Bull. Korean Math. Soc. **44** (2007), No. 3, pp. 439–445

GLOBAL ASYMPTOTIC STABILITY OF A HIGHER ORDER DIFFERENCE EQUATION

Alaa E. Hamza and R. Khalaf-Allah

Abstract. The aim of this work is to investigate the global stability, periodic nature, oscillation and the boundedness of solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + Cx_{n-2l}x_{n-2k}}, \qquad n = 0, 1, 2, \ldots,$$

where A, B, C are nonnegative real numbers and l, k are nonnegative integers, $l \leq k$.

1. Introduction

Difference equations have always played an important role in the construction and analysis of mathematical models of biology, ecology, physics, economic processes, etc. [3].

The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations. Cinar [1] examined the global asymptotic stability of all positive solutions of the rational difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}, \qquad n = 0, 1, 2, \ldots.$$

X. Yang et al [4] investigated the asymptotic behavior of solutions of the difference equation

$$x_{n+1} = \frac{ax_n + bx_{n-1}}{c + dx_n x_{n-1}}, \qquad n = 0, 1, 2, \ldots,$$

where $a \geq 0, b, c, d > 0$.

In this paper, we study the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + Cx_{n-2l}x_{n-2k}}, \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

Received May 29, 2006; Revised May 29, 2007.
2000 *Mathematics Subject Classification.* 39A11.
Key words and phrases. difference equation, periodic solution, globally asymptotically stable.

where A, B, C are nonnegative real numbers and l, k are nonnegative integers, $l \leq k$.

The following particular cases can be obtained:

(1) When $A = 0$, equation (1.1) reduces to the equation

$$x_{n+1} = 0, \qquad n = 0, 1, 2, \ldots.$$

(2) When $B = 0$, equation (1.1) reduces to the equation

$$x_{n+1} = \frac{Ax_{n-1}}{Cx_{n-2l}x_{n-2k}}, \qquad n = 0, 1, 2, \ldots.$$

This equation can be reduced to the linear difference equation

$$y_{n+1} - y_{n-1} + y_{n-2l} + y_{n-2k} = \gamma,$$

by taking

$$x_n = e^{y_n}, \gamma = \ln \frac{A}{C}.$$

(3) When $C = 0$, equation (1.1) reduces to the equation

$$x_{n+1} = \frac{A}{B} x_{n-1}, \qquad n = 0, 1, 2, \ldots$$

which is a linear difference equation.

For various values of l and k, we can get more equations.

2. Preliminaries

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \qquad n = 0, 1, 2, \ldots, \tag{2.1}$$

where $f : \mathbb{R}^{k+1} \to \mathbb{R}$.

Definition 1 ([2]). An equilibrium point for equation (2.1) is a point $\bar{x} \in \mathbb{R}$ such that $\bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x})$.

Definition 2 ([2]).

(1) An equilibrium point $\bar{x}$ for equation (2.1) is called *locally stable* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that every solution $\{x_n\}$ with initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in]\bar{x} - \delta, \bar{x} + \delta[$ is such that $x_n \in]\bar{x} - \epsilon, \bar{x} + \epsilon[$ for all $n \in \mathbb{N}$. Otherwise $\bar{x}$ is said to be *unstable*.

(2) The equilibrium point $\bar{x}$ of equation (2.1) is called *locally asymptotically stable* if it is locally stable and there exists $\gamma > 0$ such that for any initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in]\bar{x} - \gamma, \bar{x} + \gamma[$, the corresponding solution $\{x_n\}$ tends to $\bar{x}$.

(3) An equilibrium point $\bar{x}$ for equation (2.1) is called a *global attractor* if every solution $\{x_n\}$ converges to $\bar{x}$ as $n \to \infty$.

(4) The equilibrium point $\bar{x}$ for equation (2.1) is called *globally asymptotically stable* if it is locally asymptotically stable and global attractor.

The linearized equation associated with equation (2.1) is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x}) y_{n-i}, \qquad n = 0, 1, 2 \ldots. \tag{2.2}$$

The characteristic equation associated with equation (2.2) is

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x}) \lambda^{k-i} = 0. \tag{2.3}$$

Theorem 2.1 ([2]). *Assume that f is a C^1 function and let $\bar{x}$ be an equilibrium point of equation* (2.1). *Then the following statements are true:*

(1) *If all roots of equation* (2.3) *lie in the open disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable.*
(2) *If at least one root of equation* (2.3) *has absolute value greater than one, then $\bar{x}$ is unstable.*

The change of variables $x_n = \sqrt{\frac{B}{C}} y_n$ reduces equation (1.1) to the difference equation

$$y_{n+1} = \frac{\gamma y_{n-1}}{1 + y_{n-2l} y_{n-2k}}, \qquad n = 0, 1, 2, \ldots, \tag{2.4}$$

where $\gamma = \frac{A}{B}$.

3. Linearized stability analysis

In this section we study the asymptotic stability of the nonnegative equilibrium points of equation (2.4). We can see that equation (2.4) has two nonnegative equilibrium points $\bar{y} = 0$ and $\bar{y} = \sqrt{\gamma - 1}$ when $\gamma > 1$ and the zero equilibrium only when $\gamma \leq 1$. The linearized equation associated with equation (2.4) about $\bar{y}$ is

$$z_{n+1} - \frac{\gamma}{1 + \bar{y}^2} z_{n-1} + \frac{\gamma \bar{y}^2}{(1 + \bar{y}^2)^2} (z_{n-2l} + z_{n-2k}) = 0. \tag{3.1}$$

The characteristic equation associated with this equation is

$$\lambda^{2k+1} - \frac{\gamma}{1 + \bar{y}^2} \lambda^{2k-1} + \frac{\gamma \bar{y}^2}{(1 + \bar{y}^2)^2} (\lambda^{2k-2l} + 1) = 0. \tag{3.2}$$

We summarize the results of this section in the following theorem.

Theorem 3.1. (1) *If $\gamma < 1$, then the zero equilibrium point is locally asymptotically stable.*
(2) *If $\gamma > 1$, then the equilibrium points $\bar{y} = 0$ and $\bar{y} = \sqrt{\gamma - 1}$ are unstable (saddle points).*

Proof. The linearized equation associated with equation (2.4) about $\bar{y} = 0$ is

$$z_{n+1} - \gamma z_{n-1} = 0.$$

and the characteristic equation associated with this equation is

$$\lambda^{2k+1} - \gamma\lambda^{2k-1} = 0.$$

So $\lambda = 0, \pm\sqrt{\gamma}$.

(1) If $\gamma < 1$, then $|\lambda| < 1$ for all roots and $\bar{y} = 0$ is locally asymptotically stable.

(2) If $\gamma > 1$, it follows that $\bar{y} = 0$ is unstable (saddle point). The linearized equation (3.1) about $\bar{y} = \sqrt{\gamma - 1}$ becomes

$$z_{n+1} - z_{n-1} + (1 - \frac{1}{\gamma})(z_{n-2l} + z_{n-2k}) = 0, \qquad n = 0, 1, 2, \ldots.$$

The associated characteristic equation is

$$\lambda^{2k+1} - \lambda^{2k-1} + (1 - \frac{1}{\gamma})(\lambda^{2k-2l} + 1) = 0.$$

Let $f(\lambda) = \lambda^{2k+1} - \lambda^{2k-1} + (1 - \frac{1}{\gamma})(\lambda^{2k-2l} + 1)$. We can see that $f(\lambda)$ has a root in $(-\infty, -1)$. Then the point $\bar{y} = \sqrt{\gamma - 1}$ is unstable (saddle point).

□

4. Global behavior of equation (2.4)

Theorem 4.1. *If $\gamma < 1$, then the zero equilibrium point is globally asymptotically stable.*

Proof. Let $\{y_n\}$ be a solution of equation (2.4). Hence

$$y_{n+1} = \frac{\gamma y_{n-1}}{1 + y_{n-2l}y_{n-2k}} < \gamma y_{n-1}, \qquad n = 0, 1, 2, \ldots.$$

Then $\lim_{n\to\infty} y_n = 0$.

In view of Theorem 3.1, $\bar{y} = 0$ is globally asymptotically stable. □

5. Existence of prime period two solutions

This section is devoted to discuss the condition under which there exist prime period two solutions.

Theorem 5.1. *A necessary and sufficient condition for equation (2.4) to have a prime period two solution is that $\gamma = 1$. In this case the prime period two solution is of the form $\ldots, 0, \varphi, 0, \varphi, 0, \ldots$, where $\varphi > 0$. Furthermore, every solution converges to a period two solution.*

Proof. Sufficiency: let $\gamma = 1$, then for every $\varphi > 0$ we have $\ldots, 0, \varphi, 0, \varphi, 0, \ldots$ is a prime period two solution.

Necessity: assume that equation (2.4) has a prime period two solution $\ldots, \psi, \varphi, \psi, \varphi, \psi, \ldots$. Then $\varphi = \frac{\gamma\varphi}{1+\psi^2}$, $\psi = \frac{\gamma\psi}{1+\varphi^2}$. Hence we have

$$\varphi + \varphi\psi^2 = \gamma\varphi, \tag{5.1}$$

and

$$\psi + \psi\varphi^2 = \gamma\psi. \tag{5.2}$$

From equations (5.1) and (5.2), by subtracting we get

$$(\varphi - \psi) + \varphi\psi(\psi - \varphi) = \gamma(\varphi - \psi).$$

This implies

$$\varphi\psi = 1 - \gamma. \tag{5.3}$$

So $\gamma \leq 1$. Similarly, from equations (5.1) and (5.2), by adding we get

$$(\varphi + \psi) + \varphi\psi(\psi + \varphi) = \gamma(\varphi + \psi).$$

This implies

$$\varphi\psi = \gamma - 1. \tag{5.4}$$

So $\gamma \geq 1$. Then $\gamma = 1$. In this case $\varphi\psi = 0$ and the solution is of the form

$$\ldots, 0, \varphi, 0, \varphi, 0, \ldots \qquad \text{with } \varphi > 0.$$

Now let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of equation (2.4) with $\gamma = 1$. Then

$$y_{n+1} = \frac{\gamma y_{n-1}}{1 + y_{n-2l}y_{n-2k}} \leq y_{n-1}, \qquad n = 0, 1, 2, \ldots$$

and so the even terms $\{y_{2n}\}_{n=0}^{\infty}$ decreases to a limit φ and the odd terms $\{y_{2n+1}\}_{n=0}^{\infty}$ decreases to a limit ψ, where $\varphi = \frac{\varphi}{1+\psi^2}, \psi = \frac{\psi}{1+\varphi^2}$. Then $\varphi\psi^2 = 0$ and $\psi\phi^2 = 0$. Therefore, $\{y_n\}_{n=-2k}^{\infty}$ converges to the periodic solution $\ldots, 0, \varphi, 0, \varphi, 0, \ldots$ with $\varphi > 0$. □

6. Semicycle analysis

In this section, we discuss the existence of semicycles. We need the following theorem to obtain the main result of this section.

Theorem 6.1. *Assume that $f \in C([0, \infty[^{2k+1}, [0, \infty[)$ increases in the even arguments and decreases in the others. Let $\bar{y}$ be an equilibrium point for the difference equation*

$$y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-2k}), \qquad n = 0, 1, 2, \ldots \tag{6.1}$$

Let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of equation (6.1) such that either,

(C_1) $y_{-2k}, y_{-2k+2}, \ldots, y_0 > \bar{y}$ and $y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} < \bar{y}$, or

(C_2) $y_{-2k}, y_{-2k+2}, \ldots, y_0 < \bar{y}$ and $y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} > \bar{y}$.

is satisfied, then $\{y_n\}_{n=-2k}^{\infty}$ oscillates about $\bar{y}$ with semicycles of length one.

Proof. Assume that f increases in the even arguments and decreases in the others. Let f satisfy the condition (C_1), we have

$$\begin{aligned} y_1 &= f(y_0, y_{-1}, y_{-2}, \ldots, y_{-2k+1}, y_{-2k}) < f(\bar{y}, y_{-1}, \bar{y}, \ldots, y_{-2k+1}, \bar{y}) \\ &< f(\bar{y}, \bar{y}, \bar{y}, \ldots, \bar{y}, \bar{y}) = \bar{y}, \\ y_2 &= f(y_1, y_0, y_{-1}, \ldots, y_{-2k+2}, y_{-2k+1}) > f(\bar{y}, y_0, \bar{y}, \ldots, y_{-2k+2}, \bar{y}) \\ &> f(\bar{y}, \bar{y}, \bar{y}, \ldots, \bar{y}, \bar{y}) = \bar{y}. \end{aligned}$$

By induction we obtain the result. If f satisfies condition (C_2), we can prove the result similarly. □

Corollary 6.2. *Assume that $\gamma > 1$ and let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of equation (2.4) such that either (C_1) or (C_2) is satisfied. Then $\{y_n\}_{n=-2k}^{\infty}$ oscillates about the positive equilibrium point $\bar{y} = \sqrt{\gamma - 1}$ with semicycles of length one.*

Proof. The proof follows directly from the previous theorem. □

7. Existence of unbounded solutions

Finally we show that, under certain initial conditions, unbounded solution will be obtained.

Theorem 7.1. *Assume that $\gamma > 1$. Let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of equation (2.4) and $\bar{y} = \sqrt{\gamma - 1}$, the positive equilibrium point. Then the following statements are true:*

(1) *If $y_{-2k}, y_{-2k+2}, \ldots, y_0 > \bar{y}$ and $y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} < \bar{y}$, then $\{y_{2n}\}$ increases to ∞ and $\{y_{2n+1}\}$ decreases to 0.*
(2) *If $y_{-2k}, y_{-2k+2}, \ldots, y_0 < \bar{y}$ and $y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} > \bar{y}$, then $\{y_{2n}\}$ decreases to 0 and $\{y_{2n+1}\}$ increases to ∞.*

Proof. (1) Let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of equation (2.4) with initial conditions, $y_{-2k}, y_{-2k+2}, \ldots, y_0 > \bar{y}$ and $y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} < \bar{y}$. Then

$$y_{2n+2} = \frac{\gamma y_{2n}}{1 + y_{2n-2l+1} y_{2n-2k+1}} > \frac{\gamma y_{2n}}{1 + \bar{y}^2} = y_{2n}$$

and

$$y_{2n+3} = \frac{\gamma y_{2n+1}}{1 + y_{2n-2l+1} y_{2n-2k+1}} < \frac{\gamma y_{2n+1}}{1 + \bar{y}^2} = y_{2n+1}$$

and so $\{y_{2n}\}$ increases to ∞ and $\{y_{2n+1}\}$ decreases to 0.

(2) The proof is similar and will be omitted. □

Acknowledgment. Many thanks to Dr. Adel El-Tohamy for his help and support.

References

[1] C. Cinar, *On the positive solution of the difference equation* $x_{n+1} = \frac{ax_{n-1}}{1+bx_n x_{n-1}}$, Appl. Math. Comput. **156** (2004), no. 2, 587–590.

[2] V. L. Kocic and G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications*, Mathematics and its Applications, 256. Kluwer Academic Publishers Group, Dordrecht, 1993.

[3] R. E. Mickens, *Difference equations*, Theory and applications. Second edition. Van Nostrand Reinhold Co., New York, 1990.

[4] X. Yang, W. Su, B. Chen, G. M. Megson, and D. J. Evans, *On the recursive sequence* $x_{n+1} = \frac{ax_n+bx_{n-1}}{c+dx_n x_{n-1}}$, Appl. Math. Comput. **162** (2005), no. 3, 1485–1497.

Alaa E. Hamza
Department of Mathematics
Faculty of Science, Cairo University
Giza, 12211, Egypt
E-mail address: hamzaaeg2003@yahoo.com

R. Khalaf-Allah
Department of Mathematics
Faculty of Science, Helwan University
Cairo, 11795, Egypt
E-mail address: abuzead73@yahoo.com