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SOME ALGEBRA FOR GENERALIZED PLANCK RANDOM VARIABLES

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ABSTRACT. The exact distributions of $X+Y$, XY and $X/(X+Y)$ are derived when X and Y are independent generalized Planck random variables.

1. Introduction

Let X and Y be independent *generalized Planck* random variables specified by the probability density functions (pdfs)

$$f_X(x) = \frac{Cx^{\lambda-1}\exp(-ax)}{1-k\exp(-bx)} \tag{1}$$

and

$$f_Y(y) = \frac{Dy^{\mu-1}\exp(-\alpha y)}{1-j\exp(-\beta y)}, \tag{2}$$

respectively, for $x>0$ and $y>0$, where $C=C(a,b,\lambda,k)$ and $D=D(\alpha,\beta,\mu,j)$ denote normalizing constants. The parameter ranges are given by either $a>0$, $b>0$, $\lambda>0$ and $-1 \leq k < 1$ ($\alpha>0$, $\beta>0$, $\mu>0$ and $-1\leq j<1$) or $a>0$, $b>0$, $\lambda>1$ and $k=1$ ($\alpha>0$, $\beta>0$, $\mu>1$ and $j=1$).

The aim of this note is to derive the exact distributions of $X+Y$, XY and $X/(X+Y)$. The results are organized as follows: some basic properties of (1) and (2) are derived in Section 2; the exact expressions for the pdfs of the sum, product and ratio are given in Section 3; finally, the corresponding moments properties are derived in Section 4.

The calculations of this note use several special functions, including the modified Bessel function of the third kind defined by

$$K_\nu(x) = \frac{x^\nu\Gamma(1/2)}{2^\nu\Gamma(\nu+1/2)}\int_1^\infty \exp(-xt)\left(t^2-1\right)^{\nu-1/2}dt,$$

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the confluent hypergeometric function defined by

$$ {}_1F_1\left(a;b;x\right) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!}, $$

the Gauss hypergeometric function defined by

$$ {}_2F_1\left(a,b;c;x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} $$

and, the Kummer function defined by

$$ \begin{aligned} \Psi\left(a,b;x\right) = & \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \, {}_1F_1(a;b;x) \\ & + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} \, {}_1F_1(1+a-b;2-b;x), \end{aligned} $$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. The properties of the above special functions can be found in Prudnikov *et al.* [3] and Gradshteyn and Ryzhik [1].

2. Some properties of the GP distribution

A random variable X is said to have the generalized Planck (GP) distribution if its pdf is given by

$$ f(x) = \frac{C x^{\lambda-1} \exp(-ax)}{1 - k\exp(-bx)} \tag{3} $$

for $x > 0$, where $C = C(a, b, \lambda, k)$ denotes a normalizing constant. This generalizes the standard Planck distribution with pdf

$$ f(x) = \frac{a^\lambda}{\Gamma(\lambda)\eta(\lambda)} \frac{x^{\lambda-1}}{\exp(ax) - 1} $$

for $x > 0$, where $a > 0$, and $\lambda > 1$ (Section 33.6.1, Johnson and Kotz, [2]). The parameter ranges of (3) are given by either $a > 0$, $b > 0$, $\lambda > 0$ and $-1 \leq k < 1$ or $a > 0$, $b > 0$, $\lambda > 1$ and $k = 1$.

The normalizing constant C in (3) can be determined as follows. After substituting $y = bx$, the integral of $f(x)$ becomes

$$ \int_0^\infty f(x)dx = \frac{C}{b^\lambda} \int_0^\infty \frac{y^{\lambda-1}\exp\left(-ay/b\right)}{1-k\exp(-y)} dy. $$

The integral on the right can be evaluated by using equation (3.411.6) in Gradshteyn and Ryzhik [1] to yield

$$ \int_0^\infty \frac{y^{\lambda-1}\exp\left(-ay/b\right)}{1-k\exp(-y)} dy = \Gamma(\lambda)\Psi\left(k, \lambda, \frac{a}{b}\right), $$

see Section 9.55 of Gradshteyn and Ryzhik [1] for detailed properties of this special function. Thus, the normalizing constant C becomes

$$C(a, b, \lambda, k) = \frac{b^{\lambda}}{\Gamma(\lambda)\Psi\left(k, \lambda, \frac{a}{b}\right)}. \tag{4}$$

The GP distribution is very flexible. Its particular cases include: the standard Planck distribution for $a = b$ and $k = 1$, the gamma distribution for $k = 0$, and the exponential distribution for $k = 0$ and $\lambda = 1$. Using the expansion

$$\frac{1}{1 - k\exp(-bx)} = \sum_{j=0}^{\infty} k^j \exp(-jbx),$$

one can write $f(x)$ as

$$f(x) = \frac{b^{\lambda}}{\Psi\left(k, \lambda, \frac{a}{b}\right)} \sum_{j=0}^{\infty} \frac{k^j}{(a + jb)^{\lambda}} \frac{(a + jb)^{\lambda}}{\Gamma(\lambda)} x^{\lambda - 1} \exp\left\{-(a + jb)x\right\} \tag{5}$$

for $x > 0$. Thus, we also note that the GP distribution can be represented as a mixture of gamma distributions.

The standard Planck distribution and its original Planck formula (in Physics) have attracted applications in many areas, including aerospace engineering, astronomy, astrophysics, black body radiation, chemical and biochemical engineering, pharmaceutical sciences, transport barrier studies, and water science and technology. It is our hope that the generalized Planck distribution given by (3) will be adopted as a better model in these areas and more.

For the GP distribution

$$\frac{d\log f(x)}{dx} = \frac{\lambda - 1}{x} - a - \frac{kb}{\exp(bx) - k}$$

and

$$\frac{d^2 \log f(x)}{dx^2} = \frac{1 - \lambda}{x^2} + \frac{kb^2 \exp(bx)}{\left\{\exp(bx) - k\right\}^2}.$$

Standard calculations based on these derivatives show that $f(x)$ can exhibit the following shapes:

$-1 < k < 0$	$\lambda < 1$:	Maybe multi-modal, e.g. $a = 1/10$, $b = 1$, $\lambda = 99/100$, $k = -1/3$
$-1 < k \leq 0$	$\lambda \geq 1$:	Unimodal
$0 \leq k < 1$	$\lambda \leq 1$:	Strictly decreasing
$0 < k < 1$	$\lambda > 1$:	Maybe multi-modal, e.g. $a = 1$, $b = 10$, $\lambda = 3/2$, $k = 9/10$
$k = 1$,	$\lambda < 2$:	Maybe multi-modal, e.g. $a = 1$, $b = 10$, $\lambda = 7/4$, $k = 1$
$k = 1$,	$\lambda \geq 2$:	Unimodal

Since $f(x)$ is a density, $f(x) \to 0$ when $x \to \infty$. When $x \to 0$,

$$f(0) = \begin{cases} 0, & \text{if } \lambda > 1, \\ C/(1-k), & \text{if } \lambda = 1, \\ \infty, & \text{if } \lambda < 1 \end{cases}$$

for $\mid k \mid < 1$, and

$$f(0) = \begin{cases} 0, & \text{if } \lambda > 2, \\ C/b, & \text{if } \lambda = 2, \\ \infty, & \text{if } \lambda < 2 \end{cases}$$

for $k = 1$. Call x_0 a solution to the equation

$$\frac{\lambda - 1}{x} - a = \frac{kb}{\exp(bx) - k}.$$

Then $\max(0, x_0)$ is a mode (or the mode in case of unimodality) of the GP distribution. In some special cases, the expression simplifies and a closed form can be found for the mode. For instance, if $k = 0$ then $x_0 = (\lambda - 1)/a$ and if $\lambda = 1$ then $x_0 = \log\{k(1 - b/a)\}/b$ for suitable a, b and k.

Using the gamma-mixture representation (5), one can write the cdf of the GP distribution as

$$F(x) = C \sum_{j=0}^{\infty} k^j \int_0^x y^{\lambda-1} \exp\{-(a+jb)y\}\, dy.$$

On substituting $z = (a + jb)y$, the above reduces to

$$F(x) = C \sum_{j=0}^{\infty} \frac{k^j \gamma(\lambda, (a+jb)x)}{(a+jb)^{\lambda}}. \tag{6}$$

Thus, we have expressed the cdf of GP distribution as a weighted infinite sum of incomplete gamma functions – compare with equation (64) in Johnson and Kotz [2].

The nth moment of a random variable X having the GP distribution is

$$\mathrm{E}\left(X^n\right) = \frac{C}{b^{n+\lambda}} \int_0^{\infty} \frac{x^{n+\lambda-1} \exp(-ax/b)}{1 - k\exp(-x)} dx.$$

Using equation (3.411) in Gradshteyn and Ryzhik [1], the integral on the right can be evaluated as

$$\int_0^{\infty} \frac{x^{n+\lambda-1} \exp(-ax/b)}{1 - k\exp(-x)} dx = \Gamma(n+\lambda)\Psi\left(k, n+\lambda, \frac{a}{b}\right).$$

Hence, using (4) one can express the nth moment in terms of the ratio of two Lerch functions:

$$\mathrm{E}\left(X^n\right) = C b^{-n-\lambda}\Gamma(n+\lambda)\Psi\left(k, n+\lambda, \frac{a}{b}\right), \tag{7}$$

compare with equation (63) in Johnson and Kotz [2].

3. Exact distributions of the sum and ratio

Here, we determine the pdfs of the sum $R = X + Y$ and the ratio $W = X/(X+Y)$. Transform $(R, W) = (X+Y, X/(X+Y))$. Under this transformation, the joint pdf of (R, W) is

$$\begin{aligned} & f(r,w) \\ = \; & r f_X(rw) f_Y(r(1-w)) \\ = \; & \frac{CDr^{\lambda+\mu-1} w^{\lambda-1}(1-w)^{\mu-1} \exp\{-arw - \alpha r(1-w)\}}{[1 - k\exp(-brw)]\,[1 - j\exp(-\beta r(1-w))]}. \end{aligned} \tag{8}$$

Using the series expansion

$$(1-x)^{-1} = \sum_{m=0}^{\infty} (-1)^m x^m, \tag{9}$$

one can rewrite (8) as

$$\begin{aligned} & f(r,w) \\ = \; & CDr^{\lambda+\mu-1} w^{\lambda-1}(1-w)^{\mu-1} \exp\{-arw - \alpha r(1-w)\} \\ & \times \left\{ \sum_{m=0}^{\infty} k^m \exp(-mbrw) \right\} \left\{ \sum_{n=0}^{\infty} j^n \exp(-n\beta r(1-w)) \right\} \\ = \; & CDr^{\lambda+\mu-1} w^{\lambda-1}(1-w)^{\mu-1} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k^m j^n \\ & \times \exp\left[-r\{(a+mb)w + (\alpha+n\beta)(1-w)\}\right]. \end{aligned} \tag{10}$$

Thus, the pdf of R can be expressed as

$$\begin{aligned} & f_R(r) \\ = \; & CDr^{\lambda+\mu-1} \exp\{-r(\alpha+n\beta)\} \int_0^1 w^{\lambda-1}(1-w)^{\mu-1} \\ & \times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k^m j^n \exp\{-rw(a+mb-\alpha-n\beta)\}\, dw \\ = \; & CDr^{\lambda+\mu-1} \exp\{-r(\alpha+n\beta)\} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k^m j^n \\ & \times \int_0^1 w^{\lambda-1}(1-w)^{\mu-1} \exp\{-rw(a+mb-\alpha-n\beta)\}\, dw \\ = \; & CDr^{\lambda+\mu-1} \exp\{-r(\alpha+n\beta)\} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k^m j^n B(\lambda,\mu) \\ & \times {}_1F_1\left(\lambda; \lambda+\mu; -r(a+mb-\alpha-n\beta)\right), \end{aligned}$$

where the last step follows by application of equation (2.3.6.1) in Prudnikov *et al.* ([3], volume 1). Hence, the pdf of R can be written as

$$\begin{aligned} f_R(r) &= CDB(\lambda,\mu)r^{\lambda+\mu-1}\exp\{-r(\alpha+n\beta)\}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}k^m j^n \\ &\qquad \times {}_1F_1(\lambda;\lambda+\mu;-r(a+mb-\alpha-n\beta)). \end{aligned} \tag{11}$$

The pdf of W is obtained by integrating (10) with respect to r as follows:

$$\begin{aligned} f_W(w) &= CD\int_0^{\infty} r^{\lambda+\mu-1}w^{\lambda-1}(1-w)^{\mu-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}k^m j^n \\ &\qquad \times \exp\left[-r\left\{(a+mb)w+(\alpha+n\beta)(1-w)\right\}\right]dr \\ &= CDw^{\lambda-1}(1-w)^{\mu-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}k^m j^n \\ &\qquad \times\int_0^{\infty} r^{\lambda+\mu-1}\exp\left[-r\left\{(a+mb)w+(\alpha+n\beta)(1-w)\right\}\right]dr \\ &= CDw^{\lambda-1}(1-w)^{\mu-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}k^m j^n \\ &\qquad \times\Gamma(\lambda+\mu)\left\{(a+mb)w+(\alpha+n\beta)(1-w)\right\}^{-n}. \end{aligned}$$

Thus, one can express the pdf of W as

$$\begin{aligned} f_W(w) &= CD\Gamma(\lambda+\mu)w^{\lambda-1}(1-w)^{\mu-1} \\ &\qquad \times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}k^m j^n\left\{(a+mb)w+(\alpha+n\beta)(1-w)\right\}^{-n}. \end{aligned} \tag{12}$$

4. Exact distribution of the product

Here, we determine the pdf of the product $P = XY$. Transform $(X,P) = (X,XY)$. Under this transformation, the joint pdf of (X,P) can be expressed as

$$\begin{aligned} f(x,p) &= (1/x)f_X(x)f_Y(p/x) \\ &= \frac{CDx^{\lambda-\mu-1}\exp(-ax-\alpha p/x)}{[1-k\exp(-bx)]\,[1-j\exp(-\beta p/x)]}. \end{aligned} \tag{13}$$

Using the series expansion (9), one can rewrite (13) as

$$\begin{aligned} f(x,p) &= CDx^{\lambda-\mu-1}\exp\left(-ax-\alpha p/x\right) \\ &\quad \times\left\{\sum_{m=0}^{\infty} k^m \exp(-mbx)\right\}\left\{\sum_{n=0}^{\infty} j^n \exp(-n\beta p/x)\right\} \\ &= CDx^{\lambda-\mu-1}\exp\left(-ax-\alpha p/x\right) \\ &\quad \times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k^m j^n \exp\left[-mbx-n\beta p/x\right]. \end{aligned}$$

Thus, the pdf of P can be expressed as

$$\begin{aligned} & f_P(p) \\ &= CD\int_0^{\infty} x^{\lambda-\mu-1}\exp\left(-ax-\alpha p/x\right) \\ &\quad \times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k^m j^n \exp\left[-mbx-n\beta p/x\right]dx \\ &= CD\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k^m j^n \\ &\quad \times\int_0^{\infty} x^{\lambda-\mu-1}\exp\left\{-(a+mb)x-(\alpha+n\beta)p/x\right\}dx \\ &= CD\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k^m j^n\left\{\frac{(\alpha+n\beta)p}{a+mb}\right\}^{(\lambda-\mu)/2} \\ &\quad \times K_{\lambda-\mu}\left(2\sqrt{(a+mb)(\alpha+n\beta)p}\right), \end{aligned}$$

where the last step follows by application of equation (2.3.16.1) in Prudnikov *et al.* ([3], volume 1). Hence, the pdf of P can be written as

$$\begin{aligned} & f_P(p) \\ &= CDp^{(\lambda-\mu)/2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k^m j^n\left(\frac{\alpha+n\beta}{a+mb}\right)^{(\lambda-\mu)/2} \\ &\quad \times K_{\lambda-\mu}\left(2\sqrt{(a+mb)(\alpha+n\beta)p}\right). \end{aligned} \tag{14}$$

5. Moments

The moments of R, W and P can be easily determined. Using (7), the moments of R and P can be expressed as

$$E\left(R^p\right) = E\left((X+Y)^p\right)$$

$$
\begin{aligned}
&= \sum_{q=0}^{p} \binom{p}{q} E\left(X^q Y^{p-q}\right) \\
&= \sum_{q=0}^{p} \binom{p}{q} E\left(X^q\right) E\left(Y^{p-q}\right) \\
&= CDb^{-\lambda}\beta^{-\mu} \sum_{q=0}^{p} \binom{p}{q} b^{-q}\beta^{q-p}\Gamma(q+\lambda)\Gamma(p-q+\mu) \\
&\quad \times \Psi\left(k, q+\lambda, \frac{a}{b}\right) \Psi\left(j, p-q+\mu, \frac{\alpha}{\beta}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left(P^p\right) \\
&= E\left((XY)^p\right) \\
&= E\left(X^p\right) E\left(Y^p\right) \\
&= CDb^{-p-\lambda}\beta^{-p-\mu}\Gamma(p+\lambda)\Gamma(q+\mu)\Psi\left(k, p+\lambda, \frac{a}{b}\right) \Psi\left(j, p+\mu, \frac{\alpha}{\beta}\right).
\end{aligned}
$$

Using (12), the moment of W can be expressed as

$$
\begin{aligned}
E\left(W^p\right) &= \int_0^1 CD\Gamma(\lambda+\mu) w^{p+\lambda-1}(1-w)^{\mu-1} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k^m j^n \\
&\quad \times \left\{(a+mb)w + (\alpha+n\beta)(1-w)\right\}^{-n} dw \\
&= CD\Gamma(\lambda+\mu) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k^m j^n \int_0^1 w^{p+\lambda-1}(1-w)^{\mu-1} \\
&\quad \times \left\{(a+mb)w + (\alpha+n\beta)(1-w)\right\}^{-n} dw \\
&= CD\Gamma(\lambda+\mu) B(p+\lambda, \mu) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{k^m j^n}{(\alpha+\beta n)^n} \\
&\quad \times {}_2F_1\left(p+\lambda, n; p+\lambda+\mu; \frac{\alpha+\beta n - a - bm}{\alpha+\beta n}\right),
\end{aligned}
$$

where the last step follows by application of equation (2.2.6.1) in Prudnikov *et al.* ([3], volume 1).

Figures 1, 2 and 3 illustrate possible shapes of the pdfs (11), (12) and (14) for a range of values of λ, μ, j, k, a, b, α and β. The four curves in each plot correspond to selected values of μ. The effect of the parameters is evident.

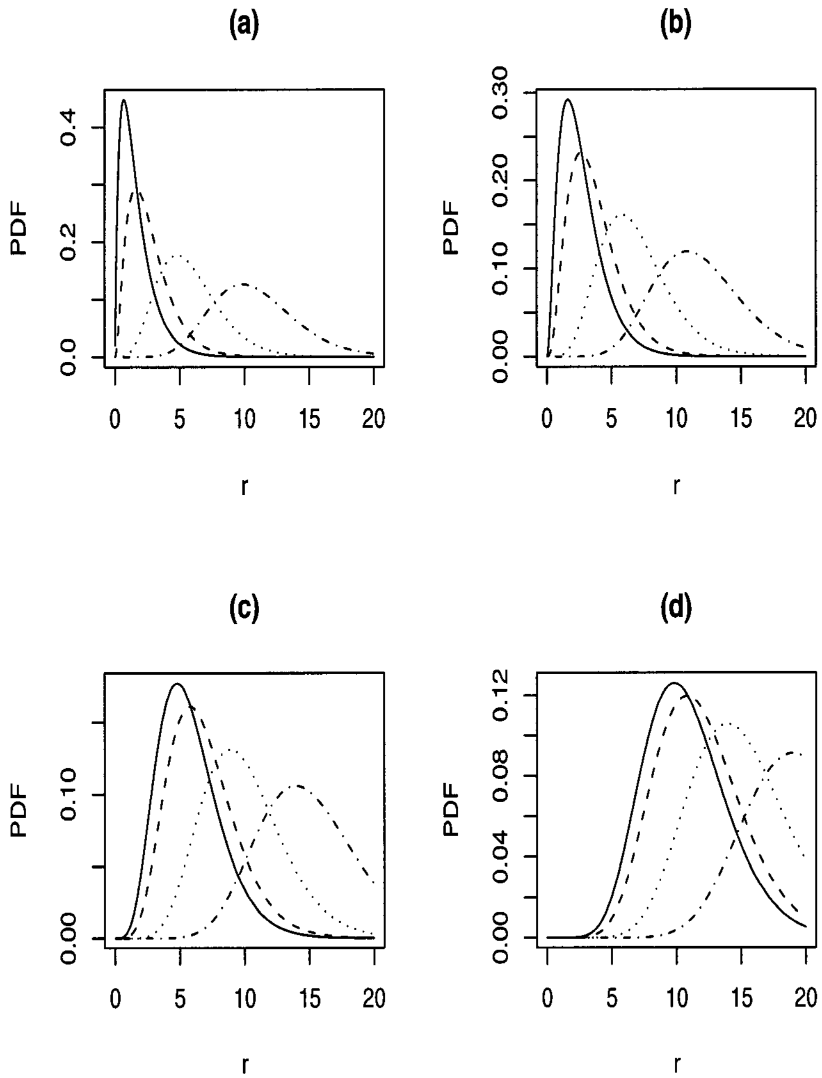

Figure 1. Plots of the pdf (11) for $a = 1$, $b = 1$, $\alpha = 1$, $\beta = 1$, $k = 0.5$, $j = 0.5$, (a): $\lambda = 1$; (b): $\lambda = 2$; (c): $\lambda = 5$; and, $\lambda = 10$. The four curves in each plot correspond to $\mu = 1$ (solid curve), $\mu = 2$ (curve of dots), $\mu = 5$ (curve of dashes), and $\mu = 10$ (curve of dots and dashes).

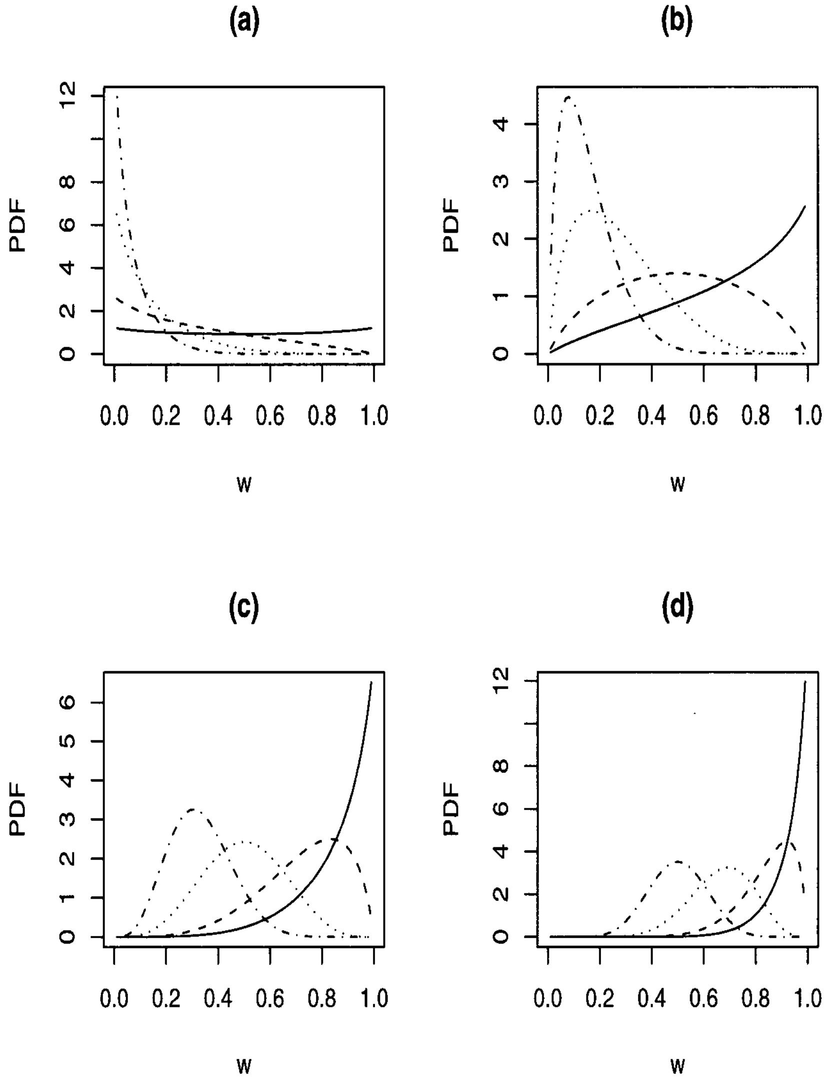

Figure 2. Plots of the pdf (12) for $a = 1$, $b = 1$, $\alpha = 1$, $\beta = 1$, $k = 0.5$, $j = 0.5$, (a): $\lambda = 1$; (b): $\lambda = 2$; (c): $\lambda = 5$; and, $\lambda = 10$. The four curves in each plot correspond to $\mu = 1$ (solid curve), $\mu = 2$ (curve of dots), $\mu = 5$ (curve of dashes), and $\mu = 10$ (curve of dots and dashes).

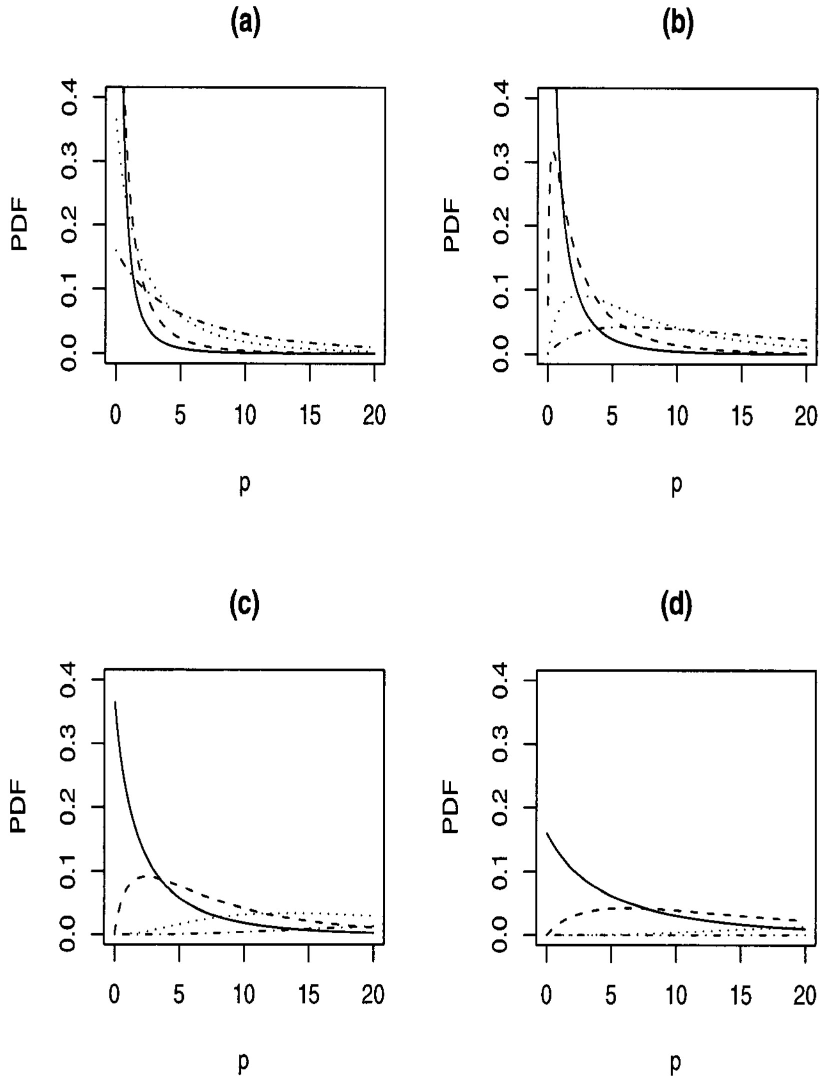

Figure 3. Plots of the pdf (14) for $a = 1$, $b = 1$, $\alpha = 1$, $\beta = 1$, $k = 0.5$, $j = 0.5$, (a): $\lambda = 1$; (b): $\lambda = 2$; (c): $\lambda = 5$; and, $\lambda = 10$. The four curves in each plot correspond to $\mu = 1$ (solid curve), $\mu = 2$ (curve of dots), $\mu = 5$ (curve of dashes), and $\mu = 10$ (curve of dots and dashes).

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