x_1, x_2, \ldots, x_n, \frac{s}{|c|^p}) \geq \alpha\right\}. \end{aligned}$$

Let $t = \frac{s}{|c|^p}$ then, $||x_1, x_2, \ldots, cx_n||_\alpha = \inf\{|c|^p t : \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha\} = |c|^p \inf\{t : \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha\} = |c|^p\ ||x_1, x_2, \ldots, x_n||_\alpha$.

(4)

$$\begin{aligned}
& \|x_1, x_2, \ldots, x_n\|_\alpha + \|x_1, x_2, \ldots, x_n'\|_\alpha \\
= & \inf\{t : \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha\} \\
& + \inf\left\{s : \mathrm{N}(x_1, x_2, \ldots, x_n', s) \geq \alpha\right\} \\
= & \inf\left\{t + s : \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha, \mathrm{N}(x_1, x_2, \ldots, x_n', s) \geq \alpha\right\} \\
\geq & \inf\left\{t + s : \mathrm{N}(x_1, x_2, \ldots, x_n + x_n', t + s) \geq \alpha\right\} \\
\geq & \inf\left\{r : \mathrm{N}(x_1, x_2, \ldots, x_n + x_n', r) \geq \alpha\right\}, r = t + s \\
= & \|x_1, x_2, \ldots, x_n + x_n'\|_\alpha.
\end{aligned}$$

Therefore, $\|x_1, x_2, \ldots, x_n + x_n'\|_\alpha \leq \|x_1, x_2, \ldots, x_n\|_\alpha + \|x_1, x_2, \ldots, x_n'\|_\alpha$. Thus $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0,1)\}$ is a quasi α-n-norm on X. Let $0 < \alpha_1 < \alpha_2$. Then

$$\begin{aligned}
& \|x_1, x_2, \ldots, x_n\|_{\alpha_1} \\
= & \inf\{t : \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha_1\} \\
& \|x_1, x_2, \ldots, x_n\|_{\alpha_2} = \inf\{t : \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha_2\}.
\end{aligned}$$

As

$$\begin{aligned}
\alpha_1 < \alpha_2 & \Rightarrow \{t : \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha_2\} \subset \{t : \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha_1\} \\
& \Rightarrow \inf\{t : \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha_2\} \\
& \geq \inf\{t : \mathrm{N}(x_1, x_2, \ldots, x_n, t) \geq \alpha_1\} \\
& \Rightarrow \|x_1, x_2, \ldots, x_n\|_{\alpha_2} \geq \|x_1, x_2, \ldots, x_n\|_{\alpha_1}.
\end{aligned}$$

Hence $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0,1)\}$, is an ascending family of quasi α-n-norms on X corresponding to the fuzzy quasi n-norm on X. □

Theorem 4.2. *Let $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0,1)\}$ be an ascending family of quasi n-norms corresponding to (X, N). Now we define a function $N' : X^n \times R \to [0,1]$ by,*

$$N'(x_1, x_2, \ldots, x_n, t) = \begin{cases} \sup\{\alpha \in (0,1) : \|x_1, x_2, \ldots, x_n\|_\alpha \leq t\}, \\ \quad \textit{when } x_1, x_2, \ldots, x_n \textit{ are linearly independent}, t \neq 0. \\ 0, \quad \textit{otherwise}. \end{cases}$$

Then (X, N') is a f-q-n-NLS.

Proof. Let us verify the six conditions of the f-q-n-NLS as follows: $(N1)$ For all $t \in R$ with $t < 0$ we have $N'(x_1, x_2, \ldots, x_n, t) = 0$, for all $(x_1, x_2, \ldots, x_n) \in X^n$, as $\{\alpha : \|x_1, x_2, \ldots, x_n\|_\alpha \leq t\} = \phi$ when $t < 0$. For $t = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, $\{\alpha : \|x_1, x_2, \ldots, x_n\|_\alpha \leq t\} = \phi \Rightarrow N'(x_1, x_2, \ldots, x_n, t) = 0$. When $x_1, x_2, \ldots, x_n$ are linearly dependent and $t = 0$ then from the definition, $N'(x_1, x_2, \ldots, x_n, t) = 0$. Thus for all $t \in R$ with $t \leq 0$, $N'(x_1, x_2, \ldots,$

$x_n, t) = 0$. $(N2)$ Let $N^{'}(x_1, x_2, \ldots, x_n, t) = 1$. That is for $t > 0$, $N^{'}(x_1, x_2, \ldots, x_n, t) = 1$. Choose any $\epsilon \in (0,1)$. Then for $t > 0$, there exists $\alpha_t \in (\epsilon, 1]$ such that $||x_1, x_2, \ldots, x_n||_{\alpha_t} \leq t$ and hence $||x_1, x_2, \ldots, x_n||_{\epsilon} \leq t$. Since $t > 0$ is arbitrary, this implies that $||x_1, x_2, \ldots, x_n||_{\epsilon} = 0 \Rightarrow x_1, x_2, \ldots, x_n$ are linearly dependent. Conversely, if $x_1, x_2, \ldots, x_n$ are linearly dependent, then for $t > 0$, $N^{'}(x_1, x_2, \ldots, x_n, t) = \sup\{\alpha : ||x_1, x_2, \ldots, x_n||_{\alpha} \leq t\} = \sup\{\alpha : \alpha \in (0,1)\} = 1$. Thus for all $t > 0, N^{'}(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent. $(N3)$ As $||x_1, x_2, \ldots, x_n||_{\alpha}$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$ we have $N^{'}(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$. $(N4)^{'}$For all $t \in R$ with $t > 0$, $c \in F$, $0 \leq p < 1$, $N^{'}(x_1, x_2, \ldots, cx_n, t) = \sup\{\alpha : ||x_1, x_2, \ldots, cx_n||_{\alpha} \leq t\}$ $= \sup\{\alpha : ||x_1, x_2, \ldots, x_n||_{\alpha} \leq \frac{t}{|c|^p}\} = N^{'}(x_1, x_2, \ldots, x_n, \frac{t}{|c|^p})$. $(N5)$ We have to show that for all $s, t \in R$,

$$N^{'}(x_1, x_2, \ldots, x_n + x_n^{'}, s + t) \geq \min\left\{N^{'}(x_1, x_2, \ldots, x_n, s), N^{'}(x_1, x_2, \ldots, x_n^{'}, t)\right\}.$$

If (a) $s + t < 0$ (b) $s = t = 0$ (c) $s + t > 0$; $s > 0, t < 0$; $s < 0$, $t > 0$, then in these cases the relation is obvious. If (d) $s > 0$, $t > 0$, let $p = N^{'}(x_1, x_2, \ldots, x_n, s), q = N^{'}(x_1, x_2, \ldots, x_n^{'}, t)$ and $p \leq q$. If $p = 0$ and $q = 0$ then obviously $(N5)$ holds. Let $0 < r < p \leq q$. Then there exists $\alpha > r$ such that $||x_1, x_2, \ldots, x_n||_{\alpha} \leq s$ and there exists $\beta > r$ such that $||x_1, x_2, \ldots, x_n^{'}||_{\alpha} \leq t$. Let $\gamma = \min\{\alpha, \beta\} > r$. Thus $||x_1, x_2, \ldots, x_n||_{\gamma} \leq ||x_1, x_2, \ldots, x_n||_{\alpha} \leq s$ and $||x_1, x_2, \ldots, x_n^{'}||_{\gamma} \leq ||x_1, x_2, \ldots, x_n^{'}||_{\alpha} \leq t$. Now $||x_1, x_2, \ldots, x_n + x_n^{'}||_{\gamma} \leq ||x_1, x_2, \ldots, x_n||_{\alpha} + ||x_1, x_2, \ldots, x_n^{'}||_{\alpha} \leq s + t$. Therefore, $N^{'}(x_1, x_2, \ldots, x_n + x_n^{'}, t + s) \geq \gamma > r$. Since $0 < r < \gamma$ is arbitrary, $N^{'}(x_1, x_2, \ldots, x_n + x_n^{'}, t + s) \geq p = \min\{N^{'}(x_1, x_2, \ldots, x_n, t), N^{'}(x_1, x_2, \ldots, x_n^{'}, s)\}$. Similarly if $p \geq q$, then also the relation holds. Thus, $N^{'}(x_1, x_2, \ldots, x_n, t + s) \geq \min\{N^{'}(x_1, x_2, \ldots, x_n, s), N^{'}(x_1, x_2, \ldots, x_n^{'}, t)\}$ $(N6)$ Let $(x_1, x_2, \ldots, x_n) \in X^n$ and $\alpha \in (0,1)$. Now $t > ||x_1, x_2, \ldots, x_n||_{\alpha} \Rightarrow N^{'}(x_1, x_2, \ldots, x_n, t) = \sup\{\beta : ||x_1, x_2, \ldots, x_n||_{\beta} \leq t\} \geq \alpha$. So, $\lim_{t \to \infty} N^{'}(x_1, x_2, \ldots, x_n, t) = 1$. If $t_1 < t_2 \leq 0$ then $N^{'}(x_1, x_2, \ldots, x_n, t_1) = N^{'}(x_1, x_2, \ldots, x_n, t_2) = 0$ for all $(x_1, x_2, \ldots, x_n) \in X^n$. If $t_2 > t_1 > 0$ then $\{\alpha : ||x_1, x_2, \ldots, x_n||_{\alpha} \leq t_1\} \subset \{\alpha : ||x_1, x_2, \ldots, x_n||_{\alpha} \leq t_2\} \Rightarrow \sup\{\alpha : ||x_1, x_2, \ldots, x_n||_{\alpha} \leq t_1\} \leq \sup\{\alpha : ||x_1, x_2, \ldots, x_n||_{\alpha} \leq t_2\}$ $\Rightarrow N^{'}(x_1, x_2, \ldots, x_n, t_1) \leq N^{'}(x_1, x_2, \ldots, x_n, t_2)$. Thus $N^{'}(x_1, x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in R$. Hence $(X, N^{'})$ is a f-q-n-NLS. □

Remark 4.3. Assume further that for $x_1, x_2, \ldots, x_n$ are linearly independent, $(N8)$ $N(x_1, x_2, \ldots, x_n, t)$ is a continuous function of $t \in R$ (R-set of real numbers) and strictly increasing in the subset $\{t : 0 < N(x_1, x_2, \ldots, x_n, t) < 1\}$ of R.

Theorem 4.4. *Let (X, N) be a f-q-n-NLS satisfying the conditions $(N7)$ and $(N8)$ and $\{||\bullet, \bullet, \ldots, \bullet||_\alpha : \alpha \in (0,1)\}$ be an ascending family of quasi n-norms corresponding to (X, N). Then for $(y_1, y_2, \ldots, y_n) \in X^n$ with $y_1, y_2, \ldots, y_n$ are linearly independent, $N(y_1, y_2, \ldots, y_n, ||y_1, y_2, \ldots, y_n||_\alpha) \geq \alpha, \alpha \in (0,1)$.*

Proof. Let $||y_1, y_2, \ldots, y_n||_\alpha = T$, then $T > 0$. Also there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $N(y_1, y_2, \ldots, y_n, t_n) \geq \alpha$ and $\lim_{n\to\infty} t_n = T$. So, $\lim_{n\to\infty} N(y_1, y_2, \ldots, y_n, t_n) \geq \alpha \Rightarrow$ By $(N8)$, $N(y_1, y_2, \ldots, y_n, \lim_{n\to\infty} t_n) \geq \alpha$. $\Rightarrow N(y_1, y_2, \ldots, y_n, ||y_1, y_2, \ldots, y_n||_\alpha) \geq \alpha$, for all $\alpha \in (0,1)$. □

Theorem 4.5. *Let (X, N) be a f-q-n-NLS satisfying the conditions $(N7)$ and $(N8)$ and $\{||\bullet, \bullet, \ldots, \bullet||_\alpha : \alpha \in (0,1)\}$ be an ascending family of quasi n-norms corresponding to (X, N). Then for $(y_1, y_2, \ldots, y_n) \in X^n$ with $y_1, y_2, \ldots, y_n$ are linearly independent, $\alpha \in (0,1)$ and $t^{'}(> 0) \in R$, $||y_1, y_2, \ldots, y_n||_\alpha = t^{'}$ if and only if $N(y_1, y_2, \ldots, y_n, t^{'}) = \alpha$.*

Proof. Let $\alpha \in (0,1)$, $y_1, y_2, \ldots, y_n$ are linearly independent and $t^{'} = ||y_1, y_2, \ldots, y_n||_\alpha = \inf\{s : N(y_1, y_2, \ldots, y_n, s) \geq \alpha\}$. Since $N(y_1, y_2, \ldots, y_n, t)$ is continuous (by $(N8)$), we have by Theorem 4.4

$$N(y_1, y_2, \ldots, y_n, t^{'}) \geq \alpha. \tag{4.1}$$

Also, $N(y_1, y_2, \ldots, y_n, t^{'}) \leq N(y_1, y_2, \ldots, y_n, s)$ if $N(y_1, y_2, \ldots, y_n, s) \geq \alpha$. If possible, let $N(y_1, y_2, \ldots, y_n, t^{'}) > \alpha$, then again by $(N8)$, there exists $t^{''} < t^{'}$ such that $N(y_1, y_2, \ldots, y_n, t^{''}) > \alpha$ which is impossible, since $t^{'} = \inf\{s : N(y_1, y_2, \ldots, y_n, s) \geq \alpha\}$. Thus

$$N(y_1, y_2, \ldots, y_n, t^{'}) \leq \alpha \tag{4.2}$$

By (4.1) and (4.2) we get $N(y_1, y_2, \ldots, y_n, t^{'}) = \alpha$. Thus

$$t^{'} = ||y_1, y_2, \ldots, y_n||_\alpha \Rightarrow N(y_1, y_2, \ldots, y_n, t^{'}) = \alpha. \tag{4.3}$$

Next if $N(y_1, y_2, \ldots, y_n, t^{'}) = \alpha$, then from the definition

$$||y_1, y_2, \ldots, y_n||_\alpha = \inf\{t : J(w, z|y_2, \ldots, y_n, t) \geq \alpha\} = t^{'}. \tag{4.4}$$

(Since $N(x_1, x_2, \ldots, x_n, t)$ is strictly increasing in $\{t : 0 < N(x_1, x_2, \ldots, x_n, t) < 1\}$.) From (4.3) and (4.4) we have, for $y_1, y_2, \ldots, y_n$ are linearly independent, $\alpha \in (0,1)$ and $t^{'}(> 0) \in R$, $||y_1, y_2, \ldots, y_n||_\alpha = t^{'}$ if and only if $N(y_1, y_2, \ldots, y_n, t^{'}) = \alpha$. □

Theorem 4.6. *Let (X, N) be a f-q-n-NLS satisfying the conditions $(N7)$ and $(N8)$. Let $||x_1, x_2, \ldots, x_n||_\alpha = \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}$, $\alpha \in (0,1)$ and $N^{'} : X^n \times R \to [0,1]$ is defined by,*

$$N^{'}(x_1, x_2, \ldots, x_n, t) = \begin{cases} \sup\{\alpha \in (0,1) : ||x_1, x_2, \ldots, x_n||_\alpha \leq t\}, \\ \quad \textit{when } x_1, x_2, \ldots, x_n \textit{ are linearly independent}, t \neq 0. \\ 0, \quad \textit{otherwise}. \end{cases}$$

Then (a) $\{||\bullet, \bullet, \ldots, \bullet||_\alpha : \alpha \in (0,1)\}$ *is an ascending family of quasi α-n-norms corresponding to* (X, N). *(b)* $(X, N^{'})$ *is a f-q-n-NLS. (c)* $N^{'} = N$.

Proof. (a) and (b) follows from Theorem 4.1 and Theorem 4.2. (c) To prove this we consider the following cases. Let $(y_1, y_2, \ldots, y_n, t_0) \in X^n \times R$ and $N(y_1, y_2, \ldots, y_n, t) = \alpha_0$. Case(i) If $y_1, y_2, \ldots, y_n$ are linearly dependent and $t_0 \leq 0$, then $N(y_1, y_2, \ldots, y_n, t_0) = N^{'}(y_1, y_2, \ldots, y_n, t_0) = 0$. Case(ii) If $y_1, y_2, \ldots, y_n$ are linearly dependent and $t_0 > 0$, then $N(y_1, y_2, \ldots, y_n, t_0) = N^{'}(y_1, y_2, \ldots, y_n, t_0) = 1$. Case(iii) If $y_1, y_2, \ldots, y_n$ are linearly independent and $t_0 \leq 0$, then $N(y_1, y_2, \ldots, y_n, t_0) = N^{'}(y_1, y_2, \ldots, y_n, t_0) = 0$. Case(iv) Suppose $y_1, y_2, \ldots, y_n$ are linearly independent and $t_0(> 0) \in R$ such that $N(y_1, y_2, \ldots, y_n, t_0) = 0$. For $\alpha \in (0,1)$, $||y_1, y_2, \ldots, y_n||_\alpha = \inf\{t : N(y_1, y_2, \ldots, y_n, t_0) \geq \alpha\}$ By Theorem 4.4, $N(y_1, y_2, \ldots, y_n, ||y_1, y_2, \ldots, y_n||_\alpha) \geq \alpha$, for all $\alpha \in (0,1)$. Since, $N(y_1, y_2, \ldots, y_n, t_0) = 0 < \alpha$ it follows that $t_0 < ||y_1, y_2, \ldots, y_n||_\alpha$, for all $\alpha > 0$. So,

$N^{'}(y_1, y_2, \ldots, y_n, t_0) = \sup\{\alpha : ||y_1, y_2, \ldots, y_n||_\alpha \leq t_0\} = \sup\{\phi\} = 0$. Therefore $N(y_1, y_2, \ldots, y_n, t_0) = N^{'}(y_1, y_2, \ldots, y_n, t_0)$. Case(v) If $y_1, y_2, \ldots, y_n$ are linearly independent and $t_0(> 0) \in R$, such that $0 < N(y_1, y_2, \ldots, y_n, t_0) < 1$. Let $N(y_1, y_2, \ldots, y_n, t_0) = \alpha_0$. Then $0 < \alpha_0 < 1$. Now $N^{'}(x_1, x_2, \ldots, x_n, t) = \sup\{\alpha : ||x_1, x_2, \ldots, x_n||_\alpha \leq t\}$, when $x_1, x_2, \ldots, x_n$ are linearly independent,

$$t \neq 0 \tag{4.5}$$

and

$$||x_1, x_2, \ldots, x_n||_\alpha = \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}, \alpha \in (0,1). \tag{4.6}$$

Since $N(y_1, y_2, \ldots, y_n, t_0) = \alpha_0$, we have from (4.6),

$$||y_1, y_2, \ldots, y_n||_{\alpha_0} \leq t_0. \tag{4.7}$$

Using (4.7) we get from (4.5),

(4.8)

$$N^{'}(y_1, y_2, \ldots, y_n, t_0) \geq \alpha_0 \Rightarrow N^{'}(y_1, y_2, \ldots, y_n, t_0) \geq N(y_1, y_2, \ldots, y_n, t_0).$$

Now from Theorem 4.5, $N(y_1, y_2, \ldots, y_n, t_0) = \alpha_0 \Leftrightarrow ||y_1, y_2, \ldots, y_n||_{\alpha_0} = t_0$. Now for $1 > \alpha > \alpha_0$, let $||y_1, y_2, \ldots, y_n||_\alpha = t^{'}$, then $t^{'} \geq t_0$. Then by Theorem 4.5, $N(y_1, y_2, \ldots, y_n, t^{'}) = \alpha$. So, $N(y_1, y_2, \ldots, y_n, t^{'}) = \alpha > \alpha_0 = N(y_1, y_2, \ldots, y_n, t_0)$. Since $N(y_1, y_2, \ldots, y_n, t)$ is strictly increasing and $N(y_1, y_2, \ldots, y_n, t^{'}) > N(y_1, y_2, \ldots, y_n, t_0)$, it follows that $t^{'} > t_0$. So for $1 > \alpha > \alpha_0$, $||y_1, y_2, \ldots, y_n||_\alpha = t^{'} \not\leq t_0$. Hence

$$N^{'}(y_1, y_2, \ldots, y_n, t_0) \leq \alpha_0 = N(y_1, y_2, \ldots, y_n, t_0). \tag{4.9}$$

By (4.8) and (4.9) we have, $N^{'}(y_1, y_2, \ldots, y_n, t_0) = N(y_1, y_2, \ldots, y_n, t_0)$. Case (vi) If $y_1, y_2, \ldots, y_n$ are linearly independent and $t_0(> 0) \in R$, such that $N(y_1, y_2, \ldots, y_n, t_0) = 1$. Then by (4.5) and (4.6) it follows that, $||y_1, y_2, \ldots, y_n||_\alpha \leq t_0 \Rightarrow N^{'}(y_1, y_2, \ldots, y_n, t_0) = 1$. Thus $N(y_1, y_2, \ldots, y_n, t_0) = N^{'}(y_1, y_2,$

$\ldots, y_n, t_0) = 1$. Hence $N(x_1, x_2, \ldots, x_n, t) = N'(x_1, x_2, \ldots, x_n, t)$ for all $(x_1, x_2, \ldots, x_n, t) \in X^n \times R$. □

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SRINIVASAN VIJAYABALAJI
DEPARTMENT OF MATHEMATICS
ANNAMALAI UNIVERSITY
ANNAMALAINAGAR-608002, TAMILNADU, INDIA
E-mail address: balaji_nandini@rediffmail.com

NATESAN THILLAIGOVINDAN
DEPARTMENT OF MATHEMATICS
ANNAMALAI UNIVERSITY
ANNAMALAINAGAR-608002, TAMILNADU, INDIA
E-mail address: thillai_n@sify.com