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EXISTENCE OF FUZZY IDEALS WITH ADDITIONAL CONDITIONS IN BCK/BCI-ALGEBRAS

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ABSTRACT. We give an answer to the following question:

Question. Let S be a subset of $[0,1]$ containing a maximal element $m > 0$ and let $C := \{I_t \mid t \in S\}$ be a decreasing chain of ideals of a BCK/BCI-algebra X. Then does there exists a fuzzy ideal μ of X such that $\mu(X) = S$ and $C_\mu = C$?

1. INTRODUCTION

The study of BCK-algebras was initiated by Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Also, Iséki introduced the notion of BCI-algebras which is a generalization of BCK-algebras. Since then a great deal of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. For the general development of BCK/BCI-algebras the ideal theory plays an important role. Zadeh introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches. In 1991, Xi [9] applied the fuzzy set to BCK-algebras, and he introduced the notion of fuzzy ideals, which has an important role for improving the theory of BCK/BCI-algebras. Since then a great deal of literature has been produced on the theory of fuzzy ideals of BCK/BCI-algebras (see [1, 2, 3, 4, 5, 6, 8]). In this paper, we discuss the existence of fuzzy ideals with additional conditions in BCK/BCI-algebras.

2. PRELIMINARIES

Throughout the paper, a partially ordered set, *poset*, $(P, \leq)$, is a nonempty set P endowed with a reflexive anti-symmetric and transitive relation $\leq$. A poset is often

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denoted by the underlying set P only. If more than one poset is considered, we denote all (generally different) order relations by the same symbol $\leq$. If $(P, \leq)$ and $(Q, \leq)$ are two posets, then the map $f : P \to Q$ is said to be *isotone* if it preserves the order, i.e., if for any $p, q \in P$

$$p \leq q \text{ implies } f(p) \leq f(q). \tag{2.1}$$

The map $f : P \to Q$ is said to be *anti-isotone* if

$$p \leq q \text{ implies } f(q) \leq f(p). \tag{2.2}$$

$(P, \leq)$ and $(Q, \leq)$ are said to be *isomorphic* if there is a bijection $f : P \to Q$ such that f and f^{-1} are isotone. If f is a bijection and f and f^{-1} are anti-isotone, then P and Q are said to be *anti-isomorphic.*

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.* If we define a relation $\leq$ on a BCK/BCI-algebra X by

$$x \leq y \text{ if and only if } x * y = 0, \tag{2.3}$$

then $(X, \leq)$ is a poset. A nonempty subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

(i) $0 \in I$,
(ii) $(\forall x \in X)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$.

3. Existence of Fuzzy Ideals

For any fuzzy set μ in a set X, the range (also called image) of μ, denoted by $\mu(X)$, is the set

$$\mu(X) = \{\mu(x) : x \in X\}. \tag{3.1}$$

The level sets of the fuzzy set μ are denoted by μ_t, $t \in [0,1]$, and are given by

$$\mu_t = \{x \in X : \mu(x) \geq t\} = \mu^{-1}[t,1]. \tag{3.2}$$

The collection of all level sets corresponding to the range $\mu(X)$ of μ is denoted by C_μ and is given by

$$C_\mu := \{\mu_t : t \in \mu(X)\}. \tag{3.3}$$

We have the following question:

Question. Let S be a subset of $[0,1]$ containing a maximal element $m > 0$ and let $C := \{I_t \mid t \in S\}$ be a decreasing chain of ideals of a BCK/BCI-algebra X. Then does there exist a fuzzy ideal μ of X such that $\mu(X) = S$ and $C_\mu = C$?

We will give an answer to the above question in this article.

Definition 3.1 ([9]). A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy ideal* of X if it satisfies

(F0) $\mu(0) \geq \mu(x)$ for all $x \in X$.
(F1) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

In what follows let X be a BCK/BCI-algebra unless otherwise specified, and denote by $FI(X)$ the set of all fuzzy ideals of X, that is,

$$FI(X) := \{\text{Fuzzy ideals of } X\}. \tag{3.4}$$

Lemma 3.2 ([9]). *Let μ be a fuzzy set in X. Then $\mu \in FI(X)$ if and only if for every $t \in [0, \mu(0)]$, the level set μ_t is an ideal of X,* which is called a *level ideal.*

Note that the collection C_μ of level ideals corresponding to the range $\mu(X)$ of the fuzzy ideal μ is a chain of ideals in the sense that it is totally ordered by inclusion.

Example 3.3. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra in which the operation $*$ is defined by the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Let μ be a fuzzy set in X given by

(3.5) $$\mu = \begin{pmatrix} 0 & a & b & c & d \\ 0.7 & 0.5 & 0.4 & 0.3 & 0.3 \end{pmatrix}.$$

Then $\mu \in FI(X)$, $\mu(X) = \{0.7, 0.5, 0.4, 0.3\}$ and

$$C_\mu = \{\mu_{0.7}, \mu_{0.5}, \mu_{0.4}, \mu_{0.3}\},$$

which is a chain because $\mu_{0.7} \subsetneq \mu_{0.5} \subsetneq \mu_{0.4} \subsetneq \mu_{0.3}$.

Theorem 3.4. *Let $\mu \in FI(X)$. Then $(C_\mu, \subseteq)$ and $(\mu(X), \leq)$ are anti-isomorphic.*

Proof. Define a map $f : \mu(X) \to C_\mu$ by $f(t) = \mu_t$ for all $t \in \mu(X)$. Obviously f is a bijection, and f and f^{-1} are anti-isotone. Hence we have the desired result. □

Let $\mu \in FI(X)$. Since $\mu(0) \geq \mu(x)$ for all $x \in X$, we have

(3.6) $$\bigcap_{t \in \mu(X)} \mu_t = \mu_{\mu(0)} \in C_\mu.$$

Note that $X \in C_\mu$ if and only if $\inf(\mu(X)) \in \mu(X)$, in this case we obtain

(3.7) $$X = \bigcup_{t \in \mu(X)} \mu_t = \mu_{\inf(\mu(X))} \in C_\mu.$$

Lemma 3.5. *For any $\mu \in FI(X)$ and $t \in \mu(X)$, we have*

(3.8) $$\bigcup_{s \in (t,1] \cap \mu(X)} \mu_s \subsetneq \mu_t.$$

Proof. Clearly, we have

(3.9) $$\bigcup_{s \in (t,1] \cap \mu(X)} \mu_s \subseteq \mu_t.$$

Since $t \in \mu(X)$ and $\mu_t = \mu^{-1}[t, 1]$, there exists $x \in \mu_t$ such that $\mu(x) = t$. If $s \in (t, 1]$, then obviously $x \notin \mu_s = \mu^{-1}[s, 1]$. Hence (3.8) is valid. □

The following example shows that there exist a BCK-algebra with a decreasing chain of ideals (indexed by subsets of [0,1]) not satisfying the equality in (3.8).

Example 3.6. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra in which the operation $*$ is defined by the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	d	0

Let μ be a fuzzy set in X given by

$$\mu = \begin{pmatrix} 0 & a & b & c & d \\ 0.9 & 0.7 & 0.6 & 0.5 & 0.3 \end{pmatrix}. \tag{3.10}$$

Then $\mu \in FI(X)$, $\mu(X) = \{0.9, 0.7, 0.6, 0.5, 0.3\}$ and

$$C_\mu = \{\mu_{0.9}, \mu_{0.7}, \mu_{0.6}, \mu_{0.5}, \mu_{0.3}\},$$

which is a decreasing chain. For $t = 0.5 \in \mu(X)$, we have

$$\bigcup_{s \in (t,1] \cap \mu(X)} \mu_s = \mu_{0.9} \cup \mu_{0.7} \cup \mu_{0.6} = \{0, a, b\} \neq \{0, a, b, c\} = \mu_t.$$

Lemma 3.7. *Let S be a subset of $[0,1]$ containing a maximal element $m > 0$ and let $C := \{I_t \mid t \in S\}$ be a decreasing chain of ideals of X. For any $\mu \in FI(X)$ such that $\mu(X) = S$ and $C_\mu = C$, we have*

$$\bigcup_{r \in (t,1] \cap S} \mu_r = \bigcup_{r \in (s,1] \cap S} I_r \tag{3.11}$$

whenever $\mu_t = I_s$ for some $t, s \in S$, and

$$\{\mu^{-1}(t) \mid t \in S\} = \left\{ I_k \setminus \bigcup_{r \in (k,1] \cap S} I_r \mid k \in S \right\}. \tag{3.12}$$

Proof. Assume that $\mu_t = I_s$ for some $t, s \in S$. Note that $C_\mu = C$ and they are decreasing chains, and so we must have $\mu_m = I_m$. Since $\mu_t = I_s$ for some $t, s \in S$, it follows that either $t = s = m$ or $t, s < m$. If $t = s = m$, then

$$\{\mu_r \mid r \in (t,1] \cap S\} = \emptyset = \{I_r \mid r \in (s,1] \cap S\}$$

and thus (3.11) is valid. Now we assume that $\mu_t = I_s$ for some $t, s \in S \setminus \{m\}$. Then

$$\begin{aligned} \{\mu_r \mid r \in (t,1] \cap S\} &= \{\mu_r \mid r \in S, \mu_r \subsetneq \mu_t\} \\ &= \{I_r \mid r \in S, I_r \subsetneq I_s\} \\ &= \{I_r \mid r \in (s,1] \cap S\}, \end{aligned}$$

which induces (3.11). Since $C_\mu = C$, for every $t \in S$ there exists $k \in S$ such that $\mu_t = I_k$. It follows from (3.11) that

$$\bigcup_{r \in (t,1] \cap S} \mu_r = \bigcup_{r \in (k,1] \cap S} I_r$$

so that

$$\mu^{-1}(t) = \mu_t \setminus \bigcup_{r \in (t,1] \cap S} \mu_r = I_k \setminus \bigcup_{r \in (k,1] \cap S} I_r.$$

Hence

$$\{\mu^{-1}(t) \mid t \in S\} \subseteq \left\{ I_k \setminus \bigcup_{r\in(k,1]\cap S} I_r \mid k \in S \right\}. \tag{3.13}$$

Similarly we prove that for every $k \in S$, there exists $t \in S$ such that

$$I_k \setminus \bigcup_{r\in(k,1]\cap S} I_r = \mu^{-1}(t).$$

Hence

$$\left\{ I_k \setminus \bigcup_{r\in(k,1]\cap S} I_r \mid k \in S \right\} \subseteq \{\mu^{-1}(t) \mid t \in S\}. \tag{3.14}$$

Combining (3.13) and (3.14) induces (3.12). □

Theorem 3.8. *Let S be a subset of $[0,1]$ containing a maximal element $m > 0$ and let $C := \{I_t \mid t \in S\}$ be a decreasing chain of ideals of X. Then there exists $\mu \in FI(X)$ satisfying $\mu(X) = S$ and $C_\mu = C$ if and only if the following conditions holds:*

(1) *For every $t \in S$,*

$$\bigcup_{r\in(t,1]\cap S} I_r \subsetneq I_t. \tag{3.15}$$

(2) *The BCK/BCI-algebra X is the disjoint union*

$$X = \bigcup_{t\in S} \left(I_t \setminus \bigcup_{r\in(t,1]\cap S} I_r \right). \tag{3.16}$$

Proof. Let $t \in S$ be fixed and suppose that there exists $\mu \in FI(X)$ satisfying $\mu(X) = S$ and $C_\mu = C$. Then there exists $s_t \in S$ such that $I_t = \mu_{s_t}$. By Lemma 3.5, we have

$$\bigcup_{r\in(t,1]\cap S} \mu_r \subsetneq \mu_t, \tag{3.17}$$

and so

$$\bigcup_{r\in(t,1]\cap S} I_r = \bigcup_{r\in(s_t,1]\cap S} \mu_r \subsetneq \mu_{s_t} = I_t \tag{3.18}$$

by (3.11). This proves (3.15). Since $\mu(X) = S$ and $C_\mu = C$, it follows from (3.12) that

$$X = \bigcup_{t\in S} \mu^{-1}(t) = \bigcup_{t\in S} \left(I_t \setminus \bigcup_{r\in(t,1]\cap S} I_r \right). \tag{3.19}$$

Thus (3.16) is valid. Conversely assume that (3.15) and (3.16) are true. Note that

$$I_m \setminus \bigcup_{r\in(m,1]\cap S} I_r = I_m \setminus \emptyset = I_m = \bigcap_{t\in S} I_t. \tag{3.20}$$

Let $s \in S$ be fixed. Since C is a decreasing chain, we have

$$\bigcup_{t\in[s,1]\cap S}\left(I_t \setminus \bigcup_{r\in(t,1]\cap S} I_r\right) \subseteq I_s. \tag{3.21}$$

Now let $x \in I_s$. Then there exists $t \in S$ such that $x \in I_t \setminus \bigcup_{r\in(t,1]\cap S} I_r$. Since $x \in I_s$ and C is a decreasing chain, it follows that $t \in [s,1] \cap S$ so that

$$x \in I_t \setminus \bigcup_{r\in(t,1]\cap S} I_r \subseteq \bigcup_{t\in[s,1]\cap S}\left(I_t \setminus \bigcup_{r\in(t,1]\cap S} I_r\right). \tag{3.22}$$

Hence

$$\bigcup_{t\in[s,1]\cap S}\left(I_t \setminus \bigcup_{r\in(t,1]\cap S} I_r\right) = I_s. \tag{3.23}$$

Let $\mu : X \to [0,1]$ be defined by $\mu(x) = t$ if $x \in I_t \setminus I_t^*$, $t \in S$, where $I_t^* = \bigcup_{r\in(t,1]\cap S} I_r$. Since the union in (3.16) is a disjoint union, it follows that μ is well defined on X. Given $t \in S$, the set $I_t \setminus I_t^*$ is nonempty by (3.15). Thus $t \in \mu(X)$, and so $\mu(X) = S$. Now for every $s \in S$, we obtain

$$\begin{aligned} \mu_s = \mu^{-1}[s,1] &= \bigcup_{t\in[s,1]\cap S} \mu^{-1}(t) \\ &= \bigcup_{t\in[s,1]\cap S}\left(I_t \setminus \bigcup_{r\in(t,1]\cap S} I_r\right) = I_s \end{aligned} \tag{3.24}$$

by (3.23). It follows that $C_\mu = C$. Finally we prove that μ is a fuzzy ideal of X. Since $0 \in I_m$, we have $\mu(0) = m \geq \mu(x)$ for all $x \in X$. Let $x, y \in X$ be such that $\mu(x * y) = t_1$ and $\mu(y) = t_2$ for $t_1, t_2 \in S$. We may assume that $t_1 \leq t_2$ without loss of generality. Then $x * y \in I_{t_1} \setminus I_{t_1}^*$ and $y \in I_{t_2} \setminus I_{t_2}^*$. Since $I_{t_2} \subseteq I_{t_1}$ and I_{t_1} is an ideal, it follows that $x \in I_{t_1} \setminus I_{t_1}^*$ so that

$$\mu(x) = t_1 = \min\{\mu(x * y), \mu(y)\}.$$

Hence μ is a fuzzy ideal of X. □

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