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ONE-PARAMETER GROUPS OF BOEHMIANS

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ABSTRACT. The space of periodic Boehmians with Δ–convergence is a complete topological algebra which is not locally convex. A family of Boehmians $\{T_\lambda\}$ such that T_0 is the identity and $T_{\lambda_1+\lambda_2} = T_{\lambda_1} * T_{\lambda_2}$ for all real numbers λ_1 and λ_2 is called a one-parameter group of Boehmians. We show that if $\{T_\lambda\}$ is strongly continuous at zero, then $\{T_\lambda\}$ has an exponential representation. We also obtain some results concerning the infinitesimal generator for $\{T_\lambda\}$.

1. Introduction

The space of Boehmians was first introduced by J. Mikusiński and P. Mikusiński [8] as a generalization of Schwartz distributions as well as T. K. Boehme's regular operators. Since the early 1980's there have been many articles by several authors published about spaces of Boehmians; see for example [1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16].

It is well known that the exponential function $f(t) = e^{\alpha t}$ is the most general function which is continuous at zero and satisfies $f(0) = 1$ and $f(t_1 + t_2) = f(t_1)f(t_2)$ for all real numbers t_1 and t_2.

In this note we will investigate one-parameter groups of periodic Boehmians. That is, we will study families $\{T_\lambda\}$ $(\lambda \in \mathbb{R})$ of Boehmians which satisfy $T_0 = \delta$ and $T_{\lambda_1+\lambda_2} = T_{\lambda_1} * T_{\lambda_2}$ for all real λ_1 and λ_2, where δ is the identity and "$*$" is convolution.

This paper is organized as follows. In Section 2, we give a brief construction of the space of periodic Boehmians $\beta(\Gamma)$. We also present some known results that will be needed in the sequel. In Section 3, we study one-parameter groups of Boehmians. We show that a one-parameter group of Boehmians which is strongly continuous at zero has an exponential representation. We will give sufficient conditions for an element $F \in \beta(\Gamma)$ to be the infinitesimal generator of a one-parameter group. We also present some necessary conditions. In Section 4, we show that a necessary and sufficient condition for a Boehmian to be an infinitesimal generator for a one-parameter group is that it is a logarithm.

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2. Preliminaries

Let $\mathcal{L}(\Gamma)$ denote the space of all Lebesgue integrable functions on the unit circle Γ. Let $C(\Gamma)$ denote the subspace of $\mathcal{L}(\Gamma)$ consisting of all complex-valued continuous functions, and $C^{\mathbb{N}}(\Gamma)$ denote the collection of sequences in $C(\Gamma)$. We make no distinction between a function on Γ and a $2\pi-$periodic function on $\mathbb{R}$.

A sequence $\{\varphi_n\}_{n=1}^{\infty}$ of nonnegative functions in $C(\Gamma)$ is called a *delta sequence* if

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_n(t)dt = 1$, for all $n \in \mathbb{N}$;

(ii) $\varphi_n(t) = 0$ for $0 < \epsilon_n < |t| < \pi$, where $\epsilon_n \to 0$.

The collection of delta sequences will be denoted by Δ.

Let $f, g \in C(\Gamma)$. The *convolution* of the two functions f and g is given by

$$(f * g)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-\sigma)g(\sigma)d\sigma. \tag{2.1}$$

Let $\mathbb{A} = \{(f_n, \varphi_n) \in C^{\mathbb{N}}(\Gamma) \times \Delta : f_n * \varphi_k = f_k * \varphi_n, \text{ for all } n, k \in \mathbb{N}\}$. $(f_n, \varphi_n) \sim (g_n, \delta_n)$ if $f_n * \delta_k = g_k * \varphi_n$, for all $n, k \in \mathbb{N}$. "$\sim$" is an equivalence relation on $\mathbb{A}$. The collection of equivalence classes will be denoted by $\beta(\Gamma)$. Elements of $\beta(\Gamma)$ are called *Boehmians*, and a typical element of $\beta(\Gamma)$ is written as $\left[\frac{f_n}{\varphi_n}\right]$.

Addition, multiplication, and scalar multiplication are defined in the natural way, and $\beta(\Gamma)$ with these operations is an algebra with identity $\delta = \left[\frac{\varphi_n}{\varphi_n}\right]$.

$$\left[\frac{f_n}{\varphi_n}\right] + \left[\frac{g_n}{\delta_n}\right] = \left[\frac{f_n * \delta_n + g_n * \varphi_n}{\varphi_n * \delta_n}\right], \tag{2.2}$$

$$\left[\frac{f_n}{\varphi_n}\right] * \left[\frac{g_n}{\delta_n}\right] = \left[\frac{f_n * g_n}{\varphi_n * \delta_n}\right], \tag{2.3}$$

$$\alpha\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{\alpha f_n}{\varphi_n}\right], \text{ where } \alpha \in \mathbb{C}. \tag{2.4}$$

$\mathcal{L}(\Gamma)$ can be identified with a subspace of $\beta(\Gamma)$ by

$$f \leftrightarrow \left[\frac{f * \varphi_n}{\varphi_n}\right].$$

Similarly, $D'(\Gamma)$, the space of Schwartz distributions [18] on the unit circle, can be identified with a subspace of $\beta(\Gamma)$.

For $f \in C(\Gamma)$, the *kth Fourier coefficient* is given by

$$\widehat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt, \quad k \in \mathbb{Z}. \tag{2.5}$$

Definition 2.1. For $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\Gamma)$, the *kth Fourier coefficient* is defined by

$$\widehat{F}(k) = \lim_{n\to\infty} \widehat{f}_n(k). \tag{2.6}$$

The limit in the above definition is independent of the representative of F.

Throughout the sequel, ω will denote a real-valued even function defined on the integers $\mathbb{Z}$ such that $0 = \omega(0) \leq \omega(n+m) \leq \omega(n) + \omega(m)$ for all $n, m \in \mathbb{Z}$ and $\sum_{n=1}^{\infty} \frac{\omega(n)}{n^2} < \infty$.

Also throughout the sequel, $\{s_n\}_{n=1}^{\infty}$ will denote a sequence in $\mathbb{Z}^+$, the set of positive integers, satisfying the following conditions.

(i) There exists a sequence of positive integers $\{t_n\}_{n=1}^{\infty}$ disjoint from $\{s_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \frac{1}{t_n} < \infty$;

(ii) $\mathbb{Z}^+ = \{s_n\} \cup \{t_n\}$.

Theorem 2.2 (see [13]). *If $\{\xi_n\}_{n=-\infty}^{\infty}$ is a sequence of complex numbers such that $\xi_{\pm s_n} = O(e^{\omega(s_n)})$ as $n \to \infty$, then $\{\xi_n\}_{n=-\infty}^{\infty}$ is the sequence of Fourier coefficients for some Boehmian.*

The next theorem is a stronger version of Theorem 3.5 in [15]. Since the proof is similar to that of Theorem 3.5, it is omitted.

Theorem 2.3. *Let $\theta(t)$ be a monotone increasing function such that*

$$\int_1^{\infty} \frac{\theta(t)}{t^2} dt = \infty.$$

Let $\{\lambda_n\}_{n=1}^{\infty}$ be an increasing sequence of positive integers such that

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D > 0.$$

Then, for each $F \in \beta(\Gamma)$, $\liminf_{n\to\infty} e^{-\theta(\lambda_n)}|\widehat{F}(\lambda_n)| = 0$.

Definition 2.4. A sequence $\{F_n\}_{n=1}^{\infty}$ in $\beta(\Gamma)$ is said to be *Δ-convergent* to $F \in \beta(\Gamma)$, denoted by $\Delta\text{-}\lim_{n\to\infty} F_n = F$, if there exists a delta sequence $\{\varphi_n\}_{n=1}^{\infty}$ such that $(F_n - F) * \varphi_n \in C(\Gamma)$ for all n, and $(F_n - F) * \varphi_n \to 0$ uniformly on Γ as $n \to \infty$.

The space $\beta(\Gamma)$ with Δ-convergence is an F-space [9]. That is, it is a complete topological vector space in which the topology is induced by an invariant metric.

The dual space $\beta'(\Gamma)$ of $\beta(\Gamma)$ is nontrivial. Indeed, $\beta'(\Gamma)$ separates points on $\beta(\Gamma)$. However, $\beta(\Gamma)$ is not locally convex [4].

Theorem 2.5 (see [14]). *For each $\Lambda \in \beta'(\Gamma)$, there exists a unique trigonometric polynomial $p(t) = \sum_{n=-m}^{m} \alpha_n e^{int}$ (for some $m \in \mathbb{N}$) such that*

$$\Lambda(F) = \sum_{n=-m}^{m} \alpha_n \widehat{F}(n), \text{ for all } F \in \beta(\Gamma). \tag{2.7}$$

Conversely, any $p(t) = \sum_{n=-m}^{m} \alpha_n e^{int}$ *defines a bounded linear functional on* $\beta(\Gamma)$ *via* (2.7).

As is the case with most spaces of generalized functions, the Fourier series for each $F \in \beta(\Gamma)$ converges to F. That is,

$$F = \Delta - \lim_{n\to\infty} \sum_{k=-n}^{n} \widehat{F}(k)e^{ikt}, \text{ for each } F \in \beta(\Gamma). \tag{2.8}$$

Theorem 2.6. *Suppose that* $F_n, F \in \beta(\Gamma)$ *for* $n \in \mathbb{N}$ *such that*

$$\Delta\text{-} \lim_{n\to\infty} F_n = F.$$

Then $\lim_{n\to\infty} \widehat{F}_n(k) = \widehat{F}(k)$ *for all* $k \in \mathbb{Z}$.

The next theorem, which is a stronger version of Theorem 3.2 in [13], gives a partial converse to the previous theorem.

Theorem 2.7. *Suppose that* $\{F_n\}_{n=1}^{\infty}$ *is a sequence of Boehmians such that*

(i) *there exists a Boehmian* G *with* $\left|\widehat{F}_n(\pm s_p)\right| \le \left|\widehat{G}(\pm s_p)\right|$ *for all* $p, n \in \mathbb{N}$;

(ii) *for each* k, $\lim_{n\to\infty} \widehat{F}_n(k) = \xi_k$.

Then $\{\xi_k\}_{k=-\infty}^{\infty}$ *is the Fourier coefficients of a Boehmian* F. *Moreover,*

$$\Delta\text{-} \lim_{n\to\infty} F_n = F.$$

Proof. By Theorem 2.2 there exist $H_1, H_2 \in \beta(\Gamma)$ having Fourier coefficients as follows: $\widehat{H}_1(\pm s_p) = 1 \quad (p \in \mathbb{N})$ and zero otherwise, and

$$\widehat{H}_2(\pm t_p) = \sup\left\{\left|\widehat{F}_n(\pm t_p)\right| : n \in \mathbb{N}\right\} \quad (p \in \mathbb{N})$$

and zero otherwise.

Let $H = G * H_1 + H_2$. Then $H \in \beta(\Gamma)$. Moreover, $\left|\widehat{F}_n(k)\right| \le \left|\widehat{H}(k)\right|$ for all $n \in \mathbb{N}$ and all $k \in \mathbb{Z}$.

To finish the proof, apply Theorem 3.2 in [13]. □

3. The main results

Let $\{T_\lambda\}, \lambda \in \mathbb{R}$, be a family of Boehmians such that

(i) $T_0 = \delta$;

(ii) $T_{\lambda_1+\lambda_2} = T_{\lambda_1} * T_{\lambda_2}$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$.

Then, $\{T_\lambda\}$ is called a *one-parameter group*.

Remark 3.1. The space $\beta(\Gamma)$ has zero divisors. However (i) and (ii) show that this is not true for a one-parameter group $\{T_\lambda\}$. Indeed, $\{T_\lambda\}$ is a group under convolution.

Example 3.2. Let $F \in \beta(\Gamma)$ such that $\widehat{F}(\pm s_n) = O(\sqrt{s_n})$ as $n \to \infty$. For each $\lambda \in \mathbb{R}$, define $T_\lambda = \Delta\text{-}\lim_{n\to\infty} \sum_{k=0}^{n} \frac{\lambda^k F^k}{k!}$, where $F^0 = \delta$ and $F^k = F*F*\cdots*F$ (k times). By Theorems 2.2 and 2.7, the above sequence converges for each λ.

Now, the family $\{T_\lambda\}$ of Boehmians clearly satisfies property (i). Also, since $\widehat{T}_{\lambda_1+\lambda_2}(k) = e^{(\lambda_1+\lambda_2)\widehat{F}(k)} = e^{\lambda_1\widehat{F}(k)}e^{\lambda_2\widehat{F}(k)} = \widehat{T}_{\lambda_1}(k)\widehat{T}_{\lambda_2}(k)$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $k \in \mathbb{Z}$, the family $\{T_\lambda\}$ satisfies property (ii). Therefore, $\{T_\lambda\}$ is a one-parameter group of Boehmians.

We will require some type of continuity condition in order to show that $\{T_\lambda\}$ has an exponential representation.

Definition 3.3. A function $Q : \mathbb{R} \to \beta(\Gamma)$ is *continuous at* λ_0 provided $\Delta\text{-}\lim_{\lambda\to\lambda_0} Q(\lambda) = Q(\lambda_0)$.

If $\{T_\lambda\}$ is a one-parameter group which is continuous at zero, then there exists a unique function $\tau : \mathbb{Z} \to \mathbb{C}$ such that $\widehat{T}_\lambda(k) = e^{\lambda\tau(k)}$, for all $\lambda \in \mathbb{R}$ and all $k \in \mathbb{Z}$.

Thus,

$$T_\lambda = \Delta - \lim_{n\to\infty} \sum_{k=-n}^{n} e^{\lambda\tau(k)} e^{ikt} \text{ for all } \lambda \in \mathbb{R}. \tag{3.1}$$

Definition 3.4. A function $Q : \mathbb{R} \to \beta(\Gamma)$ is *strongly continuous at* λ_0 if

(i) for each $k \in \mathbb{Z}$, $\widehat{Q}_\lambda(k) \to \widehat{Q}_{\lambda_0}(k)$ as $\lambda \to \lambda_0$; and

(ii) there exist $M > 0$ and $\eta > 0$ such that

$$|\widehat{Q}_\lambda(\pm s_n) - \widehat{Q}_{\lambda_0}(\pm s_n)| \le M|\lambda - \lambda_0|\,\omega(s_n), \tag{3.2}$$

for all $n \in \mathbb{N}$ and $|\lambda - \lambda_0| < \eta$.

Remark 3.5. If Q is strongly continuous at λ_0, then Q is continuous at λ_0.

Suppose that $\{T_\lambda\}$ is a one-parameter group. Let

$$F_\epsilon = \frac{T_\epsilon - \delta}{\epsilon}, \text{ for } \epsilon \neq 0. \tag{3.3}$$

If there exists a Boehmian F such that $F = \Delta\text{-}\lim_{\epsilon\to 0} F_\epsilon$, then F is called the *infinitesimal generator* for $\{T_\lambda\}$.

Theorem 3.6. *Suppose that $\{T_\lambda\}$ is a one-parameter group that is strongly continuous at zero. Then, $T_\lambda = e^{\lambda F}$, for all $\lambda \in \mathbb{R}$, where F is the infinitesimal generator for $\{T_\lambda\}$, i.e.,*

$$T_\lambda = \Delta - \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\lambda^k F^k}{k!} \textit{ for all } \lambda \in \mathbb{R}, \tag{3.4}$$

where $F^k = \underbrace{F * F * \cdots * F}_{k\ times}$.

Proof. Since $\{T_\lambda\}$ is strongly continuous at zero, there exist $M > 0$ and $\eta > 0$ such that

$$\left| \frac{\widehat{T}_\lambda(\pm s_k) - 1}{\lambda} \right| \leq M\omega(s_k), \tag{3.5}$$

for all $k \in \mathbb{N}$ and $0 < |\lambda| < \eta$. Also, for each $k \in \mathbb{Z}$, the mapping $\lambda \to \widehat{T}_\lambda(k)$ is continuous at zero, and $\lambda_1 + \lambda_2 \to \widehat{T}_{\lambda_1}(k) \cdot \widehat{T}_{\lambda_2}(k)$. Thus, for all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$,

$$\widehat{T}_\lambda(k) = e^{\lambda \alpha_k}, \tag{3.6}$$

where $\alpha_k = \frac{d}{d\lambda}\widehat{T}_\lambda(k)|_{\lambda=0}$. Thus,

$$\frac{\widehat{T}_\lambda(k) - 1}{\lambda} = \frac{e^{\lambda\alpha_k} - 1}{\lambda} \to \alpha_k \text{ as } \lambda \to 0. \tag{3.7}$$

Now,

$$F = \Delta - \lim_{\lambda \to 0} \frac{T_\lambda - \delta}{\lambda}.$$

Equations (3.5) and (3.7) imply this limit exists. Hence

$$\widehat{F}(k) = \lim_{\lambda \to 0} \frac{\widehat{T}_\lambda(k) - 1}{\lambda}, \tag{3.8}$$

for all $k \in \mathbb{Z}$.

By (3.7) and (3.8) we see that

$$\widehat{F}(k) = \alpha_k, \tag{3.9}$$

for all $k \in \mathbb{Z}$.

Thus, (3.6) and (3.9) yield

$$\widehat{T}_\lambda(k) = e^{\lambda \widehat{F}(k)}, \tag{3.10}$$

for all $k \in \mathbb{Z}$.

By using (3.5) and (3.8), we obtain

$$\sum_{k=0}^{n} \frac{|\lambda \widehat{F}(s_p)|^k}{k!} \leq \sum_{k=0}^{n} \frac{|\lambda M \omega(s_p)|^k}{k!} \leq e^{M|\lambda||\omega(s_p)|},$$

for all $n, p \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, and

$$\sum_{k=0}^{n} \frac{(\lambda \widehat{F}(p))^k}{k!} \to e^{\lambda \widehat{F}(p)} \text{ as } n \to \infty,$$

for all $p \in \mathbb{N}$ and $\lambda \in \mathbb{R}$.

Thus, for each $\lambda \in \mathbb{R}$, the sequence $\left\{\sum_{k=0}^{n} \frac{\lambda^k F^k}{k!}\right\}$ is Δ-convergent, and

$$\left(\sum_{0}^{\infty} \frac{\lambda^k F^k}{k!}\right)^{\wedge}(n) = e^{\lambda \widehat{F}(n)},$$

for all $n \in \mathbb{Z}$.

This and (3.10) give

$$T_\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k F^k}{k!}, \text{ for all } \lambda \in \mathbb{R}.$$

□

Theorem 3.6 suggests that it may be of interest to investigate infinitesimal generators in more detail.

Theorem 3.7. *If F is the infinitesimal generator for the one-parameter group $\{T_\lambda\}$, then $\widehat{F}(k) = \tau(k)$, for all $k \in \mathbb{Z}$.*

Proof. Since F is the infinitesimal generator for $\{T_\lambda\}$, $\Delta\text{-}\lim_{\lambda \to 0} \frac{T_\lambda - T_0}{\lambda}$ exists. Also, $T_\lambda = T_0 + \lambda \frac{T_\lambda - T_0}{\lambda}$, for $\lambda \neq 0$. Thus $\Delta\text{-}\lim_{\lambda \to 0} T_\lambda = T_0$ and $\{T_\lambda\}$ is continuous at $\lambda = 0$. Therefore, $T_\lambda = \Delta\text{-}\lim_{n \to \infty} \sum_{k=-n}^{n} e^{\lambda \tau(k)} e^{ikt}, \lambda \in \mathbb{R}$. This gives $\widehat{T_\lambda}(k) = e^{\lambda \tau(k)}$, for all $k \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$. Now, $\Delta\text{-}\lim_{\lambda \to 0} \frac{T_\lambda - \delta}{\lambda} = F$ implies that $\lim_{\lambda \to 0} \frac{\widehat{T_\lambda}(k) - 1}{\lambda} = \widehat{F}(k), k \in \mathbb{Z}$. However, $\lim_{\lambda \to 0} \frac{\widehat{T_\lambda}(k) - 1}{\lambda} = \lim_{\lambda \to 0} \frac{e^{\lambda \tau(k)} - 1}{\lambda} = \tau(k),\ k \in \mathbb{Z}$. Therefore, $\widehat{F}(k) = \tau(k)$, for all $k \in \mathbb{Z}$. □

Theorem 3.8 (Sufficient conditions). *If $F \in \beta(\Gamma)$ such that*

$$\sup\left\{\left|\operatorname{Re}\left[\frac{\widehat{F}(\pm s_n)}{\omega(s_n)}\right]\right| : n \in \mathbb{N}\right\} < \infty, \tag{3.11}$$

then F is the infinitesimal generator for the one-parameter group

$$T_\lambda = \Delta - \lim_{n \to \infty} \sum_{k=-n}^{n} e^{\lambda \widehat{F}(k)} e^{ikt} \text{ for all } \lambda \in \mathbb{R}. \tag{3.12}$$

Proof. By the hypothesis there exists an $M > 0$ such that

$$\left|\operatorname{Re} \widehat{F}(\pm s_n)\right| \leq M\omega(s_n), n \in \mathbb{N}. \tag{3.13}$$

Thus, $|e^{\lambda \widehat{F}(\pm s_n)}| = e^{\lambda \operatorname{Re} \widehat{F}(\pm s_n)} \leq e^{|\lambda||\operatorname{Re} \widehat{F}(\pm s_n)|} \leq e^{M|\lambda|\omega(s_n)}, \lambda \in \mathbb{R}$. Therefore, for each $\lambda \in \mathbb{R}, \{e^{\lambda \widehat{F}(k)}\}_{k=-\infty}^{\infty}$ is the Fourier coefficients for some Boehmian. Define a mapping from $\mathbb{R}$ into $\beta(\Gamma)$ by

$$T_\lambda = \Delta\text{-}\lim_{n \to \infty} \sum_{k=-n}^{n} e^{\lambda \widehat{F}(k)} e^{ikt}, \lambda \in \mathbb{R}. \tag{3.14}$$

Now, $\widehat{T}_{\lambda_1+\lambda_2}(k) = e^{(\lambda_1+\lambda_2)\widehat{F}(k)} = e^{\lambda_1 \widehat{F}(k)} e^{\lambda_2 \widehat{F}(k)} = \widehat{T}_{\lambda_1}(k)\widehat{T}_{\lambda_2}(k)$ for $\lambda_1, \lambda_2 \in \mathbb{R}$ and $k \in \mathbb{Z}$. Thus, $T_{\lambda_1+\lambda_2} = T_{\lambda_1} * T_{\lambda_2}$, for all $\lambda_1, \lambda_2 \in \mathbb{R}$. Also, $T_0 = \delta$. Therefore, $\{T_\lambda\}$ is a one-parameter group.

Now, by using the Mean Value Theorem separately on the real and imaginary parts of $\frac{e^{\lambda \widehat{F}(\pm s_n)}-1}{\lambda}$, there exist constants $A > 0$ and $\eta > 0$ such that $\left|\frac{\widehat{T}_\lambda(\pm s_n)-1}{\lambda}\right| = \left|\frac{e^{\lambda \widehat{F}(\pm s_n)}-1}{\lambda}\right| \le A|\widehat{F}(\pm s_n)|e^{M\eta\omega(s_n)}$, for all $n \in \mathbb{N}$ and $0 < |\lambda| < \eta$. Also, for each k, $\frac{\widehat{T}_\lambda(k)-1}{\lambda} = \frac{e^{\lambda \widehat{F}(k)}-1}{\lambda} \to \widehat{F}(k)$ as $\lambda \to 0$. So, by Theorems 2.2 and 2.7, $\Delta\text{-}\lim_{\lambda\to 0} \frac{T_\lambda - \delta}{\lambda} = F$. That is, F is the infinitesimal generator for $\{T_\lambda\}$. □

Theorem 3.9 (Necessary conditions). *Let F be the infinitesimal generator for the one-parameter group $\{T_\lambda\}$. Suppose that $\theta(t)$ is a monotone increasing function such that $\int_1^\infty \frac{\theta(t)}{t^2}dt = \infty$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of positive integers such that* Re $\widehat{F}(\lambda_n) > 0$ *(or* Re $\widehat{F}(\lambda_n) < 0$*) for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{n}{\lambda_n} = D > 0$. Then,*

$$\liminf_{n\to\infty} \left| \mathrm{Re}\left[\frac{\widehat{F}(\lambda_n)}{\theta(\lambda_n)}\right]\right| = 0. \tag{3.15}$$

Proof. Suppose that there exist $M > 0$ and $n_0 \in \mathbb{N}$ such that $\left|\mathrm{Re}\ \widehat{F}(\lambda_n)\right| \ge M\theta(\lambda_n)$, $n \ge n_0$. Without loss of generality assume that Re $\widehat{F}(\lambda_n) > 0$ (for if Re $\widehat{F}(\lambda_n) < 0$, consider T_{-1}). Now, $|\widehat{T}_1(\lambda_n)| = |e^{\widehat{F}(\lambda_n)}| = e^{\mathrm{Re}\widehat{F}(\lambda_n)} \ge e^{M\theta(\lambda_n)}$, for $n \ge n_0$. Thus, by Theorem 2.3, the sequence $\{\widehat{T}_1(k)\}_{k=-\infty}^\infty$ is not the Fourier coefficients of any Boehmian. Therefore, the proof is complete. □

By using Theorems 3.6 and 3.8, it can be shown that every element of $\mathcal{L}(\Gamma)$ as well as every periodic measure is an infinitesimal generator for a one-parameter group having a representation of the form (3.4).

Example 3.10. (i) - (iii) are examples of infinitesimal generators, while (iv) is an example of a Boehmian that is not an infinitesimal generator.

(i) F such that $\widehat{F}(k) = \log|k|$, for $|k| \ge 1$.
(ii) F such that $\widehat{F}(k) = i\exp(\sqrt{|k|})$, $k \in \mathbb{Z}$.
(iii) $F = \sum_{n=1}^\infty \alpha_n \exp(i2^n t)$, where $\{\alpha_n\}_{n=1}^\infty$ is any sequence of complex numbers.
(iv) F such that $\widehat{F}(k) = \frac{|k|}{\log|k|}$, for $|k| \ge 2$.

4. Logarithms

In the field of Mikusiński operators [7], the notion of a logarithm is important in the theory as well as in the applications of differential equations. The form of the general solution of an n^{th} order linear operator-valued differential equation depends on determining whether or not the roots of the characteristic equation are logarithms. No general criterion is known to determine whether or not a given operator is a logarithm.

In this section, we will show that a Boehmian F is a logarithm if and only if F is the infinitesimal generator for some group.

Definition 4.1. Let $Q : \mathbb{R} \to \beta(\Gamma)$. Then $Q'(\lambda_0)$ is defined as:

$$Q'(\lambda_0) = \Delta - \lim_{\lambda \to \lambda_0} \frac{Q(\lambda) - Q(\lambda_0)}{\lambda - \lambda_0} \text{ (provided the limit exists).} \tag{4.1}$$

Definition 4.2. Let $F \in \beta(\Gamma)$. Suppose that there exists a Boehmian-valued function $Q : \mathbb{R} \to \beta(\Gamma)$ such that $Q'(\lambda) = F * Q(\lambda)$ for all $\lambda \in \mathbb{R}$ and $Q(0) = \delta$, then F is called a *logarithm.*

Theorem 4.3. *Let $F \in \beta(\Gamma)$. Then F is an infinitesimal generator if and only if F is a logarithm.*

Proof. Let F be the infinitesimal generator for $\{T_\lambda\}$. Then, $\Delta\text{-}\lim_{\lambda\to 0} \frac{T_\lambda - \delta}{\lambda} = F$. That is, $T_0{}' = F$. Now, $T_{\lambda_0}{}' = \Delta\text{-}\lim_{\lambda\to\lambda_0} \frac{T_\lambda - T_{\lambda_0}}{\lambda-\lambda_0} = \Delta\text{-}\lim_{\lambda\to\lambda_0} T_{\lambda_0} * \left[\frac{T_{\lambda-\lambda_0} - T_0}{\lambda-\lambda_0}\right] = T_{\lambda_0} * \Delta\text{-}\lim_{\lambda\to\lambda_0} \frac{T_{\lambda-\lambda_0} - T_0}{\lambda-\lambda_0} = T_{\lambda_0} * T_0{}' = F * T_{\lambda_0}, \quad \lambda_0 \in \mathbb{R}$.

For the other direction, let $Q : \mathbb{R} \to \beta(\Gamma)$ such that

$$Q'(\lambda) = F * Q(\lambda), \ \lambda \in \mathbb{R} \tag{4.2}$$

and $Q(0) = \delta$. Since the mapping $F \to \widehat{F}(k)$ is continuous for each $k \in \mathbb{Z}$, we obtain

$$\frac{d}{d\lambda}\widehat{Q}_\lambda(k) = \widehat{F}(k)\widehat{Q}_\lambda(k), \ k \in \mathbb{Z} \text{ and } \lambda \in \mathbb{R}. \tag{4.3}$$

Thus, $\widehat{Q}_\lambda(k) = e^{\lambda \widehat{F}(k)}$. This implies that $Q_{\lambda_1+\lambda_2} = Q_{\lambda_1} * Q_{\lambda_2}$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$. Also, by (4.2), $\Delta\text{-}\lim_{\lambda\to 0} \frac{Q(\lambda)-\delta}{\lambda} = F$. Therefore, $\{Q_\lambda\}$ is a one-parameter group with infinitesimal generator F. This completes the proof. □

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