Journal of the Society of Korea Industrial and Systems Engineering
Vol, 30, No. 2, pp.83－88, June 2007.

선형함수의 곱의 형태로 표현된 비선형함수의 선형변환 기법에 관한 연구

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Convex Underestimates of Sums of Products of Linear Functions

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본 논문에서 선형함수의 곱의 형태로 표현된 비선형 함수를 목적식 또는 제약식에 가지는 비선형 최적화 문제를 새로운 변수를 추가하여 선형 Relaxation 최적화 문제로 Reformulation하는 기법을 소개한다. 특히, 선형함수의 곱의 형태를 가지는 비선형 함수를 포함하는 비선형 정수 최적화 문제를 선형 정수 최적화 문제로 Relaxation할 경우 두 최적화 문제의 해가 일치함을 보인다. 또한 소개된 Relaxation 기법을 응용하여, 추가되는 변수의 수를 증가시킴으로서, 보다 Tight한 Relaxation 문제를 도출하는 과정에 대하여 소개한다.

Keywords : Nonlinear Optimization, Linear Relaxation, Linear Reformulation, Bi-product

1. Introduction

The purpose of this paper is to derive some results of convex underestimates of sums of products of linear functions that are useful in certain applications. This paper demonstrates extensions of Al-Khayyal and Falk [1], and Sherali and Alameddine [3]. Later Al-Khayyal and Hwang [2] applied this linearization technique to solve network vehicle routing problem efficiently. We show that a particular type of non-linear mixed-integer program can be reformulated into an equivalent mixed-integer linear program under certain conditions, thereby making problem solving relatively easy. Also it compares two alternative relaxation methods and shows one is better than the other in the sense of tighter relaxation.

2. Simplification and Extensions

In this section we will discuss the linearization of the feasible region defined by a special nonlinear equation formed by the product of variables. We will preliminarily start to investigate the linearization technique for the nonlinear form of product of two variables, that is to show whenever the form of xy appears in an optimization problem – whether it is in the ob-

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※ This work was supported by Hanyang University, Korea, made in the program year of 2005.

jective function or in constraints - we can apply the linearization technique introduced hereinafter. Later we extend this technique to the form of $\frac{x}{y}$.

2.1 Relaxation of Product of Two Variables

Consider compact set B defined by $x, y \in \mathbf{R}$.

$$B := \{(x, y) \mid L_x \le x \le U_x, L_y \le y \le U_y\}.$$

It is formed by four constraints of

$$i)\ x - L_x \ge 0 \qquad ii)\ U_x - x \ge 0$$
$$iii)\ y - L_y \ge 0 \qquad iv)\ U_x - y \ge 0.$$

By expending four ways of multiplying *i*) to *iii*), *i*) to *iv*), *ii*) to *iii*), and *ii*) to *iv*), we obtain the set C_I constructed by which are called implied constraints

$$\begin{aligned} C_I := \{ (x, y) : & xy \ge L_x x + L_x y - L_x L_y, \\ & xy \ge U_y x + U_x y - U_x U_y \\ & xy \le U_y x + L_x y - L_x U_y, \\ & xy \le L_y x + U_x y - U_x L_y \}. \end{aligned}$$

Since,

$$\begin{aligned} & x - L_x \ge 0,\ y - L_y \ge 0 \\ & \Rightarrow (x - L_x)(y - L_x) \ge 0 \\ & \Rightarrow xy \ge L_y x + L_x y - L_x L_y, \end{aligned}$$

$$\begin{aligned} & U_x - x \ge 0,\ U_y - y \ge 0 \\ & \Rightarrow (U_x - x)(U_y - y) \ge 0 \\ & \Rightarrow xy \ge U_y x + U_x y - U_x U_y, \end{aligned}$$

$$\begin{aligned} & x - L_x \ge 0,\ U_y - y \ge 0 \\ & \Rightarrow (x - L_x)(U_y - y) \ge 0 \\ & \Rightarrow xy \le U_y x + L_x y - L_x U_y, \end{aligned}$$

$$\begin{aligned} & y - L_y \ge 0,\ U_x - x \ge 0 \\ & \Rightarrow (y - L_y)(U_x - x) \ge 0 \\ & \Rightarrow xy \le L_y x + U_x y - U_x L_y. \end{aligned}$$

Notice that $B \subseteq C_I$.

Now for notational simplicity, let's introduce a notation that

$$[L, U] := \{ (x, y) \mid L_x \le x \le U_x, L_x \le y \le U_y \},$$

where $L := \begin{bmatrix} L_x \\ L_y \end{bmatrix},\ U := \begin{bmatrix} U_x \\ U_y \end{bmatrix}$.

Then $B := \{ (x, y) \mid (x, y) \in [L, U] \}$.

Now consider the set B' in higher dimensional space

$$B' := \{ (x, y, z) \mid (x, y) \in [L, U], (x, y) \in C_I, z = xy \}.$$

Then, set B' has constraints as follows

$$\begin{aligned} & z = xy, \\ & L_x \le x \le U_x, \\ & L_y \le y \le U_y, \\ & z \ge L_y x + L_x y - L_x L_y, \\ & z \ge U_y x + U_x y - U_x U_y, \\ & z \le U_y x + L_x y - L_x L_y, \\ & z \le L_y x + U_x y - U_x L_y. \end{aligned}$$

Notice that the projection B' onto $x-y$ plane is exactly B itself.

Now let the projection of B' onto $x-y$ plane as

$$\mathrm{Proj}_{\mathrm{R}^2} B' := \{(x, y) : (x, y, z) \in B' \text{ for all } z\}.$$

Then, it is clear that $B \subseteq \mathrm{Proj}_{\mathrm{R}^2} B'$ which means that for all $(x, y) \in B$, there exists z such that $(x, y, z) \in B'$.

Notice that $B' \subseteq \{(x, y, z) \mid (x, y) \in C_I, z = xy\}$ so B' has constraints defining C_I which are obtained by substituting $z = xy$, that is

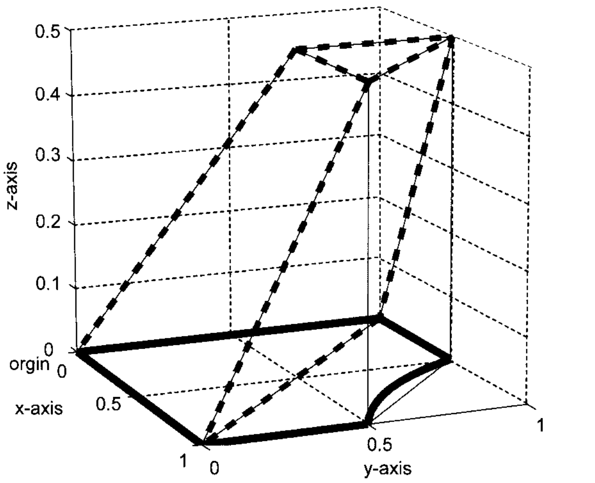
$$z \ge \max\{L_y x + L_x y - L_x L_y,\ U_y x + U_x y - U_x U_y\}, \cdots\cdots (1)$$
$$z \le \min\{U_y x + L_x y - L_x U_y,\ L_y x + U_x y - U_x L_y\}. \cdots\cdots (2)$$

Then, equation(1) and (2) represent convex and concave envelope of xy over $[L, U]$ respectively. So, whenever xy appears in a problem with bounded information for each variable, we can linearize xy by introducing the new variable $z = xy$ and adding constraints C_I with $z = xy$.

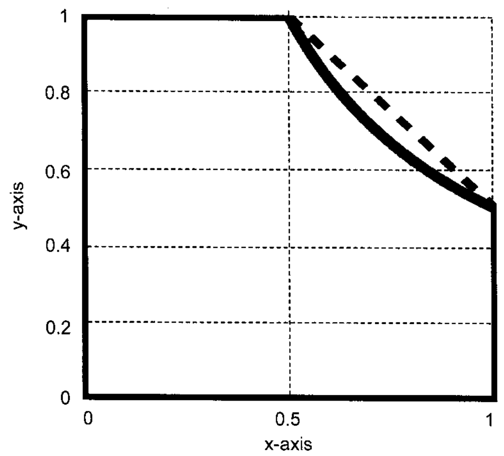
As an example, consider feasible region S with $[L, U] = [0, e]$, where $e = (1, 1)^T$, and the additional constraint $xy \le \frac{1}{2}$ then we can express S' as

$$z \ge \max\{0, x + y - 1\},\ z \le \min\{x, y\},$$
$$z \le \frac{1}{2},\ 0 \le x \le 1,\ 0 \le y \le 1.$$

Now <Figure 1> shows the feasible region S and S'. Set S is two dimensional space in the $x-y$ plane. Set S' is in 3-dimensional space of (x, y, z). As shown in the <Figure 1>, for every point $(x, y) \in S$, there exists z such that $(x, y, z) \in S'$. The projection of S' onto the $x-y$ plane is shown in <Figure 2>. It illustrates why the feasible region S is a subset of the convex set $\mathrm{Proj}_{\mathrm{R}^2} B'$.



<Figure 1> Example of nonlinear feasible region and its convex relaxation



<Figure 2> Original nonlinear feasible region and projected region

2.2 Extension to the form of $\frac{x}{y}$

Consider feasible regions F generated by $x, y \in \mathbf{R}$ for $0 < L_y \leq U_y$ and suppose there exist additional constraints with the term $\frac{x}{y}$. Then, the hyperrectangle

$$L_x \leq x \leq U_x,$$
$$L_y \leq y \leq U_y$$

is equivalent to

$$L_x \leq x \leq U_x,$$
$$\frac{1}{U_y} \leq \frac{1}{y} \leq \frac{1}{L_y}.$$

So we can apply same result of (1) and (2) as follows

$$L_x \leq x \leq U_x,\ L_y \leq y \leq U_y$$
$$z \geq \frac{x}{U_y} + \frac{L_x}{y} - \frac{L_x}{U_y},\ z \geq \frac{x}{L_x} + \frac{U_x}{y} - \frac{U_x}{L_y},$$
$$z \leq \frac{x}{L_y} + \frac{L_x}{y} - \frac{L_x}{L_y},\ z \leq \frac{x}{U_y} + \frac{U_x}{y} - \frac{U_x}{U_y}.$$

By letting $\frac{1}{y} = w$, we can rewrite it as

$$yw = 1,\ L_x \leq x \leq U_x,\ L_y \leq y \leq U_y,$$
$$z \geq \frac{x}{U_y} + L_x w - \frac{L_x}{U_y},\ z \geq \frac{x}{L_y} + U_x w - \frac{U_x}{L_y},$$
$$z \leq \frac{x}{L_y} + L_x w - \frac{L_x}{L_y},\ z \leq \frac{x}{U_y} + U_x w - \frac{U_x}{U_y}.$$

Then, substituting $z' = wy$ and applying same methods with bounds on $w \in \left[\frac{1}{U_y}, \frac{1}{L_y}\right]$, we get

$$z' = 1,\ L_x \leq x \leq U_x,\ L_y \leq y \leq U_y,$$
$$z \geq \frac{x}{U_y} + L_x w - \frac{L_x}{U_y},\ z \geq \frac{x}{L_y} + U_x w - \frac{U_x}{L_y},$$
$$z \leq \frac{x}{L_y} + L_x w - \frac{L_x}{L_y},\ z \leq \frac{x}{U_y} + U_x w - \frac{U_x}{U_y},$$
$$z' \geq L_y w + \frac{y}{L_y} - \frac{L_y}{U_y},\ z' \geq U_y w + \frac{y}{L_y} - \frac{U_y}{L_y},$$
$$z' \leq U_y w + \frac{y}{U_y} - 1,\ z' \leq L_y w + \frac{y}{L_y} - 1.$$

Substituting $z' = 1$ gives

$$L_x \leq x \leq U_x,\quad L_y \leq y \leq U_y,$$
$$z \geq \frac{x}{U_y} + L_x w - \frac{L_x}{U_y},\ z \geq \frac{x}{L_y} + U_x w - \frac{U_x}{L_y},$$
$$z \leq \frac{x}{L_y} + L_x w - \frac{L_x}{L_y},\ z \leq \frac{x}{U_y} + U_x w - \frac{U_x}{U_y},$$
$$1 \geq L_y w + \frac{y}{U_y} - \frac{L_y}{U_y},\ 1 \geq U_y w + \frac{y}{L_y} - \frac{U_y}{L_y},$$
$$2 \leq U_y w + \frac{y}{U_y},\ 2 \leq L_y w + \frac{y}{L_y}.$$

For example, suppose we have constraints with a term $\frac{x}{y}$ in such a compact set defined by

$$0 \leq x \leq 1,$$
$$1 \leq y \leq 2.$$

Applying the result, we get the convex relaxation as follows

$$x - 2z \leq 0,$$
$$x + w - z \leq 1,$$

$$x - z \geq 0,$$
$$x + 2w - 2z \geq 1,$$
$$2w + y \leq 3,$$
$$4w + y \geq 4,$$
$$w + y \geq 2,$$
$$0 \leq x \leq 1,$$
$$1 \leq y \leq 2.$$

If we have a constraint $x + \frac{x}{y} \leq 2$ and an objective function to maximize $x + y$, then we add the constraint $x + z \leq 2$ by substituting $\frac{x}{y}$ to z. The optimal solution to the original problem is $(x^*, y^*) = (1, 2)$ and the optimal solution to the relaxed problem is determined at $(x^*, y^*, z^*, w^*) = (1, 2, \frac{1}{2}, \frac{1}{2})$.

3. Development

In Section 2, we have seen the relaxation technique for the form of product of variables. In this section we will first investigate product form of linear functions as a generalization of the result from Section 2.

Later in this section, we will show that some cases of relaxation give exact reformulation and others give tighter relaxations.

3.1 Product of Linear Functions

Given $f(x)$ and $g(y)$ are linear functions of vector x and y. Consider the feasible region F generated by following constraints

$$f(x)g(y) \leq U,$$
$$L_f \leq f(x) \leq U_f,$$
$$L_g \leq g(y) \leq U_g.$$

Consider the linearization of $f(x)g(y)$ with substituting $f(x)g(y) = z$ as follows and define the feasible region F' as

$$z \leq U,$$
$$L_f \leq f(x) \leq U_f,$$
$$L_g \leq g(y) \leq U_g,$$
$$z \geq L_g f(x) + L_f g(y) - L_g L_f,$$
$$z \geq U_g f(x) + U_f g(y) - U_f U_g,$$
$$z \leq U_g f(x) + L_f g(y) - L_f U_g,$$
$$z \leq L_g f(x) + U_f g(y) - U_f L_g.$$

By the same reasoning of (1) and (2), we have

$$f(x)g(y) \geq \max\{L_g f(x) + L_f g(y) - L_f L_g, \; U_g f(x) + U_f g(y) - U_f U_g\}, \quad \cdots\cdots (3)$$

and

$$f(x)g(y) \leq \min\{U_g f(x) + L_f g(y) - L_f U_g, \; L_g f(x) + U_f g(y) - L_g U_f\}. \quad \cdots\cdots (4)$$

Combining (3) and (4), we see that the product $f(x)g(y)$ of two linear functions is bounded below by a piecewise linear convex function and bounded above by a piecewise linear concave function.

It is clear that linearization of the product of functions can also be extended to the case of $\frac{f(x)}{g(x)}$. The basic idea of Section 2 can be applied directly to such extension and the desired relaxation can be easily found.

3.2 Exactness of Convex Relaxation

Now we want to investigate exactness of this relaxation under certain conditions. Following ***Proposition* 1** is a well known result that commonly arises in optimization problems in the situation when something happens $(x = 1)$ then other condition should follow that is represented by $f(y) = 0$.

***Proposition* 1**. *Consider the set*

$$S := \{(x, y) \mid xf(x) = 0, x \in \{0, 1\}, y \in \mathbf{Y}\},$$

where $\{f(y) \mid y \in \mathbf{Y}\}$ *is compact; i.e, there exist bounds* $[L, U]$ *such that* $L \leq f(y) \leq U$ *for all* $y \in \mathbf{Y}$. *Then, set* S *is equivalent to* :

$$S' := \{(x, y) \mid L(1-x) \leq f(y) \leq U(1-x), \; x \in \{0, 1\}, y \in \mathbf{Y}\}.$$

Proof. Suppose $x = 1$, then $f(y) = 0$ for both set S and S'. If $x = 0$ then any y satisfies $L \leq f(y) \leq U$ is in the set S and S'. □

The set S' can be derived exactly from applying relaxation technique of Section 3.1. Now we will state a more general result of exactness which follows readily from the foregoing technique.

***Proposition* 2**. *Consider the following nonlinear feasible region* P_1, *where* $L \leq U$ *and* $l \leq u$, *and its relaxation defined by* P_2.

$$P_1 := \{(x, y) \mid a \leq xf(y) \leq b, L \leq f(y) \leq U, x \in \{l, u1\}\},$$

$$P_2 := \{(x, y, z) \mid a \le z \le b,\ L \le f(y) \le U,\ x \in \{l, u\},$$
$$z \ge lf(y) + Lx - Ll,\ z \ge uf(y) + Ux - Uu,$$
$$z \le uf(y) + Lx - Lu,\ z \le lf(y) + Ux - Ul\}.$$

If $(x, y, z) \in P_2$, then $z = xf(y)$ and $(x, y) \in P_1$.

Proof. We can divide P_2 into two cases that is $x = l$ and $x = u$. If $x = l$ then the last 4 equations in P_2 are

$$z \ge lf(y) + Ll - L \Rightarrow z \ge lf(y),$$
$$z \ge uf(y) + Ul - Uu \Rightarrow z - uf(y) \ge U(l-u)$$
$$z \le uf(y) + Ll - Lu \Rightarrow z - uf(y) \le L(l-u)$$
$$z \le lf(y) + Ul - Ul \Rightarrow z \le lf(y).$$

Then $z = lf(y)$ and
$U(l-u) \le z - uf(y) \le L(l-u) \Rightarrow L \le f(y) \le U$ because $l \le u$ and $z - uf(y) = f(y)(l-u)$. Now if $x = u$ then

$$z \ge lf(y) + Lu - Ll \Rightarrow z - lf(y) \ge L(u-l),$$
$$z \ge uf(y) + Uu - Uu \Rightarrow z \ge uf(y),$$
$$z \le uf(y) + Lu - Lu \Rightarrow z \le uf(y),$$
$$z \le lf(y) + Uu - Ul \Rightarrow z - lf(y) \le U(u-l)$$

gives $z = uf(y)$ and $L \le f(y) \le U$. Therefore, if $(x, y, z) \in P_2$, then $z = xf(y)$ and $(x, y) \in P_1$. □

If $f(y)$ is discrete and x continuous, we have the analogous result.

Proposition 3. For given $L \le U$ and $l \le u$, define the sets

$$P_1' := \{(x, y) \mid a \le xf(y) \le b,\ l \le x \le u,\ f(y) \in \{f(L), f(U)\}$$
$$P_2' := \{(x, y, z) \mid a \le z \le b,\ l \le x \le u,\ f(y) \in \{f(L), f(U)\}\},$$
$$z \ge f(L)x + lf(y) - lf(L),$$
$$z \ge f(U) + uf(y) - uf(U),$$
$$z \le f(y)x + lf(y) - lf(U),$$
$$z \le f(L)x + uf(y) - uf(L)\}.$$

If $(x, y, z) \in P_2'$, then $z = xf(y)$ and $(x, y) \in P_1'$.

Taking *Proposition 2* and *3* together, we have the following

Theorem 1. *Consider optimization problem (P) which has terms $xf(y)$, where (x, y) is constrained to be in either P_1 or P_1', and the corresponding relaxed problem (P_R) obtained by replacing $xf(y)$ with z and respectively, P_1 with P_2(P_1' with P_2'). The (x, y) component of the optimal solution of problem (P_R) is optimal for problem (P).*

Remark. Thus the relaxation of (P) given by (P_R) is exact in the sense that it will always produce an optimal solution for the original problem.

Immediate result follows that

Corollary 1. *Consider optimization problem (P) which has terms $xf(y)$, where (x, y) is constrained to be in either P_1 or P_1' whose binary terms in P_1 and P_1' are linearly relaxed, and the corresponding relaxed problem (P_R) obtained by replacing $xf(y)$ with z and respectively, P_1 with P_2(P_1' with P_2'). If optimal solution (x^*, y^*, z^*) to (P_R) satisfies $x^* \in \{l, u\}$ or $f(y^*) \in \{F(L), F(U)\}$ then (x^*, y^*) is optimal for problem (P).*

Remark. Thus the relaxation of (P) given by (P_R) is exact in the sense that if any one variable of optimal solution of (P_R) is at the boundary point, then it will always produce an optimal solution for the original problem.

3.3 Tighter Relaxation

Now we want to show one of two alternatives of relaxation is tighter than the other. Both ways are exact when one of the variable or function is binary by the exactness shown above.

Proposition 4. *Consider the following nonlinear feasible region*

$$S := \{(x, y, z) \mid (f_1(x) - f_2(y))\, g(z) \le U,$$
$$L_{f_1} \le f_1(x) \le U_{f_1},$$
$$L_{f_2} \le f_2(x) \le U_{f_2},$$
$$L_g \le g(z) \le U_g\}.$$

Let S_1 be the projection of the reformulation onto the space of S obtained by linearizing $(f_1(x) - f_2(y))\, g(z)$ by a single variable, and let S_2 be projection of the reformulation onto the space of S obtained by linearizing $f_1(x) g(z)$ and $f_2(y) g(z)$ using two separate variables. Then S_2 is a tighter reformulation than S_1, i.e.

$$S_2 \subset S_1.$$

Proof. Consider the nonlinear function

$$(f_1(x) - f_2(y))\, g(z)$$

over a domain such that each component function has known lower and upper bounds over its domain or subset of interest; i.e.,

$$L_{f_1} \le f_1(x) \le U_{f_1},$$
$$L_{f_2} \le f_2(y) \le U_{f_2},$$
$$L_g \le g(z) \le U_g.$$

Define $w = f_1(x) - f_2(y)$ which has bounds

$$L_{f_1} - U_{f_2} \le w \le U_{f_1} - L_{f_2}.$$

Let $u = wg(z)$, then

$$u \ge (L_{f_1} - U_{f_2})g(z) + L_g[f_1(x) - f_2(y)] - L_g(L_{f_1} - U_{f_2}) = \alpha,$$
$$u \ge (U_{f_1} - L_{f_2})g(z) + U_g[f_1(x) - f_2(y)] - U_g(U_{f_1} - L_{f_2}) = \beta,$$
$$u \le (U_{f_1} - L_{f_2})g(z) + L_g[f_1(x) - f_2(y)] - L_g(U_{f_1} - L_{f_2}) = \gamma,$$
$$u \le (L_{f_1} - U_{f_2})g(z) + U_g[f_1(x) - f_2(y)] - U_g(L_{f_1} - U_{f_2}) = \delta.$$

The latter four inequalities can be summarized as

$$\max\{\alpha, \beta\} \le u \le \min\{\gamma, \delta\}. \quad (5)$$

Now, let $v_1 = f_1(x)g(z)$ and $v_2 = f_2(y)g(z)$. Then

$$v_1 \ge L_g f_1(x) + L_{f_1} g(z) - L_{f_1} L_g = \alpha_1,$$
$$v_1 \ge U_{f_1} g(z) + U_g f_1(x) - U_{f_1} U_g = \beta_1,$$
$$v_1 \ge U_g f_1(x) + L_{f_1} g(z) - L_{f_1} U_g = \gamma_1,$$
$$v_1 \le L_g f_1(x) + U_{f_1} g(z) - L_g U_{f_1} = \delta_1,$$
$$v_2 \ge L_g f_2(y) + L_{f_2} g(z) - L_{f_2} L_g = \alpha_2,$$
$$v_2 \ge U_{f_2} g(z) + U_g f_2(z) - U_{f_2} U_g = \beta_2,$$
$$v_2 \le U_g f_2(z) + L_{f_2} g(z) - L_{f_2} U_g = \gamma_2,$$
$$v_2 \le L_g f_2(x) + U_{f_2} g(z) - L_g U_{f_2} = \delta_2.$$

We can summarize the latter eight inequalities as

$$\max\{\alpha_1, \beta_1\} \le v_1 \le \min\{\gamma_1, \delta_1\}, \quad (6)$$
$$\max\{\alpha_2, \beta_2\} \le v_2 \le \min\{\gamma_2, \delta_2\}. \quad (7)$$

Since $u = v_1 - v_2$, the bounds on $v_1 - v_2$ can be determined from (6) and (7) as

$$\max\{\alpha_1, \beta_1\} - \min\{\gamma_2, \delta_2\} \le v_1 - v_2 \le \min\{\gamma_1, \delta_1\} - \max\{\alpha_2, \beta_2\}. \quad (8)$$

Moreover, since $\alpha = \alpha_1 - \delta_2$, $\beta = \beta_1 - \gamma_2$, $\gamma = \delta_1 - \alpha_2$, and $\delta = \gamma_1 - \beta_2$, the bounds given by (5) can be written as

$$\max\{\alpha_1 - \delta_2, \beta_1 - \gamma_2\} \le v_1 - v_2 \le \min\{\delta_1 - \alpha_2, \gamma_1 - \beta_2\}. (9)$$

We will now show that the lower bounds specified by (8) are always greater than or equal to the lower bounds determined by (9) and the upper bounds of (8) are always less than or equal to the upper bounds of (9) ; i.e., the bounds on $v_1 - v_2$ given by (8) are tighter than those given by (9).

For lower bound, there are two cases to consider.

case 1 $\alpha_1 - \delta_2 \ge \beta_1 - \gamma_2$. Consider the four sub-cases

(*i*) $\alpha_1 \ge \beta_1, \gamma_2 \ge \delta_2$,
(*ii*) $\alpha_1 \le \beta_1, \gamma_2 \ge \delta_2$,
(*iii*) $\alpha_1 \ge \beta_1, \gamma_2 \le \delta_2$,
(*iv*) $\alpha_1 \le \beta_1, \gamma_2 \le \delta_2$.

For each of these subcases, it follows that

$$\max\{\alpha_1, \beta_1\} - \min\{\gamma_2, \delta_2\} \ge \alpha_1 - \delta_2 = \max\{\alpha_1 - \delta_2, \beta_1 - \gamma_2\}$$

case 2 $\beta_1 - \gamma_2 \ge \alpha_1 - \delta_2$. For the same foregoing subcases $\overline{(i)}$ through $\overline{(iv)}$, it follows that

$$\max\{\alpha_1, \beta_1\} - \min\{\gamma_2, \delta_2\} \ge \beta_1 - \gamma_2 = \max\{\alpha_1 - \delta_2, \beta_1 - \gamma_2\}. \quad \square$$

4. Concluding Remark

We have discussed useful convex relaxation technique which can be applied to various extensions to the form of product of variables and/or functions. Also we found that if one of variable or the function is optimal at the boundary point under the convex relaxation, then the projection of such an optimal point onto the domain of the original problem produces an optimal solution to the original problem. In the sense of better bound, we suggested a tighter relaxation of the product form by introducing more variables.

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