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POSITION VECTORS OF A SPACELIKE W-CURVE IN MINKOWSKI SPACE $\mathbb{E}_1^3$

Kazim İlarslan and Özgür Boyacioğlu

Abstract. In this paper, we study the position vectors of a spacelike W-curve (or a helix), i.e., curve with constant curvatures, with spacelike, timelike and null principal normal in the Minkowski 3-space $\mathbb{E}_1^3$. We give some chracterizations for spacelike W- curves whose image lies on the pseudohyperbolical space $\mathbb{H}_0^2$ and Lorentzian sphere $\mathbb{S}_1^2$ by using the positions vectors of the curve.

1. Introduction

In the Euclidean space $\mathbb{E}^3$, it is well-known that to each unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields T, N and B called respectively the tangent, the principal normal and the binormal vector fields. At each point $\alpha(s)$ of the curve α, the planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are known respectively as the osculating plane, the rectifying plane and the normal plane. The curves $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ for which the position vector α always lie in their rectifying plane, are for simplicity called *rectifying curves*. Characterization of rectifying curve is given in [3] and these curves are studied in Minkowski space $\mathbb{E}_1^3$ in [7]. Similarly, the curves for which the position vector α always lie in their osculating plane, are for simplicity called *osculating curves* and finally, the curves for which the position vector always lie in their normal plane, are for simplicity called *normal curves*. Characterization of normal curve in $\mathbb{E}_1^3$ is given [5], [6].

A curve α is called a W-curve (or a helix), if it has constant Frenet curvatures. W-curves in the Euclidean space $\mathbb{E}^n$ have been studied intensively. The simplest examples are circles as planar W-curves and helices as non-planar W-curves in $\mathbb{E}^3$.

All W-curves in the Minkowski 3-space are completely classified by Walrave in [13]. For example, the only planar spacelike W-curves are circles and hyperbolas. All spacelike W-curves in the Minkowski space-time $\mathbb{E}_1^4$ are studied in

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[11]. The examples of null W-curves in the Minkowski space-time are given in [1]. Timelike W-curves in the same space have been studied in [12].

In this paper, we obtain the position vectors of a spacelike W-curve with spacelike, timelike and null principal normal N and by using position vectors we give some characterization for the spacelike W-curve whose image lies on the pseudohyperbolical space $\mathbb{H}_0^2$ and Lorentzian sphere $\mathbb{S}_1^2$ in $\mathbb{E}_1^3$.

2. Preliminaries

The Minkowski 3-space $\mathbb{E}_1^3$ is the Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$.

Since g is an indefinite metric, recall that a vector $v \in \mathbb{E}_1^3$ can have one of three Lorentzian causal characters: it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike). Denote by $\{T, N, B\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space $\mathbb{E}_1^3$. For an arbitrary curve $\alpha(s)$ in the space $\mathbb{E}_1^3$, the following Frenet formulae are given in [4, 5, 13].

If α is a spacelike curve with a spacelike or timelike principal normal N, then the Frenet formulae read

$$(1) \qquad \begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\epsilon k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where $g(T,T) = 1, g(N,N) = \epsilon = \pm 1, g(B,B) = -\epsilon, g(T,N) = 0, g(T,B) = 0, g(N,B) = 0$.

If α is a spacelike curve with a null (lightlike) principal normal N, the Frenet formulae are

$$(2) \qquad \begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ 0 & k_2 & 0 \\ -k_1 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where $g(T,T) = 1$, $g(N,N) = 0$, $g(B,B) = 0$, $g(T,N) = 0$, $g(T,B) = 0$, $g(N,B) = 1$. In this case, k_1 can take only two values: $k_1 = 0$ when α is a straight line, or $k_1 = 1$ in all other cases.

Let m be a fixed point in $\mathbb{E}_1^3$ and $r > 0$ be a constant. The pseudo-Riemannian hyperbolical space is defined by

$$\mathbb{H}_0^2(m,r) = \{u \in \mathbb{E}_1^3 : g(u-m, u-m) = -r^2\}, \ r \in \mathbb{R}^+,$$

and the Lorentzian sphere is defined by

$$\mathbb{S}_1^2(m,r) = \{u \in \mathbb{E}_1^3 : g(u-m, u-m) = r^2\}, \ r \in \mathbb{R}^+.$$

3. Position vectors of a spacelike W-curve in $\mathbb{E}_1^3$

In this section, we obtain position vectors of a spacelike W-curve with spacelike, timelike and null principal normal.

Case 1. If $\alpha(s)$ is a spacelike W- curve with spacelike and timelike principal normal. Then we can write its position vector as follows:

$$\alpha(s) = \lambda(s)T(s) + \mu(s)N(s) + \gamma(s)B(s), \tag{3}$$

for some differentiable functions λ, μ and γ of $s \in I \subset \mathbb{R}$.

Differentiating (3) with respect to s and by using the corresponding Frenet equations (1), we find

$$\lambda'(s) - \epsilon\mu(s)k_1 = 1, \lambda(s)k_1 + \mu'(s) + \gamma(s)k_2 = 0, \mu(s)k_2 + \gamma'(s) = 0. \tag{4}$$

From (4), we get the following differential equation:

$$\mu''(s) + \mu(s)(\epsilon k_1^2 - k_2^2) + k_1 = 0. \tag{5}$$

Now we have distinguish the following cases:

Case 1.1. If $\alpha(s)$ is a spacelike W-curve with timelike principal normal N. Thus $\epsilon = -1$, therefore from (5) we get

$$\mu''(s) - \mu(s)(k_1^2 + k_2^2) + k_1 = 0. \tag{6}$$

The solution of the above differential equation is:

$$\mu(s) = c_1 e^{-s\sqrt{k_1^2+k_2^2}} + c_2 e^{s\sqrt{k_1^2+k_2^2}} + \frac{k_1}{k_1^2+k_2^2}, \tag{7}$$

where $c_1, c_2 \in \mathbb{R}$.

From (4) and using $\epsilon = -1$, we get $\lambda'(s) = 1 - \mu(s)k_1$. By using (7) we find the solution of this equation as follows,

$$\lambda(s) = s + \frac{c_1 k_1}{\sqrt{k_1^2+k_2^2}} e^{-s\sqrt{k_1^2+k_2^2}} - \frac{c_2 k_1}{\sqrt{k_1^2+k_2^2}} e^{s\sqrt{k_1^2+k_2^2}} - \frac{k_1^2 s}{k_1^2+k_2^2}, \tag{8}$$

and from (4) we have $\gamma'(s) = -\mu(s)k_2$. By using (7), we find the solution of this equation as follows,

$$\gamma(s) = \frac{c_1 k_2}{\sqrt{k_1^2+k_2^2}} e^{-s\sqrt{k_1^2+k_2^2}} - \frac{c_2 k_2}{\sqrt{k_1^2+k_2^2}} e^{s\sqrt{k_1^2+k_2^2}} - \frac{k_1 k_2 s}{k_1^2+k_2^2}. \tag{9}$$

Thus the position vector of the spacelike W-curve with timelike N is:

$$\begin{aligned}\alpha(s) = & \left(s + \tfrac{c_1 k_1}{\sqrt{k_1^2+k_2^2}} e^{-s\sqrt{k_1^2+k_2^2}} - \tfrac{c_2 k_1}{\sqrt{k_1^2+k_2^2}} e^{s\sqrt{k_1^2+k_2^2}} - \tfrac{k_1^2 s}{k_1^2+k_2^2}\right) T(s) \\ & + \left(c_1 e^{-s\sqrt{k_1^2+k_2^2}} + c_2 e^{s\sqrt{k_1^2+k_2^2}} + \frac{k_1}{k_1^2+k_2^2}\right) N(s) \\ & + \left(\frac{c_1 k_2}{\sqrt{k_1^2+k_2^2}} e^{-s\sqrt{k_1^2+k_2^2}} - \frac{c_2 k_2}{\sqrt{k_1^2+k_2^2}} e^{s\sqrt{k_1^2+k_2^2}} - \frac{k_1 k_2 s}{k_1^2+k_2^2}\right) B(s).\end{aligned} \tag{10}$$

Corollary 3.1. *Let $\alpha = \alpha(s)$ be a unit speed spacelike W-curve in $\mathbb{E}_1^3$ with timelike principal normal N and with curvatures $k_1(s) > 0$, $k_2(s) \neq 0$ for each $s \in I \subset \mathbb{R}$. Then position vector of the curve is given by the equation* (10).

Case 1.2. If $\alpha(s)$ is a spacelike W-curve with spacelike principal normal N. Thus $\epsilon = 1$, therefore from (5) we get

$$\mu''(s) + \mu(s)(k_1^2 - k_2^2) + k_1 = 0. \tag{11}$$

In this case we have the following subcases:

Case 1.2.1. If $k_1^2 = k_2^2$. The solution of the equation (11) is

$$\mu(s) = \frac{-k_1 s^2}{2} + c_1 s + c_2, \tag{12}$$

where $c_1, c_2 \in \mathbb{R}$. From (4) and using $\epsilon = 1$, we get $\lambda'(s) = 1 + \mu(s)k_1$. By using (12) we find the solution of this equation as follows,

$$\lambda(s) = \frac{-k_1^2 s^3}{6} + \frac{c_1 k_1 s^2}{2} + (c_2 k_1 + 1)s, \tag{13}$$

and from (4) we have $\gamma'(s) = -\mu(s)k_2$. By using (12), we find the solution of this equation as follows,

$$\gamma(s) = \frac{k_1 k_2 s^3}{6} - \frac{c_1 k_2 s^2}{2} - c_2 k_2 s. \tag{14}$$

Thus we find the position vector as;

$$\begin{aligned} \alpha(s) = & \left(\frac{-k_1^2 s^3}{6} + \frac{c_1 k_1 s^2}{2} + (c_2 k_1 + 1)s\right) T(s) \\ & + \left(\frac{-k_1 s^2}{2} + c_1 s + c_2\right) N(s) \\ & + \left(\frac{k_1 k_2 s^3}{6} - \frac{c_1 k_2 s^2}{2} - c_2 k_2 s\right) B(s). \end{aligned} \tag{15}$$

Case 1.2.2. If $k_1^2 > k_2^2$. The solution of the equation (11) is

$$\mu(s) = c_1 \cos(s\sqrt{k_1^2 - k_2^2}) + c_2 \sin(s\sqrt{k_1^2 - k_2^2}) - \frac{k_1}{k_1^2 - k_2^2}, \tag{16}$$

where $c_1, c_2 \in \mathbb{R}$. From $\lambda'(s) = 1 + \mu(s)k_1$ and using (16) we find the solution of this equation as follows,

$$\begin{aligned} \lambda(s) \; = \; & s + \frac{c_1 k_1}{\sqrt{k_1^2 - k_2^2}} \sin(s\sqrt{k_1^2 - k_2^2}) \\ & - \frac{c_2 k_1}{\sqrt{k_1^2 - k_2^2}} \cos(s\sqrt{k_1^2 - k_2^2}) - \frac{k_1^2 s}{k_1^2 - k_2^2}. \end{aligned} \tag{17}$$

From $\gamma'(s) = -\mu(s)k_2$. By using (16), we find the solution of this equation as follows,
(18)
$$\gamma(s) = \frac{-c_1k_2}{\sqrt{k_1^2-k_2^2}}\sin(s\sqrt{k_1^2-k_2^2}) + \frac{c_2k_2}{\sqrt{k_1^2-k_2^2}}\cos(s\sqrt{k_1^2-k_2^2}) + \frac{k_1k_2s}{k_1^2-k_2^2}.$$

Thus we find the position vector as;
(19)
$$\begin{aligned}\alpha(s) &= \left(s + \frac{c_1k_1}{\sqrt{k_1^2-k_2^2}}\sin(s\sqrt{k_1^2-k_2^2}) - \frac{c_2k_1}{\sqrt{k_1^2-k_2^2}}\cos(s\sqrt{k_1^2-k_2^2}) - \frac{k_1^2s}{k_1^2-k_2^2}\right)T(s)\\ &+ \left(c_1\cos(s\sqrt{k_1^2-k_2^2}) + c_2\sin(s\sqrt{k_1^2-k_2^2}) - \frac{k_1}{k_1^2-k_2^2}\right)N(s)\\ &+ \left(\frac{-c_1k_2}{\sqrt{k_1^2-k_2^2}}\sin(s\sqrt{k_1^2-k_2^2}) + \frac{c_2k_2}{\sqrt{k_1^2-k_2^2}}\cos(s\sqrt{k_1^2-k_2^2}) + \frac{k_1k_2s}{k_1^2-k_2^2}\right)B(s).\end{aligned}$$

Case 1.2.3. If $k_1^2 < k_2^2$. The solution of the equation (11) is

$$\mu(s) = c_1e^{-s\sqrt{k_2^2-k_1^2}} + c_2e^{s\sqrt{k_2^2-k_1^2}} + \frac{k_1}{k_2^2-k_1^2}, \tag{20}$$

where $c_1, c_2 \in \mathbb{R}$. From $\lambda'(s) = 1 + \mu(s)k_1$ and using (20) we find the solution of this equation as follows,

$$\lambda(s) = s - \frac{c_1k_1}{\sqrt{k_2^2-k_1^2}}e^{-s\sqrt{k_2^2-k_1^2}} + \frac{c_2k_1}{\sqrt{k_2^2-k_1^2}}e^{s\sqrt{k_2^2-k_1^2}} + \frac{k_1^2s}{k_2^2-k_1^2}. \tag{21}$$

From $\gamma'(s) = -\mu(s)k_2$ and using (20), we find the solution of this equation as follows,

$$\gamma(s) = \frac{c_1k_2}{\sqrt{k_2^2-k_1^2}}e^{-s\sqrt{k_2^2-k_1^2}} - \frac{c_2k_2}{\sqrt{k_2^2-k_1^2}}e^{s\sqrt{k_2^2-k_1^2}} - \frac{k_1k_2s}{k_2^2-k_1^2}. \tag{22}$$

Thus the position vector of the spacelike W-curve with timelike N is:

(23)
$$\begin{aligned}\alpha(s) = &\left(s - \frac{c_1k_1}{\sqrt{k_2^2-k_1^2}}e^{-s\sqrt{k_2^2-k_1^2}} + \frac{c_2k_1}{\sqrt{k_2^2-k_1^2}}e^{s\sqrt{k_2^2-k_1^2}} + \frac{k_1^2s}{k_2^2-k_1^2}\right)T(s)\\ &+ \left(c_1e^{-s\sqrt{k_2^2-k_1^2}} + c_2e^{s\sqrt{k_2^2-k_1^2}} + \frac{k_1}{k_2^2-k_1^2}\right)N(s)\\ &+ \left(\frac{c_1k_2}{\sqrt{k_2^2-k_1^2}}e^{-s\sqrt{k_2^2-k_1^2}} - \frac{c_2k_2}{\sqrt{k_2^2-k_1^2}}e^{s\sqrt{k_2^2-k_1^2}} - \frac{k_1k_2s}{k_2^2-k_1^2}\right)B(s).\end{aligned}$$

Corollary 3.2. *Let $\alpha = \alpha(s)$ be a unit speed spacelike W-curve in $\mathbb{E}_1^3$ with spacelike principal normal N and with curvatures $k_1(s) > 0$, $k_2(s) \neq 0$ for each $s \in I \subset \mathbb{R}$.*

(i) *If $k_1^2 = k_2^2$, then the position vector of the curve $\alpha = \alpha(s)$ is given by the equation* (15),

(ii) *If $k_1^2 > k_2^2$, then the position vector of the curve $\alpha = \alpha(s)$ is given by the equation* (19),

(iii) *If $k_1^2 < k_2^2$, then the position vector of the curve $\alpha = \alpha(s)$ is given by the equation* (23).

Case 2. If $\alpha(s)$ is a spacelike curve with null principal normal N. Differentiating (4) with respect to s and by using the corresponding Frenet equations (2), we find

$$\lambda^{'}(s) - \gamma(s)k_1 = 1, \lambda(s)k_1 + \mu^{'}(s) + k_2\mu(s) = 0, \gamma^{'}(s) - k_2\gamma(s) = 0. \tag{24}$$

In this case there are only two values of the first curvature k_1: $k_1 = 0$ when α is a straight line, or $k_1 = 1$ in all other cases. Since α is a space curve then we take $k_1 = 1$. Thus from (24) we get $\gamma(s) = c_1 e^{k_2 s}$, $\lambda(s) = \frac{c_1}{k_2} e^{k_2 s} + s$, and $\mu(s) = \frac{-c_1}{2k_2^2} e^{k_2 s} + c_2 e^{-k_2 s} - \frac{s}{k_2} + \frac{1}{k_2^2}$, where $c_1, c_2 \in \mathbb{R}$. Finally the position vector of a spacelike W-curves with null principal normal N is

$$\begin{aligned}\alpha(s) = {} & \left(\frac{c_1}{k_2} e^{k_2 s} + s\right) T(s) + \left(\frac{-c_1}{2k_2^2} e^{k_2 s} + c_2 e^{-k_2 s} - \frac{s}{k_2} + \frac{1}{k_2^2}\right) N(s) \\ & + \left(c_1 e^{k_2 s}\right) B(s).\end{aligned} \tag{25}$$

Corollary 3.3. *Let $\alpha = \alpha(s)$ be a unit speed spacelike W-curve in $\mathbb{E}_1^3$ with null (lightlike) principal normal N and with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$ for each $s \in I \subset \mathbb{R}$. Then position vector of the curve is given by the equation* (25).

4. Spacelike W-curves on $\mathbb{H}_0^2$ and $\mathbb{S}_1^2$ in $\mathbb{E}_1^3$

In this section, we give some characterization for the spacelike W-curves whose image lies on the pseudohyperboical space $\mathbb{H}_0^2$ and Lorentzian sphere $\mathbb{S}_1^2$ in $\mathbb{E}_1^3$.

Theorem 4.1. *Let $\alpha(s)$ be a unit speed spacelike W-curve with timelike principal normal N in $\mathbb{E}_1^3$ with the curvatures $k_1(s) \neq 0$ and $k_2(s) \neq 0$. The image of the curve lies on a pseudohyperbolic space $\mathbb{H}_0^2$ if and only if for each $s \in I \subset \mathbb{R}$ the curvatures satisfy the following equality:*

$$s + \frac{c_1 k_1}{\sqrt{k_1^2 + k_2^2}} e^{-s\sqrt{k_1^2+k_2^2}} - \frac{c_2 k_1}{\sqrt{k_1^2 + k_2^2}} e^{s\sqrt{k_1^2+k_2^2}} - \frac{k_1^2 s}{k_1^2 + k_2^2} = 0, \tag{26}$$

$$c_1 e^{-s\sqrt{k_1^2+k_2^2}} + c_2 e^{s\sqrt{k_1^2+k_2^2}} + \frac{k_1}{k_1^2 + k_2^2} = \frac{1}{k_1}, \tag{27}$$

$$\frac{c_1 k_2}{\sqrt{k_1^2+k_2^2}} e^{-s\sqrt{k_1^2+k_2^2}} - \frac{c_2 k_2}{\sqrt{k_1^2+k_2^2}} e^{s\sqrt{k_1^2+k_2^2}} - \frac{k_1 k_2 s}{k_1^2+k_2^2} = 0, \tag{28}$$

where $c_1, c_2 \in \mathbb{R}$.

Proof. By assumption we have

$$g(\alpha, \alpha) = -r^2$$

for every $s \in I \subset \mathbb{R}$.

Differentiation in s gives,

$$g(T, \alpha) = 0. \tag{29}$$

By a new differentiation, we find that

$$g(N, \alpha) = -\frac{1}{k_1}. \tag{30}$$

Then one more differentiation in s gives

$$g(B, \alpha) = 0, \tag{31}$$

By using equations (29), (30) and (31) in the equation (10), we find the equations (26), (27) and (28). Conversely, we assume that the equations (26), (27) and (28) holds for each $s \in I \subset \mathbb{R}$, then from the equation (10) we find the position vector of the curve $\alpha = \frac{1}{k_1} N$ which satisfy the equation $g(\alpha, \alpha) = -r^2$, which means that the curve lies on the pseudohyperbolical space $\mathbb{H}_0^2$. □

Under the light of the Theorem 4.1, we give the following corollary.

Corollary 4.2. *There are no spacelike W-curve with the timelike principal normal N with the curvature $k_1(s) \neq 0$ and $k_2(s) \neq 0$ whose image lies on Lorentzian sphere $\mathbb{S}_1^2$ in $\mathbb{E}_1^3$.*

Theorem 4.3. *Let $\alpha(s)$ be a unit speed spacelike W-curve with spacelike principal normal N in $\mathbb{E}_1^3$ with the curvatures $k_1(s) \neq 0$ and $k_2(s) \neq 0$ and $k_1^2 = k_2^2$. The image of the curve lies on a Lorentzian sphere $\mathbb{S}_1^2$ if and only if for each $s \in I \subset \mathbb{R}$ the curvatures satisfy the following equality:*

$$\frac{-k_1^2 s^3}{6} + \frac{c_1 k_1 s^2}{2} + (c_2 k_1 + 1)s = 0, \tag{32}$$

$$\frac{-k_1 s^2}{2} + c_1 s + c_2 = -\frac{1}{k_1}, \tag{33}$$

$$\frac{k_1 k_2 s^3}{6} - \frac{c_1 k_2 s^2}{2} - c_2 k_2 s = 0, \tag{34}$$

where $c_1, c_2 \in \mathbb{R}$.

Proof. By assumption we have

$$g(\alpha,\alpha) = r^2$$

for every $s \in I \subset \mathbb{R}$.

Differentiation in s gives,

$$g(T,\alpha) = 0. \tag{35}$$

By a new differentiation, we find that

$$g(N,\alpha) = -\frac{1}{k_1}. \tag{36}$$

Then one more differentiation in s gives

$$g(B,\alpha) = 0. \tag{37}$$

By using equations (35), (36) and (37) in the equation (15), we find the equations (32), (33) and (34). Conversely, we assume that the equations (32), (33) and (34) holds for each $s \in I \subset \mathbb{R}$, then from the equation (10) we find the position vector of the curve $\alpha = -\frac{1}{k_1}N$ which satisfy the equation $g(\alpha,\alpha) = r^2$. which means that the curve lies on the Lorentzian sphere $\mathbb{S}_1^2$. □

Theorem 4.4. *Let $\alpha(s)$ be a unit speed spacelike W-curve with spacelike principal normal N in $\mathbb{E}_1^3$ with the curvatures $k_1(s) \neq 0$ and $k_2(s) \neq 0$ and $k_1^2 > k_2^2$. The image of the curve lies on a Lorentzian sphere $\mathbb{S}_1^2$ if and only if for each $s \in I \subset \mathbb{R}$ the curvatures satisfy the following equality:*

$$s + \frac{c_1 k_1}{\sqrt{k_1^2 - k_2^2}}\sin(s\sqrt{k_1^2 - k_2^2}) - \frac{c_2 k_1}{\sqrt{k_1^2 - k_2^2}}\cos(s\sqrt{k_1^2 - k_2^2}) - \frac{k_1^2 s}{k_1^2 - k_2^2} = 0, \tag{38}$$

$$c_1\cos(s\sqrt{k_1^2 - k_2^2}) + c_2\sin(s\sqrt{k_1^2 - k_2^2}) - \frac{k_1}{k_1^2 - k_2^2} = \frac{-1}{k_1}, \tag{39}$$

$$\frac{-c_1 k_2}{\sqrt{k_1^2 - k_2^2}}\sin(s\sqrt{k_1^2 - k_2^2}) + \frac{c_2 k_2}{\sqrt{k_1^2 - k_2^2}}\cos(s\sqrt{k_1^2 - k_2^2}) + \frac{k_1 k_2 s}{k_1^2 - k_2^2} = 0, \tag{40}$$

where $c_1, c_2 \in \mathbb{R}$.

Theorem 4.5. *Let $\alpha(s)$ be a unit speed spacelike W-curve with spacelike principal normal N in $\mathbb{E}_1^3$ with the curvatures $k_1(s) \neq 0$ and $k_2(s) \neq 0$ and $k_1^2 < k_2^2$. The image of the curve lies on a Lorentzian sphere $\mathbb{S}_1^2$ if and only if for each $s \in I \subset \mathbb{R}$ the curvatures satisfy the following equality:*

$$s - \frac{c_1 k_1}{\sqrt{k_2^2 - k_1^2}}e^{-s\sqrt{k_2^2 - k_1^2}} + \frac{c_2 k_1}{\sqrt{k_2^2 - k_1^2}}e^{s\sqrt{k_2^2 - k_1^2}} + \frac{k_1^2 s}{k_2^2 - k_1^2} = 0, \tag{41}$$

$$c_1 e^{-s\sqrt{k_2^2 - k_1^2}} + c_2 e^{s\sqrt{k_2^2 - k_1^2}} + \frac{k_1}{k_2^2 - k_1^2} = \frac{-1}{k_1}, \tag{42}$$

$$\frac{c_1 k_2}{\sqrt{k_2^2-k_1^2}} e^{-s\sqrt{k_2^2-k_1^2}} - \frac{c_2 k_2}{\sqrt{k_2^2-k_1^2}} e^{s\sqrt{k_2^2-k_1^2}} - \frac{k_1 k_2 s}{k_2^2-k_1^2} = 0, \tag{43}$$

where $c_1, c_2 \in \mathbb{R}$.

The proof of the Theorems 4.4 and 4.5 is analogues to the proof of the Theorem 4.3.

Corollary 4.6. *There are no spacelike W-curve with the spacelike principal normal N with the curvature $k_1(s) \neq 0$ and $k_2(s) \neq 0$ whose image lies on pseudohyperbolic space $\mathbb{H}_0^2$ in $\mathbb{E}_1^3$.*

Theorem 4.7. *There are no spacelike W-curve with the null principal normal N with the curvature $k_1(s) \neq 0$ and $k_2(s) \neq 0$ whose image lies on pseudohyperbolic space $\mathbb{H}_0^2$ in $\mathbb{E}_1^3$.*

Proof. We assume that the spacelike W-curve with the null principal normal N with the curvature $k_1(s) = 1$ and $k_2(s) \neq 0$ lies on pseudohyperbolic space $\mathbb{H}_0^2$ in $\mathbb{E}_1^3$. Then the position vector of the curve satisfy

$$g(\alpha, \alpha) = -r^2$$

for every $s \in I \subset \mathbb{R}$.

Differentiation in s and using the corresponding Frenet equations (2), gives,

$$g(T, \alpha) = 0. \tag{44}$$

By a new differentiation, we find that

$$g(N, \alpha) = -1. \tag{45}$$

Then one more differentiation in s gives

$$k_2 g(N, \alpha) = 0, \tag{46}$$

from the last equation we find $k_2 = 0$, which is a contradiction. Thus there are no spacelike W-curve with the null principal normal N with the curvature $k_1(s) \neq 0$ and $k_2(s) \neq 0$ pseudohyperbolic space $\mathbb{H}_0^2$ in $\mathbb{E}_1^3$. □

Remark. In [10], The authors show that if α is a spacelike curve with the null principal normal, then α lies on a pseudohyperbolic space $\mathbb{H}_0^2$ if and only if α is a planar curve.

Corollary 4.8. *Let $\alpha(s)$ be a unit speed spacelike W-curve with spacelike or timelike principal normal N in $\mathbb{E}_1^3$ with the curvatures $k_1(s) \neq 0$ and $k_2(s) \neq 0$. The image of the curve lies on a pseudohyperbolic space $\mathbb{H}_0^2$ or Lorentzian sphere $\mathbb{S}_1^2$ of radius $r \in \mathbb{R}^+$ and with the center at origin, if and only if α is a normal curve, i.e., curves with position vector always lying in its normal plane.*

Corollary 4.9. *Let $\alpha(s)$ be a unit speed spacelike W-curve with spacelike or timelike principal normal N in $\mathbb{E}_1^3$ with the curvatures $k_1(s) \neq 0$ and $k_2(s) \neq 0$. If α is a pseudohyperbolical curves or a Lorentzian spherical curves then the radius of $\mathbb{H}_0^2$ or $\mathbb{S}_1^2$ is $r = \dfrac{1}{k_1}$.*

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Kazim İlarslan
Kirikkale University
Department of Mathematics
Faculty of Sciences and Arts
Kirikkale-Turkey
E-mail address: kilarslan@yahoo.com

Özgür Boyacioğlu
Kocatepe University
Department of Mathematics
Faculty of Sciences and Arts
A.N. Sezer Campus
Afyonkarahisar - Turkey
E-mail address: bozgur@aku.edu.tr