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A DISTRIBUTION ON $\mathbb{Z}_p$

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ABSTRACT. In this paper we explicitly compute the p-adic order of

$\log_p(1-\zeta_{p^n})$.

1. Introduction

The p-adic L functions interpolate the special values of the Dirichlet L-function and provide the analytic side of Iwasawa Theory. These p-adic L-functions can be constructed by Gamma transform of measures on $\mathbb{Z}_p$ (See Sinnott [1]). A distribution on $\mathbb{Z}_p$ with values in $\mathbb{C}_p$is a finitely additive function on the collection of compact open subsets of $\mathbb{Z}_p$ and a measure on $\mathbb{Z}_p$ is a distribution on $\mathbb{Z}_p$ with bounded values in $\mathbb{C}_p$. Let a be an integer which is not a multiple of p. If we define $\mu(a+p^n\mathbb{Z}_p)=\log_p(1-\zeta_{p^n}^a)$ and $\mu(p\mathbb{Z}_p)=0$, then, by a simple computation, we see that μ becomes a distribution on $\mathbb{Z}_p$. In this paper we will prove that the distribution μ is not a measure. In other words, we will show that the p-adic values of the distribution μ are not bounded.

Throughout this paper, p is an odd prime number, ζ_p is a primitive p-th root of unity and ord_p is an order on $\mathbb{C}_p$ such that $\text{ord}_p(p)=1$.

Theorem 1. *Let $p>3$ and n be a positive integer. Then we have*

$$\text{ord}_p(\log_p(1-\zeta_{p^n}))=\frac{2}{p-1}-n+1.$$

This holds also for $p=3$ when $n=1$.

2. Proof of theorems

Proof. First we prove Theorem 1 when $n=1$. Assume that $p>3$.
For $i=2,4,\ldots,p-3$, we have the following formula [2]

$$\sum_{a=1}^{p-1}\omega^{p-1-i}(a)\log_p(1-\zeta_p{}^a)=\frac{-p}{\tau(\omega^i)}L_p(1,\omega^i).$$

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Moreover we have

$$\sum_{a=1}^{p-1} \log_p(1-\zeta_p{}^a) = 0.$$

Adding up the above formula, we have

$$\sum_{i=2}^{p-1}(\sum_{a=1}^{p-1} \omega^{p-1-i}(a)\log_p(1-\zeta_p{}^a) = \sum_{i=2}^{p-3} \frac{-p}{\tau(\omega^i)} L_p(1,\omega^i).$$

Note that the left hand side of the above formula becomes

$$\frac{p-1}{2}\log_p(1-\zeta_p).$$

Hence we have the following formula

$$\log_p(1-\zeta_p) = \frac{-2p}{p-1}\sum_{i=2}^{p-3} \frac{1}{\tau(\omega^i)} L_p(1,\omega^i).$$

Moreover we have

$$\mathrm{ord}_p(\tau(\omega^i)) = \mathrm{ord}_p(\tau(\omega^{-(p-1-i)})) = \frac{p-1-i}{p-1}.$$

Hence we have the following formula

$$\begin{aligned}\log_p(1-\zeta_p) =& u_2(1-\zeta_p)^2 L_p(1,\omega^2)\\ &+ u_4(1-\zeta_p)^4 L_p(1,\omega^2) + \cdots + u_{p-3}(1-\zeta_p)^{p-3} L_p(1,\omega^{p-3}),\end{aligned}$$

where u_i is a unit for $i = 2, 4, \ldots, p-3$. We know that $\mathrm{ord}_p(L_p(1,\omega^i)) \geq 0$ for $i = 2, 4, \ldots, p-3$ and

$$L_p(1,\omega^2) \overset{\mathrm{mod}\ p}{\equiv} L_p(-1,\omega^2) = -(1-p)\frac{B_2}{2} = \frac{1-p}{12},$$

which concludes the proof for $p > 3$. For $p = 3$, write $1-\zeta_3 = \sqrt{3}i(1+\sqrt{3}\zeta_3 i)$. Then

$$\begin{aligned}&\log_3(1-\zeta_3) = \log_3(1+\sqrt{3}\zeta_3 i)\\ &= \sqrt{3}\zeta_3 i - \frac{(\sqrt{3}\zeta_3 i)^2}{2} + \frac{(\sqrt{3}\zeta_3 i)^3}{3} + \cdots.\end{aligned}$$

Note that

$$\mathrm{ord}_3(\frac{(\sqrt{3}\zeta_3 i)^n}{n}) \geq 2$$

for $n \geq 4$. Hence

$$\begin{aligned}&\mathrm{ord}_3(\log_3(1-\zeta_3))\\ &= \mathrm{ord}_3(\sqrt{3}\zeta_3 i - \frac{(\sqrt{3}\zeta_3 i)^2}{2} + \frac{(\sqrt{3}\zeta_3 i)^3}{3})\\ &= \mathrm{ord}_3(\sqrt{3}(1-\zeta_3)(\frac{\zeta_3{}^2 i}{2})) = 1,\end{aligned}$$

which completes the proof for $p = 3$.

Next we prove Theorem 1 for $n \geq 1$. Now assume that $p > 3$. First consider the following formula

$$(1-\zeta_{p^n})^{p^{n-1}} = (1-\zeta_p) + p\alpha,$$

where $\operatorname{ord}_p(\alpha) \geq 0$. Write $1-\zeta_p = \pi$. Note that

$$\operatorname{ord}_p(\log_p(1+x)) = \operatorname{ord}_p(x)$$

when $\operatorname{ord}_p(x) > \frac{1}{p-1}$. Hence we have

$$\begin{aligned} p^{n-1}\log_p(1-\zeta_{p^n}) &= \log_p(\pi + \pi^{p-1}\beta) \\ &= \log_p(\pi) + \log_p(1+\pi^{p-2}\beta) = \pi^2 u_1 + \pi^{p-2}\gamma, \end{aligned}$$

where $\operatorname{ord}_p(\beta), \operatorname{ord}_p(\gamma) \geq 0, \operatorname{ord}_p(u_1) = 0$, which completes the proof of Theorem 1. □

References

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