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PERIODIC SOLUTIONS OF A DISCRETE TIME NON-AUTONOMOUS RATIO-DEPENDENT PREDATOR-PREY SYSTEM WITH CONTROL

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ABSTRACT. With the help of the coincidence degree and the related continuation theorem, we explore the existence of at least two periodic solutions of a discrete time non-autonomous ratio-dependent predator-prey system with control. Some easily verifiable sufficient criteria are established for the existence of at least two positive periodic solutions.

1. Introduction

In mathematical ecology literatures, there are two different types of predator-prey models: the classical prey-dependent ones and the ratio-dependent ones. The classical prey-dependent predator-prey system often takes the general form

$$\begin{aligned} x'(t) &= xf(x) - cyp(x), \\ y'(t) &= (p(x) - d)y, \end{aligned} \tag{1.1}$$

where x, y stand for prey and predator density, respectively, $p(x)$ is the so-called predator functional response, and its various concrete forms have received great attention and have been well studied, for example, the traditional model with Holling type II functional response

$$\begin{aligned} x'(t) &= x\,(r - kx) - \frac{cxy}{m + x}, \\ y'(t) &= -dy + \frac{fxy}{m + x}. \end{aligned} \tag{1.2}$$

Unfortunately, the prey-dependent functional response fails to model the interference among predators. To overcome the shortcoming, Arditi and Ginzburg [1] proposed the following ratio-dependent predator-prey model

$$\begin{aligned} x'(t) &= x\,(a - bx) - \frac{cxy}{my + x}, \\ y'(t) &= -dy + \frac{fxy}{my + x}, \end{aligned} \tag{1.3}$$

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which incorporates mutual interference by predator. For detailed justification of (1.3) and its merits versus system (1.2), we can refer to [1]. In addition, system (1.3) and its nonautonomous model have been studied by many authors and seen great progress, see, for example, [2, 3, 8, 13].

In real life, biological controls have been successfully and frequently implemented by nature and human. Therefore, control variables are introduced to the mathematical ecological models. Generally, the model with control can be depicted by the following

$$\dot{x} = xf(x, y) + u_1, \ \dot{y} = yg(x, y) + u_2, \tag{1.4}$$

where u_1, u_2 stand for control variables. Particularly, a non-autonomous ratio-dependent predator-prey model with control based on system (1.3) is described as follows

$$\begin{aligned} x'(t) &= x[a(t) - b(t)x] - \frac{c(t)xy}{m(t)y + x} - u(t), \\ y'(t) &= -d(t)y + \frac{f(t)xy}{m(t)y + x}, \end{aligned} \tag{1.5}$$

where a, c, d, f, m, u denote the prey intrinsic rate, capture rate, death rate of predator, conversion rate, half saturation parameter and exploited term (e.g. harvest), respectively. It is noted that the control is only imposed on the prey and the continuous system (1.5) has been well studied by Tian and Zeng [12]. However, many authors have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations, see [4, 7, 10]. Therefore, the major objective of this paper is to propose a discrete analogue of nonautonomous system (1.5) and investigate the existence of periodic solutions due to the various seasonal effects present in real life situation.

2. Preliminaries

Let us begin by introducing some terminology and results.

Let $\mathbf{Z}, \mathbf{Z}^+, \mathbf{R}, \mathbf{R}^+$ and $\mathbf{R}^2$ denote the sets of all integers, nonnegative integers, real numbers, nonnegative real numbers, and two-dimensional Euclidean vector space, respectively. In the sequel, we will use the notations

$$I_\omega = \{0, 1, \ldots, \omega - 1\}, \ \bar{g} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} g(k), \ g^L = \min_{k\in I_\omega} g(k), \ g^M = \max_{k\in I_\omega} g(k),$$

where $\{g(k)\}$ is an ω-periodic sequence of real numbers defined for $k \in \mathbf{Z}$.

The major objective of this paper is to study the existence of positive periodic solutions of discrete-time form of system (1.5). To this end, we first discretize system (1.5). Using homogeneous techniques as can be found in [2],

we obtain the following model
(2.1)

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\left\{a(k) - b(k)x_1(k) - \frac{c(k)x_2(k)}{m(k)x_2(k)+x_1(k)} - \frac{u(k)}{x_1(k)}\right\} \\ x_2(k+1) &= x_2(k)\exp\left\{-d(k) + \frac{f(k)x_1(k)}{m(k)x_2(k)+x_1(k)}\right\}, k = 0, 1, \ldots, \end{aligned}$$

where $x_i(k), i = 1, 2$ denotes the density of prey and predator at time k, respectively.

In system (2.1), we always assume that $a, d : \mathbf{Z} \to \mathbf{R}$ and $b, c, m, f, u : \mathbf{Z} \to \mathbf{R}^+$ are ω periodic and $\bar{a} > 0, \bar{d} > 0$, where ω, a fixed positive integer, denotes the prescribed common period of the parameters in system (2.1). Moreover, for biological reasons, we only consider solutions $(x_1(t), x_2(t))$ with $x_1(0) > 0, x_2(0) > 0$.

For the reader's convenience, we now recall Mawhin's coincidence degree which our study is based upon.

Let X, Y be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Y$ a linear mapping, $N : X \to Y$ is a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L, \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L|\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω be an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Theorem A. (Continuation Theorem [6]) *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose*

(i) For each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;

(ii) $QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$ and

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.$$

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

The following lemma given in [2].

Lemma 2.1. *Let $g : \mathbf{Z} \to \mathbf{R}$ be ω-periodic, i.e., $g(k + \omega) = g(k)$, then for any fixed $k_1, k_2 \in I_\omega$, and any $k \in \mathbf{Z}$, we have*

$$g(k) \le g(k_1) + \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|,$$
$$g(k) \ge g(k_2) - \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|.$$

Lemma 2.2. *If $\bar{f} > \bar{d}$ and $(a-c/m)^L > 2\sqrt{b^M u^M}$, then the following algebraic equations*

$$
\begin{aligned}
&\bar{a} - \bar{b}\exp\{v_1\} - \frac{1}{\omega}\sum_{k=0}^{\omega-1}\frac{c(k)\exp\{v_2\}}{m(k)\exp\{v_2\}+\exp\{v_1\}} - \frac{\bar{u}}{\exp\{v_1\}} = 0\\
&-\bar{d} + \frac{1}{\omega}\sum_{k=0}^{\omega-1}\frac{f(k)\exp\{v_1\}}{m(k)\exp\{v_2\}+\exp\{v_1\}} = 0
\end{aligned}
\tag{2.2}
$$

has two solutions.

Proof. Consider the function

$$f(z) = -\bar{d} + \frac{1}{\omega}\sum_{k=0}^{\omega-1}\frac{f(k)}{m(k)z+1}, \quad z \geq 0.$$

It is easily seen that $f(z)$ is decreasing with z and

$$f(0) = \bar{f} - \bar{d} > 0, \quad \lim_{z\to+\infty} f(z) = -\bar{d} < 0,$$

then it follows that there exists a unique $z^\star$ such that $f(z^\star) = 0$.

Substituting $z^\star = \exp\{v_2 - v_1\}$ into the first equation in (2.2), we have

$$\bar{a} - \bar{b}\exp\{v_1\} - \frac{1}{\omega}\sum_{k=0}^{\omega-1}\frac{c(k)z^\star}{m(k)z^\star+1} - \frac{\bar{u}}{\exp\{v_1\}} = 0. \tag{2.3}$$

Obviously, it is a quadratic equation with respect to $\exp\{v_1\}$, then it has two solutions, denote by v_1^1 and v_1^2 ($v_1^1 < v_1^2$). Moreover, one can easily see that

$$\bar{a} - \bar{b}\exp\{v_1\} - \overline{\left(\frac{c}{m}\right)} - \frac{\bar{u}}{\exp\{v_1\}} < 0.$$

Solving the inequality produces

$$
\begin{aligned}
\exp\{v_1^1\} &< \frac{\bar{a} - \overline{(c/m)} - \sqrt{[\bar{a} - \overline{(c/m)}]^2 - 4\bar{b}\bar{u}}}{2\bar{b}},\\
\exp\{v_1^2\} &> \frac{\bar{a} - \overline{(c/m)} + \sqrt{[\bar{a} - \overline{(c/m)}]^2 - 4\bar{b}\bar{u}}}{2\bar{b}},
\end{aligned}
$$

which implies (2.2) has two solutions and this completes the proof. □

3. Main results

In this section, we devote ourselves to establishing easily verifiable sufficient criteria for the existence of at least two positive periodic solutions of the system (2.1) by employing the coincidence degree and the related continuation theorem introduced in previous section.

Define $l_2 = \{y = \{y(k)\} : y(k) \in \mathbf{R}^2, k \in \mathbf{Z}\}$ and $|h| = \max\{h_1, h_2\}$, for any $h = (h_1, h_2)^T \in \mathbf{R}^2$. Let $l^\omega \subset l_2$ stand for the subspace of all ω periodic sequences with the supremum norm $\|y\| = \max_{k\in I_\omega} |y(k)|$, for any $y \in l^\omega$.

Thus, from [11], we know that l^ω is a normed space with this norm and every Cauchy sequence is convergent in l^ω, which imply that l^ω is a finite-dimensional Banach space.

Let

$$l_0^\omega = \{y = \{y(k)\} \in l^\omega : \sum_{k=0}^{\omega-1} y(k) = 0\},$$
$$l_c^\omega = \{y = \{y(k)\} \in l^\omega : y(k) = h \in \mathbf{R}^2, k \in \mathbf{Z}\}.$$

Then it follows that l_0^ω and l_c^ω are both closed linear subspace of l^ω and $l^\omega = l_0^\omega \bigoplus l_c^\omega, \dim l_c^\omega = 2$.

Lemma 3.1. [11] *Let $T : X \to Y$ be a linear operator, where X and Y are normed spaces, and X has finite dimension, then T is bounded, and hence continuous.*

We now state and prove our main results.

Theorem 3.2. *If $\bar{f} > \bar{d}$ and $(a - c/m)^L > 2\sqrt{b^M u^M}$, then system* (2.1) *has at least two positive ω-periodic solutions.*

Proof. From the system (2.1), we can see that every solution is positive with initial values $x_1(0) > 0, x_2(0) > 0$. Let $x_i(k) = \exp\{y_i(k)\}, i = 1, 2$. Then system (2.1) reduces to

(3.1)
$$y_1(k+1) - y_1(k) = a(k) - b(k)\exp\{y_1(k)\} - \frac{c(k)\exp\{y_2(k)\}}{m(k)\exp\{y_2(k)\} + \exp\{y_1(k)\}} - \frac{u(k)}{\exp\{y_1(k)\}},$$
$$y_2(k+1) - y_2(k) = -d(k) + \frac{f(k)\exp\{y_1(k)\}}{m(k)\exp\{y_2(k)\} + \exp\{y_1(k)\}}.$$

By this media, it is trivial to show that if system (3.1) has a ω-periodic solution $y^* = (y_1^*, y_2^*)^T$, then $x^* = (x_1^*, x_2^*)^T = (\exp\{y_1^*\}, \exp\{y_2^*\})^T$ is a positive ω-periodic solution of system (2.1). To this end, it suffices to prove that system (3.1) has at least two ω-periodic solutions.

For $\lambda \in (0, 1)$, we consider the following system

(3.2)
$$y_1(k+1) - y_1(k) = \lambda\left[a(k) - b(k)\exp\{y_1(k)\} - \frac{c(k)\exp\{y_2(k)\}}{m(k)\exp\{y_2(k)\} + \exp\{y_1(k)\}} - \frac{u(k)}{\exp\{y_1(k)\}}\right],$$
$$y_2(k+1) - y_2(k) = \lambda\left[-d(k) + \frac{f(k)\exp\{y_1(k)\}}{m(k)\exp\{y_2(k)\} + \exp\{y_1(k)\}}\right].$$

Suppose that $y = \{y(k)\} = \{(y_1(k), y_2(k))^T\}$ is an arbitrary ω-periodic solution of system (3.2) for a certain $\lambda \in (0, 1)$. Summing on both sides of

(3.2) from 0 to $\omega-1$ with respect to k, leads to

(3.3)
$$\bar{a}\omega=\sum_{k=0}^{\omega-1}\left[b(k)\exp\{y_1(k)\}+\frac{c(k)\exp\{y_2(k)\}}{m(k)\exp\{y_2(k)\}+\exp\{y_1(k)\}}+\frac{u(k)}{\exp\{y_1(k)\}}\right],$$

and

$$\bar{d}\omega=\sum_{k=0}^{\omega-1}\left[\frac{f(k)\exp\{y_1(k)\}}{m(k)\exp\{y_2(k)\}+\exp\{y_1(k)\}}\right]. \tag{3.4}$$

From (3.2) – (3.4), it follows that

$$\begin{aligned}&\sum_{k=0}^{\omega-1}|y_1(k+1)-y_1(k)|\\ &\le\sum_{k=0}^{\omega-1}\left[|a(k)|+b(k)\exp\{y_1(k)\}+\frac{c(k)\exp\{y_2(k)\}}{m(k)\exp\{y_2(k)\}+\exp\{y_1(k)\}}\right.\\ &\qquad\left.+\frac{u(k)}{\exp\{y_1(k)\}}\right]\\ &:=(\bar{A}+\bar{a})\omega,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\sum_{k=0}^{\omega-1}|y_2(k+1)-y_2(k)|\\ &\le\sum_{k=0}^{\omega-1}\left[|d(k)|+\frac{f(k)\exp\{y_1(k)\}}{m(k)\exp\{y_2(k)\}+\exp\{y_1(k)\}}\right]:=(\bar{D}+\bar{d})\omega.\end{aligned} \tag{3.6}$$

Choose $\xi_i,\eta_i\in I_\omega, i=1,2$, such that

$$y_i(\xi_i)=\min_{k\in I_\omega}\{y_i(k)\},\ y_i(\eta_i)=\max_{k\in I_\omega}\{y_i(k)\},\ i=1,2. \tag{3.7}$$

By (3.3) and (3.7), we obtain

$$\bar{a}\omega>\sum_{k=0}^{\omega-1}b(k)\exp\{y_1(k)\}\ge\sum_{k=0}^{\omega-1}b(k)\exp\{y_1(\xi_1)\}=\exp\{y_1(\xi_1)\}\bar{b}\omega,$$

which reduces to $y_1(\xi_1)<\ln\{\frac{\bar{a}}{\bar{b}}\}$. This, together with (3.5) and Lemma 2.1, we get

$$y_1(k)\le y_1(\xi_1)+\sum_{s=0}^{\omega-1}|y_1(s+1)-y_1(s)|<\ln\{\frac{\bar{a}}{\bar{b}}\}+(\bar{A}+\bar{a})\omega:=\rho_1. \tag{3.8}$$

Again from (3.3) and (3.7), it follows that

$$\bar{a}\omega>\sum_{k=0}^{\omega-1}\frac{u(k)}{\exp\{y_1(k)\}}>\sum_{k=0}^{\omega-1}\frac{u(k)}{\exp\{y_1(\eta_1)\}}=\bar{u}\omega\exp\{-y_1(\eta_1)\},$$

which implies

$$y_1(\eta_1) > \ln\{\frac{\bar{u}}{\bar{a}}\}.$$

Therefore, by Lemma 2.1 and (3.5), we obtain

$$y_1(k) \geq y_1(\eta_1) - \sum_{s=0}^{\omega-1} |y_1(s+1) - y_1(s)| > \ln\{\frac{\bar{u}}{\bar{a}}\} - (\bar{A} + \bar{a})\omega := \rho_2. \tag{3.9}$$

From (3.4), (3.7) and (3.8), we can derive that

$$\begin{aligned}
\bar{d}\omega &< \sum_{k=0}^{\omega-1} \frac{f(k)\exp\{y_1(k)\}}{m(k)\exp\{y_2(k)\}} < \sum_{k=0}^{\omega-1} \frac{f(k)\exp\{y_1(k)\}}{m(k)\exp\{y_2(\xi_2)\}} \\
&< \frac{1}{\exp\{y_2(\xi_2)\}} \frac{\bar{a}}{\bar{b}} \overline{\left(\frac{f}{m}\right)} \exp\{(\bar{A} + \bar{a})\omega\}\omega,
\end{aligned}$$

then

$$y_2(\xi_2) < \ln\left\{\frac{\bar{a}}{\bar{b}\bar{d}} \overline{\left(\frac{f}{m}\right)}\right\} + (\bar{A} + \bar{a})\omega. \tag{3.10}$$

From (3.6), (3.10) and using Lemma 2.1, one can easily obtain

$$\begin{aligned}
y_2(k) &\leq y_2(\xi_2) + \sum_{s=0}^{\omega-1} |y_2(s+1) - y_2(s)| \\
&< \ln\left\{\frac{\bar{a}}{\bar{b}\bar{d}} \overline{\left(\frac{f}{m}\right)}\right\} + (\bar{A} + \bar{a} + \bar{D} + \bar{d})\omega := \rho_3.
\end{aligned} \tag{3.11}$$

Moreover, from (3.4) and (3.9), we get

$$\begin{aligned}
\bar{d}\omega &> \sum_{k=0}^{\omega-1} \frac{f(k)\exp\{y_1(k)\}}{m^M\exp\{y_2(\eta_2)\} + \exp\{y_1(k)\}} \\
&> \sum_{k=0}^{\omega-1} \frac{f(k)(\bar{u}/\bar{a})\exp\{-(\bar{A} + \bar{a})\omega\}}{m^M\exp\{y_2(\eta_2)\} + (\bar{u}/\bar{a})\exp\{-(\bar{A} + \bar{a})\omega\}},
\end{aligned}$$

which leads to

$$y_2(\eta_2) > \ln\left\{\frac{(\bar{f} - \bar{d})\bar{u}}{m^M\bar{a}\bar{d}}\right\} - (\bar{A} + \bar{a})\omega.$$

Consequently,

$$\begin{aligned}
y_2(k) &\geq y_2(\eta_2) - \sum_{s=0}^{\omega-1} |y_2(s+1) - y_2(s)| \\
&> \ln\left\{\frac{(\bar{f} - \bar{d})\bar{u}}{m^M\bar{a}\bar{d}}\right\} - (\bar{A} + \bar{a} + \bar{D} + \bar{d})\omega := \rho_4.
\end{aligned} \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$|y_2(k)| < |\rho_3| + |\rho_4| + \rho := B_1, \tag{3.13}$$

where ρ is a positive constant. Clearly, $\rho_1, \rho_2, B_1, \delta_-, \delta_+$ are independent of λ.

From (3.7) and the first equality of (3.1), we also have

$$a(\eta_1)-b(\eta_1)\exp\{y_1(\eta_1)\}-\frac{c(\eta_1)\exp\{y_2(\eta_1)\}}{m(\eta_1)\exp\{y_2(\eta_1)\}+\exp\{y_1(\eta_1)\}}-\frac{u(\eta_1)}{\exp\{y_1(\eta_1)\}}<0,$$

which reduces to

$$b(\eta_1)\exp\{2y_1(\eta_1)\}-\left(a(\eta_1)-\frac{c(\eta_1)}{m(\eta_1)}\right)\exp\{y_1(\eta_1)\}+u(\eta_1)>0.$$

Solving the inequality, we have

$$\exp\{y_1(\eta_1)\}<\frac{(a-c/m)^L-\sqrt{[(a-c/m)^L]^2-4b^Mu^M}}{2b^M}:=\delta_-,$$

or

$$\exp\{y_1(\eta_1)\}>\frac{(a-c/m)^L+\sqrt{[(a-c/m)^L]^2-4b^Mu^M}}{2b^M}:=\delta_+.$$

Similarly, we can obtain $\exp\{y_1(\xi_1)\}<\delta_-$ or $\exp\{y_1(\xi_1)\}>\delta_+$. Then, by (3.8) and (3.9), we get

$$\rho_2<y_1(k)<\ln\delta_-,\ \text{or}\ \ln\delta_+<y_1(k)<\rho_1,\ k\in I_\omega. \tag{3.14}$$

By Lemma 2.2, there exists two solutions of (3.1), denoted by $(y_1^1,y_2^1)^T$, $(y_1^2,y_2^2)^T$ $(y_1^1<y_1^2)$. Then by (3.14), we have

$$\rho_2<y_1^1<\ln\delta_-,\ \text{or}\ \ln\delta_+<y_1^2<\rho_1,\ k\in I_\omega. \tag{3.15}$$

Now let us take $X=Y=l^\omega$, $(Ly)(k)=y(k+1)-y(k)$, and

$$(Ny)(k)=\begin{bmatrix} a(k)-b(k)\exp\{y_1(k)\}-\dfrac{c(k)\exp\{y_2(k)\}}{m(k)\exp\{y_2(k)\}+\exp\{y_1(k)\}} \\ -\dfrac{u(k)}{\exp\{y_1(k)\}}-d(k)+\dfrac{f(k)\exp\{y_1(k)\}}{m(k)\exp\{y_2(k)\}+\exp\{y_1(k)\}}\end{bmatrix},$$

for any $y\in X$ and $k\in\mathbf{Z}$. Then by Lemma 3.1, L is a bounded linear operator and $\operatorname{Ker}L=l_c^\omega$, $\operatorname{Im}L=l_0^\omega$. Let $\operatorname{Coker}=Y/\operatorname{Im}L$ be the quotient space of Y under the equivalence relation $z\sim z'\iff z-z'\in\operatorname{Im}L$. Thus, $\operatorname{Coker}L=\{z+\operatorname{Im}L: z\in Y\}$ and $\dim\operatorname{Coker}L=\operatorname{codim}\operatorname{Im}L$. So $\dim\operatorname{Ker}L=2=\operatorname{codim}\operatorname{Im}L$. Therefore, L is a Fredholm operator of index zero by previous definition.

Define projectors P and Q by

$$Py=\frac{1}{\omega}\sum_{s=0}^{\omega-1}y(s),\ y\in X,\ Qz=\frac{1}{\omega}\sum_{s=0}^{\omega-1}z(s),\ z\in Y.$$

Obviously, $\operatorname{Im}P=\operatorname{Ker}L$ and $\operatorname{Im}L=\operatorname{Ker}Q=\operatorname{Im}(I-Q)$. Furthermore, the generalized inverse (to L) is as follows

$$K_P:\operatorname{Im}L\to\operatorname{Dom}L\cap\operatorname{Ker}P,\ K_P(z)=\sum_{s=0}^{\omega-1}z(s)-\frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega-s)z(s).$$

Take $B_2 = |y_2^1| + |y_2^2|$, and define

$$\Omega_1 = \{y = (y_1, y_2)^T \in X : \rho_2 < y_1(k) < \ln \delta_-, \|y_2\| < B_1 + B_2\},$$
$$\Omega_2 = \{y = (y_1, y_2)^T \in X : \ln \delta_+ < y_1(k) < \rho_1, \|y_2\| < B_1 + B_2\}.$$

Both Ω_1 and Ω_2 are open subsets of X. Since $\delta_- < \delta_+$, we have $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \phi$. From (3.15), we see that $(y_1^1, y_2^1)^T \in \Omega_1, (y_1^2, y_2^2)^T \in \Omega_2$.

Note that both QN and $K_P(I-Q)N$ are continuous. Since X is a finite dimensional Banach space and both QN and $K_P(I-Q)N$ map bounded continuous functions to bounded continuous functions, then by the Arzela-Ascoli theorem, we see that $QN(\Omega_i)$ and $K_P(I-Q)N(\Omega_i)$, $i = 1, 2$, are relatively compact for any open bounded set $\Omega_i \in X$. Therefore, N is L-compact on $\Omega_i, i = 1, 2$.

Since we are concerned with the periodic solutions, $y = (y_1, y_2)^T$ confined in $\operatorname{Dom} L$, system (3.2) can be regarded as the following operator equation $Ly = \lambda Ny$, which is system (3.1) when $\lambda = 1$. According to the previous estimation of periodic solution of (3.2), we have proven the requirement (i) of Theorem A.

When $y \in \partial\Omega_i \cap \ker L, i = 1, 2, y = \{(y_1, y_2)^T\}$ and $(y_1, y_2)^T$ is a constant vector in $\mathbf{R}^2$. From (2.3) and (3.14) and Lemma 2.1, it follows that

$$QNy = \begin{bmatrix} \bar{a} - \bar{b}\exp\{y_1\} - \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \dfrac{c(k)\exp\{y_2\}}{m(k)\exp\{y_2\} + \exp\{y_1\}} - \dfrac{\bar{u}}{\exp\{y_1\}} \\ \\ -\bar{d} + \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \dfrac{f(k)\exp\{y_1\}}{m(k)\exp\{y_2\} + \exp\{y_1\}} \end{bmatrix} \neq 0.$$

Moreover, direct calculation shows that

$$\deg(JQN, \Omega_i \cap \operatorname{Ker} L, 0) \neq 0, \ i = 1, 2,$$

where $\deg(\cdot)$ is the Brouwer degree and the J is the identity mapping since $\operatorname{Im} Q = \operatorname{Ker} L$.

By now, we have proved that each $\Omega_i (i = 1, 2)$ satisfies all the requirements of Theorem A. Hence, system (3.1) has at least one ω-periodic solution in each of Ω_1 and Ω_2. The proof is completed. □

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