Bull. Korean Math. Soc. 44 (2007), No. 3, pp. 395–406

RICCI CURVATURE OF INTEGRAL SUBMANIFOLDS OF AN $\mathcal{S}$-SPACE FORM

Jeong-Sik Kim, Mohit Kumar Dwivedi, and Mukut Mani Tripathi

Abstract. Involving the Ricci curvature and the squared mean curvature, we obtain a basic inequality for an integral submanifold of an $\mathcal{S}$-space form. By polarization, we get a basic inequality for Ricci tensor also. Equality cases are also discussed. By giving a very simple proof we show that if an integral submanifold of maximum dimension of an $\mathcal{S}$-space form satisfies the equality case, then it must be minimal. These results are applied to get corresponding results for C-totally real submanifolds of a Sasakian space form and for totally real submanifolds of a complex space form.

1. Introduction

One of the most fundamental problems in submanifold theory is the following: Establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. In [7], B.-Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for a submanifold in a Riemannian space form with arbitrary codimension. In [8], he gave the corresponding version of this inequality for totally real submanifolds in a complex space form. We find corresponding results for C-totally real submanifolds of a Sasakian space form in [10], [11] and [12].

The concept of framed metric structure unifies the concepts of almost Hermitian and almost contact metric structures. In particular, an $\mathcal{S}$-structure generalizes Kaehler and Sasakian structure. In [1], D. Blair discusses principal toroidal bundles and generalizes the Hopf fibration to give a canonical example of an $\mathcal{S}$-manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry. An $\mathcal{S}$-manifold

Received July 30, 2005.

2000 *Mathematics Subject Classification.* Primary 53C40, 53C15, 53C25.

Key words and phrases. $\mathcal{S}$-space form, integral submanifold, C-totally real submanifold, totally real submanifold, Lagrangian submanifold, Ricci curvature, k-Ricci curvature, scalar curvature.

Jeong-Sik Kim would like to acknowledge financial support from Korea Science and Engineering Foundation Grant(R05-2004-000-11588). Mohit Kumar Dwivedi is grateful to University Grants Commission, New Delhi for financial support in the form of Junior Research Fellowship.

of constant f-sectional curvature c is called an $\mathcal{S}$-space form $\widetilde{M}(c)$ [5], which generalizes the complex space form and Sasakian space form.

Motivated by the result of Chen in [7], recently in [9], a general basic inequality involving the Ricci curvature and the squared mean curvature of a submanifold in any Riemannian manifold is established and its several applications are presented. Using this inequality, in the present paper, we find a basic inequality for integral submanifolds of an $\mathcal{S}$-space form $\widetilde{M}(c)$ and apply this to recover the already known inequalities for totally real submanifolds in complex space forms and C-totally real submanifolds in Sasakian space forms. The paper is organized as follows. In section 2, we recall a brief account of Ricci curvature, k-Ricci curvature, scalar curvature in a Riemannian manifold and basic formulas and definitions for a submanifold. Then, we recall the result of [9] giving a general basic inequality involving the Ricci curvature and the squared mean curvature of a submanifold in any Riemannian manifold. Section 3 presents a brief account of framed metric manifold leading to $\mathcal{S}$-space forms. In section 4, we give a very simple way to present a basic inequality for integral submanifolds of an $\mathcal{S}$-space form $\widetilde{M}(c)$. Then, the already known inequalities for totally real submanifolds in complex space forms and C-totally real submanifolds in Sasakian space forms become direct consequences. In section 5, we mainly prove that an integral submanifold of maximum dimension of an $\mathcal{S}$-space form $\widetilde{M}(c)$ satisfying the equality case becomes minimal. Then, we derive the same conclusion for Lagrangian submanifold of a complex space form and C-totally real submanifold of maximum dimension of a Sasakian space form.

2. Ricci curvature of submanifolds

Let M be an n-dimensional Riemannian manifold. Let $\{e_1, \ldots, e_k\}$, $2 \leq k \leq n$, be an orthonormal basis of a k-plane section Π_k of T_pM. If $k = n$ then $\Pi_n = T_pM$; and if $k = 2$ then Π_2 is a plane section of T_pM. For a fixed $i \in \{1, \ldots, k\}$, a *k-Ricci curvature* of Π_k at e_i, denoted $\mathrm{Ric}_{\Pi_k}(e_i)$, is defined by [7]

$$\mathrm{Ric}_{\Pi_k}(e_i) = \sum_{j \neq i}^{k} K_{ij}, \tag{1}$$

where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j. An *n-Ricci curvature* $\mathrm{Ric}_{T_pM}(e_i)$ is the usual *Ricci curvature* of e_i, denoted $\mathrm{Ric}(e_i)$. Thus for any orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM and for a fixed $i \in \{1, \ldots, n\}$, we have

$$\mathrm{Ric}_{T_pM}(e_i) \equiv \mathrm{Ric}(e_i) = \sum_{j \neq i}^{n} K_{ij}.$$

The scalar curvature $\tau(\Pi_k)$ of the k-plane section Π_k is given by

$$\tau(\Pi_k) = \sum_{1\le i<j\le k} K_{ij}. \tag{2}$$

Geometrically, $\tau(\Pi_k)$ is the scalar curvature of the image $\exp_p(\Pi_k)$ of Π_k at p under the exponential map at p. The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space T_pM of M at p, that is, $\tau(p) = \tau(T_pM)$.

Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$ equipped with a Riemannian metric $\widetilde{g}$. We use the inner product notation $\langle\ ,\ \rangle$ for both the metrics $\widetilde{g}$ of $\widetilde{M}$ and the induced metric g on the submanifold M. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \quad \text{and} \quad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and σ is the second fundamental form related to the shape operator A by $\langle\sigma(X,Y),N\rangle = \langle A_N X, Y\rangle$. The equation of Gauss is given by

$$\begin{aligned} R(X,Y,Z,W) &= \widetilde{R}(X,Y,Z,W) + \langle\sigma(X,W),\sigma(Y,Z)\rangle \\ &- \langle\sigma(X,Z),\sigma(Y,W)\rangle \end{aligned} \tag{3}$$

for all $X,Y,Z,W \in TM$, where $\widetilde{R}$ and R are the Riemann curvature tensors of $\widetilde{M}$ and M respectively. The curvature tensor $R^{\perp}$ of the normal bundle of M is defined by

$$R^{\perp}(X,Y)N = \nabla^{\perp}_X\nabla^{\perp}_Y N - \nabla^{\perp}_Y\nabla^{\perp}_X N - \nabla^{\perp}_{[X,Y]}N$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$. If $R^{\perp} = 0$, then the normal connection $\nabla^{\perp}$ of M is said to be *flat*.

The mean curvature vector H is given by $H = \frac{1}{n}\mathrm{trace}(\sigma)$. The submanifold M is *totally geodesic* in $\widetilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma(X,Y) = g(X,Y)H$ for all $X, Y \in TM$, then M is *totally umbilical.*

The *relative null space* of M at p is defined by [7]

$$\mathcal{N}_p = \{X \in T_pM \mid \sigma(X,Y) = 0 \text{ for all } Y \in T_pM\},$$

which is also known as the *kernel of the second fundamental form* at p [8].

Now, let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r belongs to an orthonormal basis $\{e_{n+1},\ldots,e_m\}$ of the normal space $T_p^{\perp}M$. We put

$$\sigma^r_{ij} = \langle\sigma(e_i,e_j),e_r\rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n}\langle\sigma(e_i,e_j),\sigma(e_i,e_j)\rangle.$$

Let K_{ij} and $\widetilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\widetilde{M}$ respectively. Thus, we can say that K_{ij} and $\widetilde{K}_{ij}$ are the "intrinsic" and "extrinsic" sectional curvature of the Span$\{e_i, e_j\}$ at p. In view of (3), we get

$$K_{ij} = \widetilde{K}_{ij} + \sum_{r=n+1}^{m} \left(\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2\right). \tag{4}$$

From (4) it follows that

$$2\tau(p) = 2\widetilde{\tau}\left(T_pM\right) + n^2 \left\|H\right\|^2 - \left\|\sigma\right\|^2, \tag{5}$$

where $\widetilde{\tau}(T_pM)$ denotes the scalar curvature of the n-plane section T_pM in the ambient manifold $\widetilde{M}$. Thus, we can say that $\tau(p)$ and $\widetilde{\tau}\left(T_pM\right)$ are the "intrinsic" and "extrinsic" scalar curvature of the submanifold at p respectively.

We denote the set of unit vectors in T_pM by T_p^1M; thus

$$T_p^1M = \{X \in T_pM \mid \langle X, X\rangle = 1\}.$$

Now, we recall the following result from [9].

Theorem 2.1. *Let M be an n-dimensional submanifold of a Riemannian manifold $\widetilde{M}$. Then the following statements are true.*

(a) *For $X \in T_p^1M$ we have*

$$\mathrm{Ric}(X) \leq \frac{n^2}{4} \left\|H\right\|^2 + \widetilde{\mathrm{Ric}}_{(T_pM)}(X), \tag{6}$$

where $\widetilde{\mathrm{Ric}}_{(T_pM)}(X)$ is the n-Ricci curvature of T_pM at $X \in T_p^1M$ with respect to the ambient manifold $\widetilde{M}$.

(b) *The equality case of (6) is satisfied by $X \in T_p^1M$ if and only if*

$$\sigma(X, X) = \frac{n}{2}\, H(p) \quad \text{and} \quad \sigma(X, Y) = 0 \tag{7}$$

for all $Y \in T_pM$ such that $\langle X, Y\rangle = 0$.

(c) *The equality case of (6) holds for all $X \in T_p^1M$ if and only if either* **(1)** *p is a totally geodesic point or* **(2)** *$n = 2$ and p is a totally umbilical point.*

From Theorem 2.1, we immediately have the following

Corollary 2.2. *Let M be an n-dimensional submanifold of a Riemannian manifold. For $X \in T_p^1M$ any two of the following three statements imply the remaining one.*

(a) *The mean curvature vector $H(p)$ vanishes.*

(b) *The unit vector X belongs to the relative null space $\mathcal{N}_p$.*

(c) *The unit vector X satisfies the equality case of (6), namely*

$$\mathrm{Ric}(X) = \frac{1}{4}\, n^2 \|H\|^2 + \widetilde{\mathrm{Ric}}_{(T_pM)}(X). \tag{8}$$

3. $\mathcal{S}$-space forms

Let $\widetilde{M}$ be a $(2m+s)$-dimensional *framed metric manifold* [17] (also known as *framed f-manifold* [13] or *almost r-contact metric manifold* [15]) with a framed metric structure $(f, \xi_\alpha, \eta^\alpha, \widetilde{g})$, $\alpha \in \{1, \ldots, s\}$, that is, f is a $(1,1)$ tensor field defining an f-structure of rank $2m$; $\xi_1, \ldots, \xi_s$ are vector fields; $\eta^1, \ldots, \eta^s$ are 1-forms and $\widetilde{g}$ is a Riemannian metric on $\widetilde{M}$ such that for all $X, Y \in T\widetilde{M}$ and $\alpha, \beta \in \{1, \ldots, s\}$

$$f^2 = -I + \eta^\alpha \otimes \xi_\alpha, \quad \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad f(\xi_\alpha) = 0, \quad \eta^\alpha \circ f = 0, \tag{9}$$

$$\langle fX, fY \rangle = \langle X, Y \rangle - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y), \tag{10}$$

$$\Omega(X, Y) \equiv \langle X, fY \rangle = -\Omega(Y, X), \quad \langle X, \xi_\alpha \rangle = \eta^\alpha(X), \tag{11}$$

where $\langle \, , \rangle$ denotes the inner product of the metric $\widetilde{g}$. A framed metric structure is an $\mathcal{S}$-*structure* [1] if the Nijenhuis tensor of f equals $-2d\eta^\alpha \otimes \xi_\alpha$ and $\Omega = d\eta^\alpha$ for all $\alpha \in \{1, \ldots, s\}$.

When $s = 1$, a framed metric structure is an almost contact metric structure, while an $\mathcal{S}$-structure is a Sasakian structure. When $s = 0$, a framed metric structure is an almost Hermitian structure, while an $\mathcal{S}$-structure is a Kaehler structure. If a framed metric structure on $\widetilde{M}$ is an $\mathcal{S}$-structure then it is known [1] that

$$(\widetilde{\nabla}_X f)Y = \sum_\alpha \left(\langle fX, fY \rangle \xi_\alpha + \eta^\alpha(Y) f^2 X \right), \tag{12}$$

$$\widetilde{\nabla}\xi_\alpha = -f, \qquad \alpha \in \{1, \ldots, s\}. \tag{13}$$

The converse may also be proved. In case of Sasakian structure (that is, $s = 1$), (12) implies (13). In Kaehler case (that is, $s = 0$), we get $\widetilde{\nabla} f = 0$. For $s > 1$, examples of $\mathcal{S}$-structures are given in [1], [2] and [4]. Thus, the bundle space of a principal toroidal bundles over a Kaehler manifold with certain conditions is an $\mathcal{S}$-manifold. Thus, a generalization of the Hopf fibration $\pi' : S^{2m+1} \to PC^m$ is a canonical example of an $\mathcal{S}$-manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry.

A plane section in $T_p\widetilde{M}$ is a f-*section* if there exists a vector $X \in T_p\widetilde{M}$ orthogonal to $\xi_1, \ldots, \xi_s$ such that $\{X, fX\}$ span the section. The sectional curvature of a f-*section* is called a f-*sectional curvature*. It is known that [5] in an $\mathcal{S}$-manifold of constant f-sectional curvature c

$$\begin{aligned} \widetilde{R}(X,Y)Z & \\ = & \sum_{\alpha,\beta}\{\eta^\alpha(X)\eta^\beta(Z)f^2Y - \eta^\alpha(Y)\eta^\beta(Z)f^2X \\ & - \langle fX,fZ\rangle \eta^\alpha(Y)\xi_\beta + \langle fY,fZ\rangle \eta^\alpha(X)\xi_\beta\} \\ & + \frac{c+3s}{4}\{-\langle fY,fZ\rangle f^2X + \langle fX,fZ\rangle f^2Y\} \\ & + \frac{c-s}{4}\{\langle X,fZ\rangle fY - \langle Y,fZ\rangle fX + 2\langle X,fY\rangle fZ\} \end{aligned} \tag{14}$$

for all $X,Y,Z \in T\widetilde{M}$, where $\widetilde{R}$ is the curvature tensor of $\widetilde{M}$. An $\mathcal{S}$-manifold of constant f-sectional curvature c is called an *$\mathcal{S}$-space form* $\widetilde{M}(c)$.

When $s=1$, an $\mathcal{S}$-space form $\widetilde{M}(c)$ reduces to a Sasakian space form $\widetilde{M}(c)$ [3] and (14) reduces to

$$\begin{aligned} \widetilde{R}(X,Y)Z = & \frac{c+3}{4}\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} \\ & + \frac{c-1}{4}\{\langle X,fZ\rangle fY - \langle Y,fZ\rangle fX + 2\langle X,fY\rangle fZ \\ & \quad + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & \quad + \langle X,Z\rangle \eta(Y)\xi - \langle Y,Z\rangle \eta(X)\xi\}, \end{aligned}$$

where $\xi_1 \equiv \xi$ and $\eta^1 \equiv \eta$. When $s=0$, an $\mathcal{S}$-space form $\widetilde{M}(c)$ becomes a complex space form and (14) moves to

$$\begin{aligned} 4\widetilde{R}(X,Y)Z = & \; c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y \\ & + \langle X,fZ\rangle fY - \langle Y,fZ\rangle fX + 2\langle X,fY\rangle fZ\}. \end{aligned}$$

4. Ricci curvature of integral submanifolds

Let $\widetilde{M}$ be an $\mathcal{S}$-manifold equipped with an $\mathcal{S}$-structure $(f,\xi_\alpha,\eta^\alpha,\widetilde{g})$. A submanifold M of $\widetilde{M}$ is an *integral submanifold* if $\eta_\alpha(X)=0$, $\alpha=1,\ldots,s$, for every tangent vector X. A submanifold M of $\widetilde{M}$ is an *anti-invariant submanifold* if $f(TM) \subseteq T^\perp M$. An integral submanifold is identical with an anti-invariant submanifold normal to the structure vector fields $\xi_1,\ldots,\xi_s$. In particular case of $s=1$, an integral submanifold M of a Sasakian manifold is a *C-totally real submanifold* [16]. It is known that [6] an n-dimensional integral submanifold M, of an $\mathcal{S}$-manifold $\widetilde{M}$ of dimension $(2n+s)$, is of constant curvature s if and only if the normal connection is flat.

First, we give the following Lemma.

Lemma 4.1. *Let M be an n-dimensional integral submanifold of an $\mathcal{S}$-space form $\widetilde{M}(c)$. Let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of the tangent space T_pM.*

Then

$$\widetilde{K}_{ij} = \frac{1}{4}(c+3s), \tag{15}$$

$$\widetilde{\mathrm{Ric}}_{(T_pM)}(e_i) = \frac{1}{4}(n-1)(c+3s), \tag{16}$$

$$\widetilde{\tau}(T_pM) = \frac{1}{8}n(n-1)(c+3s). \tag{17}$$

Proof. Equation (15) follows from (14). Using $\widetilde{\mathrm{Ric}}_{(T_pM)}(e_i) = \sum_{j\neq i}^{n} \widetilde{K}_{ij}$ in (15), we get (16). Next, using $2\widetilde{\tau}(T_pM) = \sum_{i=1}^{n} \widetilde{\mathrm{Ric}}_{(T_pM)}(e_i)$, from (16) we get (17). □

Now, we have the following Theorem.

Theorem 4.2. *If M is an n-dimensional integral submanifold of an $\mathcal{S}$-space form $\widetilde{M}(c)$, then the following statements are true.*

(a) *For $X \in T_p^1M$, it follows that*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3s)\right\}. \tag{18}$$

(b) *The equality case of (18) is satisfied by $X \in T_p^1M$ if and only if (7) is true. If $H(p) = 0$, $X \in T_p^1M$ satisfies equality in (18) if and only if $X \in \mathcal{N}_p$.*

(c) *The equality case of (18) holds for all $X \in T_p^1M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. Using (16) in (6), we find the inequality (18). Rest of the proof is straightforward. □

By polarization, from Theorem 4.2, we derive

Theorem 4.3. *Let M be an n-dimensional integral submanifold of an $\mathcal{S}$-space form $\widetilde{M}(c)$. Then the Ricci tensor S satisfies*

$$S \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3s)\right\} g, \tag{19}$$

where g is the induced Riemannian metric on M. The equality case of (19) is true if and only if either M is a totally geodesic submanifold or M is a totally umbilical surface.

When $s = 0$, we have the following two results.

Theorem 4.4. *If M is an n-dimensional totally real submanifold (or isotropic submanifold) of a complex space form $\widetilde{M}(c)$, then the following statements are true.*

(a) *It follows that*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)c\right\}, \qquad X \in T_p^1M. \tag{20}$$

(b) *The equality case of* (20) *is satisfied by* $X \in T_p^1M$ *if and only if* (7) *is true. If* $H(p) = 0$, $X \in T_p^1M$ *satisfies equality in* (20) *if and only if* $X \in \mathcal{N}_p$.

(c) *The equality case of* (20) *holds for all* $X \in T_p^1M$ *if and only if either* p *is a totally geodesic point or* $n = 2$ *and* p *is a totally umbilical point.*

Theorem 4.5. *If* M *is an* n*-dimensional totally real submanifold (or isotropic submanifold) of a complex space form* $\widetilde{M}(c)$*, then the following statements are true.*

(a) *It follows that*

$$S \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)c\right\} g. \tag{21}$$

(b) *The equality case of* (21) *holds identically if and only if either* M *is totally geodesic submanifold or* M *is a totally umbilical surface.*

For $s = 1$, we again have the following two results.

Theorem 4.6. *If* M *is an* n*-dimensional* C*-totally real submanifold of a Sasakian space form* $\widetilde{M}(c)$*, then the following statements are true.*

(a) *It follows that*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3)\right\}, \qquad X \in T_p^1M. \tag{22}$$

(b) *The equality case of* (22) *is satisfied by* $X \in T_p^1M$ *if and only if* (7) *is true. If* $H(p) = 0$, $X \in T_p^1M$ *satisfies equality in* (22) *if and only if* $X \in \mathcal{N}_p$.

(c) *The equality case of* (22) *holds for all* $X \in T_p^1M$ *if and only if either* p *is a totally geodesic point or* $n = 2$ *and* p *is a totally umbilical point.*

(d) *The equality case of* (23) *holds identically if and only if either* M *is totally geodesic submanifold or* M *is a totally umbilical surface.*

Theorem 4.7. *If* M *is an* n*-dimensional* C*-totally real submanifold of a Sasakian space form* $\widetilde{M}(c)$*, then the following statements are true.*

(a) *It follows that*

$$S \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3)\right\} g. \tag{23}$$

(b) *The equality case of* (23) *holds identically if and only if either* M *is totally geodesic submanifold or* M *is a totally umbilical surface.*

It is known that (Theorem 4, [14]) if M is an n-dimensional compact minimal C-totally real submanifold of a Sasakian space form $M^{2n+1}(c)$, $c > -3$, such that M has positive sectional curvature, then M is totally geodesic. Therefore, in view of Theorem 4.7, we have the following

Theorem 4.8. *An n-dimensional compact minimal C-totally real submanifold of a Sasakian space form $M^{2n+1}(c)$, $c > -3$ with positive sectional curvature is an Einstein manifold and satisfies $4S = (n-1)(c+3)g$.*

The inequality (22) is the inequality (2.1) in Theorem 2.1 of [12]. The inequality (23) is the inequality (9) in Theorem 3.1 of [10]. The inequality (21) is the inequality (2.1) in Theorem 1 of [8]. Here, we find the proofs very much simplified.

5. Minimality of integral submanifolds of maximum dimension

We already know the following result [6]. If M is an n-dimensional integral submanifold of any $(2m+s)$-dimensional $\mathcal{S}$-space form $\widetilde{M}(c)$, then the following four statements are equivalent: (i) M is totally geodesic. (ii) M is of constant curvature $\frac{1}{4}(c+3s)$. (iii) The Ricci tensor is $\frac{1}{4}(n-1)(c+3s)g$. (iv) The scalar curvature is $\frac{1}{4}n(n-1)(c+3s)$. In Theorem 5.2, we find a condition for minimality.

Now, we begin with the following

Theorem 5.1. *Let M be an n-dimensional integral submanifold of a $(2n+s)$-dimensional $\mathcal{S}$-space form $\widetilde{M}(c)$. If a unit vector of T_pM satisfies the equality case of* (18), *then $H(p) = 0$.*

Proof. Choose an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM such that e_1 satisfies the equality case of (18). Then, $\{e_{n+1}, \dots, e_{2n}, e_{2n+1} = \xi_1, \dots, e_{2n+s} = \xi_s\}$ is an orthonormal basis of $T_p^\perp M$ such that $e_{n+j} = fe_j$, $j \in \{1, \dots, n\}$. We then have $A_{\xi_\alpha} = 0$ for all $\alpha \in \{1, \dots, s\}$ and $A_{fX}Y = A_{fY}X$ for $X, Y \in TM$. Using these two facts alongwith (7), for any $Y = \sum_{j=1}^n a_j e_{n+j} + \sum_{\alpha=1}^s a_\alpha \xi_\alpha \in T_p^\perp M$, we have

$$\begin{aligned}
\langle \sigma(e_1,e_1), Y\rangle &= a_1 \langle \sigma(e_1,e_1), fe_1\rangle \\
&\quad + \sum_{j=2}^n a_j \langle \sigma(e_1,e_1), fe_j\rangle + \sum_{\alpha=1}^s a_\alpha \langle \sigma(e_1,e_1), \xi_\alpha\rangle \\
&= a_1 \left\langle \sum_{j=2}^n \sigma(e_j,e_j), fe_1 \right\rangle + \sum_{j=2}^n a_j \langle \sigma(e_1,e_1), fe_j\rangle + 0 \\
&= a_1 \sum_{j=2}^n \langle \sigma(e_1,e_j), fe_j\rangle + \sum_{j=2}^n a_j \langle \sigma(e_1,e_j), fe_1\rangle \\
&= 0 + 0 = 0.
\end{aligned}$$

Hence in view of (7), $H(p) = 0$. □

The maximum Ricci curvature function ([8]) on a Riemannian manifold M, denoted $\overline{\mathrm{Ric}}$, is defined as

$$\overline{\mathrm{Ric}}(p) = \max\left\{\mathrm{Ric}(X) \mid X \in T_p^1 M\right\}.$$

Now, in view of Theorem 5.1, we immediately have the following

Theorem 5.2. *Let M be an n-dimensional integral submanifold of a $(2n+s)$-dimensional $\mathcal{S}$-space form $\widetilde{M}(c)$. Then*

$$\overline{\mathrm{Ric}} \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3s)\right\}. \tag{24}$$

If M satisfies the equality case of (24) identically, then M is a minimal submanifold and

$$\overline{\mathrm{Ric}} = \frac{1}{4}(n-1)(c+3s). \tag{25}$$

When $s = 0$, from Theorem 5.2 we have the following

Theorem 5.3. ([8], Theorem 2) *Let M be a Lagrangian submanifold of a $2n$-dimensional complex space form $\widetilde{M}(c)$. Then*

$$\overline{\mathrm{Ric}} \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)c\right\}.$$

If M satisfies the equality case of (24) identically, then M is a minimal submanifold and

$$\overline{\mathrm{Ric}} = \frac{1}{4}(n-1)c.$$

When $s = 1$, from Theorem 5.2 we have the following (Theorem 4.1 of [10] or Theorem 3.1 of [11])

Theorem 5.4. ([10], Theorem 4.1 or Theorem 3.1 of [11]) *Let M be an n-dimensional C-totally real submanifold of a $(2n+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then*

$$\overline{\mathrm{Ric}} \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3)\right\}.$$

If M satisfies the equality case of (24) identically, then M is a minimal submanifold and

$$\overline{\mathrm{Ric}} = \frac{1}{4}(n-1)(c+3).$$

Following the arguments as in [8], we can prove

Theorem 5.5. *Let M be an n-dimensional minimal integral submanifold of a $(2n+s)$-dimensional $\mathcal{S}$-space form $\widetilde{M}(c)$. Then the following statements are true.*

(1) *The submanifold M satisfies the equality case of (24) if and only if $\dim(\mathcal{N}_p) \geq 1$.*

(2) *If* $\dim(\mathcal{N}_p)$ *is a positive constant* d*, then* $\mathcal{N}_p$ *is completely integral distribution and* M *is* d*-ruled, that is, for each* $p \in M$*,* M *contains a* d*-dimensional totally geodesic submanifold* M' *of* $\widetilde{M}(c)$ *passing through* p.

(3) *If the submanifold* M *is also ruled, then it satisfies the equality case of* (24) *identically if and only if, for each ruling* M' *in* M*, the normal bundle* $T^{\perp}M$ *restricted to* M' *is a parallel normal subbundle of the normal bundle* $T^{\perp}M'$ *along* M'.

References

[1] D. E. Blair, *Geometry of manifolds with structural group* $\mathcal{U}(n)\times\mathcal{O}(s)$, J. Diff. Geometry **4** (1970), 155–167.

[2] ———, *On a generalization of the Hopf fibration*, An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.) **17** (1971), 171–177.

[3] ———, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, 203. Birkhauser Boston, Inc., Boston, MA, 2002.

[4] D. E. Blair, G. D. Ludden, and K. Yano, *Differential geometric structures on principal toroidal bundles*, Trans. Amer. Math. Soc. **181** (1973), 175–184.

[5] J. L. Cabrerizo, L. M. Fernandez, and M. Fernandez, *The curvature of submanifolds of an* S*-space form*, Acta Math. Hungar. **62** (1993), no. 3-4, 373–383.

[6] ———, *On certain anti-invariant submanifolds of an* S*-manifold*, Portugal. Math. **50** (1993), no. 1, 103–113.

[7] B.-Y. Chen, *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*, Glasg. Math. J. **41** (1999), no. 1, 33–41.

[8] ———, *On Ricci curvature of isotropic and Langrangian submanifolds in complex space forms*, Arch. Math. (Basel) **74** (2000), no. 2, 154–160.

[9] S. P. Hong and M. M. Tripathi, *On Ricci curvature of submanifolds*, Int. J. Pure Appl. Math. Sci. **2** (2005), no. 2, 227–245.

[10] X. Liu, *On Ricci curvature of* C*-totally real submanifolds in Sasakian space forms*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **17** (2001), no. 3, 171–177.

[11] K. Matsumoto, I. Mihai, *Ricci tensor of* C*-totally real submanifolds in Sasakian space forms*, Nihonkai Math. J. **13** (2002), no. 2, 191–198.

[12] I. Mihai, *Ricci curvature of submanifolds in Sasakian space forms*, J. Aust. Math. Soc. **72** (2002), no. 2, 247–256.

[13] H. Nakagawa, *On framed* f*-manifolds*, Kodai Math. Sem. Rep. **18** (1966) 293–306.

[14] D. Van Lindt, P. Verheyen, and L. Verstraelen, *Minimal submanifolds in Sasakian space forms*, J. Geom. **27** (1986), no. 2, 180–187.

[15] J. Vanžura, *Almost* r*-contact structures*, Ann. Scuola Norm. Sup. Pisa (3) **26** (1972), 97–115.

[16] S. Yamaguchi, M. Kon, and T. Ikawa, C*-totally real submanifolds*, J. Differential Geometry **11** (1976), no. 1, 59–64.

[17] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, **3**. World Scientific Publishing Co., Singapore, 1984.

Jeong-Sik Kim
Department of Applied Mathematics
Chonnam National University
Yosu 550-749, Korea
E-mail address: sunchon@yosu.ac.kr

Mohit Kumar Dwivedi
Department of Mathematics and Astronomy
Lucknow University
Lucknow 226 007, India

Mukut Mani Tripathi
Department of Mathematics and Astronomy
Lucknow University
Lucknow 226 007, India
E-mail address: mmtripathi66@yahoo.com