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(L, M)-NEIGHBORHOOD SPACES

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Abstract. We introduce the notions of (L, M)-neighborhood spaces and $(2, M)$-fuzzifying neighborhood spaces. We investigate the relations among (L, M)-neighborhood spaces, (L, M)-topological spaces and $(2, M)$-fuzzifying neighborhood spaces.

1. Introduction and preliminaries

Höhle [8-11] introduced the notions of L-fuzzy topology and L-filters on a completely quasi-monoidal lattice (including GL-monoid) L instead of a completely distributive lattice or a unit interval as the extensions of fuzzy topologies [3,16,18] and fuzzy filters [1,2,4-7]. Kotzé[14] introduced an (L, M)-topological space as a general approach where L and M are frames with 0 and 1. Kim et al.[12] introduced notions of (L, M)-topological spaces as an extension of that of Kotzé [11]. Here, L is a completely distributive lattice with 0 and 1 and M is a strictly two-sided, commutative quantale as an extension of a frame.

In this paper, we introduce notions of (L, M)-neighborhood spaces and $(2, M)$-fuzzifying neighborhood spaces with respect to Kim [12] as an extension of Demirci [4]. We investigate the relations among (L, M)-neighborhood spaces, (L, M)-topological spaces and $(2, M)$-fuzzifying neighborhood spaces.

In this paper, let X be a nonempty set and $L = (L, \leq, \vee, \wedge, ')$ a completely distributive lattice with the least element 0 and the greatest element 1 in L with an order reversing involution $'$. The family L^X denotes the set of all fuzzy subsets of a given set X. For each $\alpha \in L$,

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let $\overline{\alpha}$ denote the constant fuzzy sets of X. We denote the characteristic function of a subset A of X by 1_A. A fuzzy point x_t for $t \in L(t \neq 0)$ is an element of L^X such that

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

The set of all fuzzy points in X is denoted by $Pt(X)$. We say that $x_t \, q \, \lambda$ if $x_t \not\leq \lambda'$. If $x_t \leq \lambda'$, we denote $x_t \, \overline{q} \, \lambda$.

Let $M = (M, \leq, \vee, \wedge, \perp, \top)$ be a completely distributive lattice with the least element $\perp$ and the greatest element $\top$ in M.

Definition 1.1 ([8-11,17]). A triple $(M, \leq, \odot)$ is called a *strictly two-sided, commutative quantale* (stsc-quantale, for short) iff it satisfies the following properties:

(M1) $(M, \odot)$ is a commutative semigroup,

(M2) $a = a \odot \top$, for each $a \in M$,

(M3) $\odot$ is distributive over arbitrary joins, i.e.,

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b).$$

Remark 1.2. [8-11,13,17](1) Each frame is a stsc-quantale. In particular, the unit interval $([0,1], \leq, \wedge, 0, 1)$ is a stsc-quantale .

(2) Every left continuous t-norm t on $([0,1], \leq, t)$ with $\odot = t$ is a stsc-quantale.

(3) Every GL-monoid is a stsc-quantale.

Definition 1.3 ([12,14]). A map $\mathcal{T} : L^X \to M$ is called an (L, M)-*topology* on X if it satisfies the following conditions:

(LO1) $\mathcal{T}(\overline{0}) = \mathcal{T}(\overline{1}) = \top$,

(LO2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \odot \mathcal{T}(\mu_2)$, for all $\mu_1, \mu_2 \in L^X$,

(LO3) $\mathcal{T}(\bigvee_{i\in\Lambda} \mu_i) \geq \bigwedge_{i\in\Lambda} \mathcal{T}(\mu_i)$, for any $\{\mu_i\}_{i\in\Lambda} \subset L^X$.

The pair $(X, \mathcal{T})$ is called an (L, M)-*topological space.*

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be (L, M)-topological spaces. A map $\phi : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called *LF-continuous* iff $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\phi^{\leftarrow}(\lambda))$, for all $\lambda \in L^Y$.

Remark 1.4 ([12]). Let $L = \{0, 1\}$ be given and $2^X \cong P(X)$ in a sense $1_A \in 2^X$ iff $A \in P(X)$. A map $\tau : P(X) \to M$ is called a $(2, M)$-fuzzifying topology on X if it satisfies the following conditions:

(O1) $\tau(X) = \tau(\emptyset) = \top$,

(O2) $\tau(A \cap B) \geq \tau(A) \odot \tau(B)$,for all $A, B \in P(X)$,

(O3) $\tau(\bigcup_{i\in\Lambda} A_i) \geq \bigwedge_{i\in\Lambda} \tau(A_i)$, for any $\{A_i\}_{i\in\Lambda} \subset P(X)$.

The pair (X, τ) is called a $(2, M)$-fuzzifying topological space.

Let (X, τ_1) and (Y, τ_2) be $(2, M)$-fuzzifying topological spaces. A map $\phi : (X, \tau_1) \to (Y, \tau_2)$ is called *fuzzifying continuous* iff $\tau_2(A) \leq \tau_1(\phi^{-1}(A))$, $\forall A \in P(Y)$.

2. (L, M)-filter spaces

Definition 2.1. A map $\mathcal{F} : L^X \to M$ is called an (L, M)-filter on X if it satisfies the following conditions:

(LF1) $\mathcal{F}(\overline{0}) = \perp$ and $\mathcal{F}(\overline{1}) = \top$.

(LF2) $\mathcal{F}(\lambda \wedge \mu) \geq \mathcal{F}(\lambda) \odot \mathcal{F}(\mu)$ for all $\lambda, \mu \in L^X$.

(LF3) If $\lambda \leq \mu$, then $\mathcal{F}(\lambda) \leq \mathcal{F}(\mu)$ for all $\lambda, \mu \in L^X$.

The pair $(X, \mathcal{F})$ is called an (L, M)-filter space. Let $(X, \mathcal{F}_1)$ and $(Y, \mathcal{F}_2)$ be (L, M)-filter spaces. A map $\phi : (X, \mathcal{F}_1) \to (Y, \mathcal{F}_2)$ is called a *filter map* iff $\mathcal{F}_2(\mu) \leq \mathcal{F}_1(\phi^{\leftarrow}(\mu))$, for all $\mu \in L^Y$.

Remark 2.2. In the sense in Remark 1.4, a map $F : P(X) \to M$ is called a $(2, M)$-fuzzifying filter on X if it satisfies the following conditions:

(F1) $F(X) = \top$ and $F(\emptyset) = \perp$,

(F2) $F(A \cap B) \geq F(A) \odot F(B)$, for all $A, B \in P(X)$,

(F3) If $A \subset B$, then $F(A) \subset F(B)$ for any $A, B \in P(X)$.

The pair (X, F) is called an $(2, M)$-fuzzifying filter spaces. Let (X, F_1) and (Y, F_2) be $(2, M)$-fuzzifying filter spaces. A map $\phi : (X, F_1) \to (Y, F_2)$ is called a *fuzzifying filter map* iff $F_2(A) \leq F_1(\phi^{-1}(A))$, for all $A \in P(Y)$.

Remark 2.3. (1) If $L = ([0, 1], \wedge)$ and $M = \{0, 1\}$, (L, M)-filter space is the concept of Chang [3].

(2) If $L = \{0, 1\}$ and $M = ([0, 1], \odot = \wedge)$, (L, M)-filter space is the concept of generalised filter [1,2].

(3) If L and M are frames with 0 and 1, (L, M)-filter space is the concept of Gähler [5,6].

THEOREM 2.4. *Let $(X, \mathcal{F})$ be an (L, M)-filter space. We define a function $\mathcal{T}_{\mathcal{F}} : L^X \to M$ as follows:*

$$\mathcal{T}_{\mathcal{F}}(\lambda) = \begin{cases} \mathcal{F}(\lambda), & \text{if } \lambda \neq \overline{0}, \\ \top, & \text{if } \lambda = \overline{0}. \end{cases}$$

Then $(X, \mathcal{T}_{\mathcal{F}})$ is an (L, M)-topological space.

Proof. We only show the condition (LO3). For $\lambda_j \in L^X$, since $\lambda_j \leq \bigvee_{j \in J} \lambda_j$ for all $j \in J$, we have $\mathcal{F}(\lambda_j) \leq \mathcal{F}(\bigvee_{j \in J} \lambda_j)$, so

$$\bigwedge_{j \in J} \mathcal{T}_{\mathcal{F}}(\lambda_j) \leq \mathcal{T}_{\mathcal{F}}(\bigvee_{j \in J} \lambda_j).$$

□

THEOREM 2.5. *Let (X, F) be a $(2, M)$-fuzzifying filter space. We define a function $\mathcal{F}_F : L^X \to M$ as follows:*

$$\mathcal{F}_F(\lambda) = \bigwedge_{r \in L} F(\lambda_r),$$

where $\lambda_r = \{x \in X : \lambda(x) \geq r\}$ for $r \in L - \{0\}$. Then $\mathcal{F}_F$ is an (L, M)-filter.

Proof. (LF1) Clear.
(LF2) For each $\lambda, \mu \in L^X$, we have

$$\begin{aligned} \mathcal{F}_F(\lambda \wedge \mu) &= \bigwedge_{r \in L} F\Big((\lambda \wedge \mu)_r\Big) = \bigwedge_{r \in L} F(\lambda_r \cap \mu_r) \\ &\geq \bigwedge_{r \in L} \Big(F(\lambda_r) \odot F(\mu_r)\Big) \geq \bigwedge_{r \in L} F(\lambda_r) \odot \bigwedge_{r \in L} F(\mu_r) \\ &= \mathcal{F}_F(\lambda) \odot \mathcal{F}_F(\mu). \end{aligned}$$

(LF3) If $\lambda \leq \mu$, then $\lambda_r \subset \mu_r$. Thus

$$\mathcal{F}_F(\lambda) = \bigwedge_{r \in L} F(\lambda_r) \leq \bigwedge_{r \in L} F(\mu_r) = \mathcal{F}_F(\mu)$$

□

Lemma 2.5. *Let $A \in P(X)$ and $\alpha \in L - \{0\}$. Then $\mathcal{F}_F(\alpha \cdot 1_A) = F(A)$.*

Theorem 2.6. *Let (X, F_1), (Y, F_2) be $(2, M)$-fuzzifying filter spaces. A map $\phi : (X, F_1) \to (Y, F_2)$ is a fuzzifying filter map iff $\phi : (X, \mathcal{F}_{F_1}) \to (Y, \mathcal{F}_{F_2})$ is a filter map.*

Proof. For each $\mu \in L^Y$, we have

$$\mathcal{F}_{F_1}(\phi^{\leftarrow}(\mu)) = \bigwedge_{r \in L} F_1((\phi^{\leftarrow}(\mu))_r) \geq \bigwedge_{r \in L} F_2(\mu_r) = \mathcal{F}_{F_2}(\mu).$$

Conversely, suppose there exists $A \in P(X)$ such that $F_1(\phi^{-1}(A)) \ngeq F_2(A)$. It implies $\mathcal{F}_{F_1}(1_{\phi^{-1}(A)}) = F_1(\phi^{-1}(A)) \ngeq F_2(A) = \mathcal{F}_{F_2}(1_A)$. □

Example 2.7. Let $X = \{x, y, z\}$ be a set . Define a binary operation $\otimes$ on $M = [0, 1]$ by $x \otimes y = \max\{0, x + y - 1\}$. Then $M = ([0, 1], \leq, \otimes)$ is a stsc-quantale. Define a $(2, M)$-fuzzifying topology $F : P(X) \to [0, 1]$ as follows:

$$F(A) = \begin{cases} 1, & \text{if } A = X \\ 0.8, & \text{if } A = \{x, y\}, \\ 0.6, & \text{if } A = \{y\}, \\ 0.7, & \text{if } B = \{y, z\}, \\ 0, & \text{otherwise.} \end{cases}$$

For $\lambda, \mu \in [0, 1]^X$ with

$$\lambda(x) = 0.3, \lambda(y) = 0.7, \lambda(z) = 0.5, \quad \mu(x) = 0.7, \mu(y) = 0.2, \mu(z) = 0.5,$$

we have

$$(\lambda)_r \in \{\{y\}, \{y, z\}, X\}, \quad (\mu)_r \in \{\{x\}, \{x, z\}, X\}.$$

Hence $\mathcal{F}_F(\lambda) = 0.6$ and $\mathcal{F}_F(\mu) = 0$.

3. (L, M)-neighborhood spaces.

DEFINITION 3.1. An (L, M)-*neighborhood system* on X is a set $\mathcal{Q} = \{\mathcal{Q}_{x_t} \mid x_t \in Pt(X)\}$ of maps $\mathcal{Q}_{x_t} : L^X \to M$ such that for each $\lambda, \mu \in L^X$,we have

(LN1) $\mathcal{Q}_{x_t}$ is an (L, M)-filter on X.

(LN2) $\mathcal{Q}_{x_t}(\lambda) > \perp$ implies $x_t q \lambda$.

(LN3) $\mathcal{Q}_{x_t}(\lambda) = \bigvee_{x_t q \mu \leq \lambda} \Big(\bigwedge_{y_t q \mu} \mathcal{Q}_{y_t}(\mu) \Big)$.

The pair $(X, \mathcal{Q})$ is called an (L, M)-*neighborhood space.*

Let $(X, \mathcal{Q}_1)$ and $(Y, \mathcal{Q}_2)$ be (L, M)-neighborhood spaces. A function $\phi : (X, \mathcal{Q}_1) \to (Y, \mathcal{Q}_2)$ is called an LN-map if $(\mathcal{Q}_1)_{x_t}(\phi^{\leftarrow}(\lambda)) \geq (\mathcal{Q}_2)_{\phi^{\to}(x_t)}(\lambda)$ for all $\lambda \in L^Y$ and for all $x_t \in Pt(X)$.

REMARK 3.2. By the sense of Remark 2.2, since $x_1 \, q \, 1_A$ iff $x \in A$, a map $\mathcal{N}_x : P(X) \to M$ is called a $(2, M)$-*fuzzifying neighborhood* of $x \in X$ if it satisfies the following conditions:

(N1) $\mathcal{N}_x$ is a $(2, M)$-filter on X.

(N2) $\mathcal{N}_x(A) > \perp$ implies $x \in A$.

(N3)$\mathcal{N}_x(A) = \bigvee_{x \in B \subset A} (\bigwedge_{y \in B} \mathcal{N}_y(B))$.

A set $\mathcal{N} = \{\mathcal{N}_x \mid x \in X\}$ is called a $(2, M)$-*fuzzifying neighborhood system* on X. A map $f : (X, \mathcal{N}_1) \to (Y, \mathcal{N}_2)$ is called a N-fuzzifying map if for each $A \in P(Y)$ and for each $x \in X$, $(N_1)_x(f^{-1}(A))) \geq (N_2)_{f(x)}(A)$.

THEOREM 3.3. *Let $(X, \mathcal{T})$ be an (L, M)-topological space and $x_t \in Pt(X)$. Define a map $\mathcal{Q}^{\mathcal{T}}_{x_t} : L^X \to M$ as:*

$$\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda) = \begin{cases} \bigvee \{\mathcal{T}(\mu) \mid x_t \; q \; \mu \leq \lambda\} & \text{if } x_t \; q \; \lambda, \\ 0 & \text{if } x_t \overline{q} \; \lambda. \end{cases}$$

Then (1) $\mathcal{Q}^{\mathcal{T}} = \{\mathcal{Q}^{\mathcal{T}}_{x_t} \mid x_t \in Pt(X)\}$ *is an (L, M)-neighborhood system on X,*

(2) *if $t < s$ for $t, s \in L$, then $\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda) \leq \mathcal{Q}^{\mathcal{T}}_{x_s}(\lambda)$.*

Proof. (1) (LF1) and (LF3) are easily proved.

(LF2) Suppose there exist $\lambda, \mu \in L^X$ such that

$$\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda \wedge \mu) \not\geq \mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda) \odot \mathcal{Q}^{\mathcal{T}}_{x_t}(\mu).$$

By the definition of $\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda)$ and (M3) of Definition 1.1, there exists $\lambda_1 \in L^X$ with $x_t \ q \ \lambda_1 \leq \lambda$ such that

$$\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda \wedge \mu) \not\geq \mathcal{T}(\lambda_1) \odot \mathcal{Q}^{\mathcal{T}}_{x_t}(\mu).$$

Again, by the definition of $\mathcal{Q}^{\mathcal{T}}_{x_t}(\mu)$ and (M3) of Definition 1.1, there exists $\mu_1 \in L^X$ with $x_t \ q \ \mu_1 \leq \mu$ such that

$$\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda \wedge \mu) \not\geq \mathcal{T}(\lambda_1) \odot \mathcal{T}(\mu_1).$$

Since $x_t \ q \ (\lambda_1 \wedge \mu_1) \leq \lambda \wedge \mu$, we have

$$\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda \wedge \mu) \geq \mathcal{T}(\lambda_1 \wedge \mu_1) \geq \mathcal{T}(\lambda_1) \odot \mathcal{T}(\mu_1).$$

It is a contradiction. Hence

$$\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda \wedge \mu) \geq \mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda) \odot \mathcal{Q}^{\mathcal{T}}_{x_t}(\mu), \ \ \forall \lambda, \mu \in L^X$$

So, $\mathcal{Q}_{x_t}$ is an (L, M)-filter on X.

(LN2) It is easy from the definition of $\mathcal{Q}^{\mathcal{T}}_{x_t}$.

(LN3) For all $\lambda \in L^X$ with $x_t q \mu \leq \lambda$, we have

$$\mathcal{T}(\mu) \leq \bigwedge \{\mathcal{Q}^{\mathcal{T}}_{y_s}(\mu) \mid y_s \ q \ \mu\} \leq \mathcal{Q}^{\mathcal{T}}_{x_t}(\mu) \leq \mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda).$$

Therefore,

$$\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda) = \bigvee_{x_t q \mu \leq \lambda} \mathcal{T}(\mu) \leq \bigvee_{x_t q \mu \leq \lambda} \Big(\bigwedge_{y_s \ q \ \mu} \mathcal{Q}^{\mathcal{T}}_{y_s}(\mu) \Big) \leq \mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda).$$

This means that $\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda) = \bigvee_{x_t q \mu \leq \lambda} \Big(\bigwedge_{y_s \ q \ \mu} \mathcal{Q}^{\mathcal{T}}_{y_s}(\mu) \Big)$.

(2) For $t < s$ with $t, s \in L$ and $\forall \lambda \in L^X$, since

$$\{\mu \in L^X \mid x_t \ q \ \mu \leq \lambda\} \subset \{\rho \in L^X \mid x_s \ q \ \rho \leq \lambda\},$$

we have $\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda) \leq \mathcal{Q}^{\mathcal{T}}_{x_s}(\lambda)$. □

EXAMPLE 3.4. Let $X = \{x, y\}$ be a set and $L = M = [0,1]$ a completely distributive lattice. Define a binary operation $\otimes$ on $M = [0,1]$ by $x \otimes y = \max\{0, x+y-1\}$. Then $([0,1], \leq, \otimes)$ is a stsc-quantale. Let $\mu, \rho \in [0,1]^X$ be defined as follows:

$$\mu(x) = 0.6,\ \mu(y) = 0.3 \quad \rho(x) = 0.5,\ \rho(y) = 0.7.$$

We define an (L, M)-topology $\mathcal{T} : [0,1]^X \to [0,1]$ as follows:

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{1}, \overline{0} \\ 0.8, & \text{if } \lambda = \mu, \\ 0.3, & \text{if } \lambda = \rho, \\ 0.7, & \text{if } \lambda = \mu \vee \rho, \\ 0.2, & \text{if } \lambda = \mu \wedge \rho, \\ 0, & \text{otherwise.} \end{cases}$$

We obtain $\mathcal{Q}^{\mathcal{T}}_{x_{0.5}}, \mathcal{Q}^{\mathcal{T}}_{y_{0.8}} : [0,1]^X \to [0,1]$ as:

$$\mathcal{Q}^{\mathcal{T}}_{x_{0.5}}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{1}, \\ 0.8 & \text{if } \mu \leq \lambda \neq \overline{1}, \\ 0 & \text{otherwise.} \end{cases} \mathcal{Q}^{\mathcal{T}}_{y_{0.8}}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{1}, \\ 0.8 & \text{if } \mu \leq \lambda \neq \overline{1}, \\ 0.3 & \text{if } \rho \leq \lambda \not\geq \mu, \\ 0.2 & \text{if } \rho \wedge \mu \leq \lambda \not\geq \rho, \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 3.3, we can obtain the following corollary.

COROLLARY 3.5. *Let (X, τ) be a $(2, M)$-fuzzifying topological space. We define a function $(\mathcal{N}_\tau)_x : P(X) \to M$ by*

$$(\mathcal{N}_\tau)_x(A) = \bigvee_{x \in B \subset A} \tau(B).$$

Then $(\mathcal{N}_\tau)_x$ is a $(2, M)$-fuzzifying neighborhood of $x \in X$.

THEOREM 3.6. *Let $\mathcal{Q} = \{\mathcal{Q}_{x_t} : L^X \to M \mid x_t \in Pt(X)\}$ be a family of $\mathcal{Q}_{x_t}$ satisfying (LN1) and (LN2) of Definition 3.1. We define a map $\mathcal{T}^{\mathcal{Q}} : L^X \to M$ as follows:*

$$\mathcal{T}^{\mathcal{Q}}(\lambda) = \begin{cases} \bigwedge\{\mathcal{Q}_{x_t}(\lambda) \mid x_t \; q \; \lambda\} & \text{if } \lambda \neq \overline{0} \\ \top & \text{if } \lambda = \overline{0}. \end{cases}$$

Then we have the following properties.

(1) *$\mathcal{T}^{\mathcal{Q}}$ is an (L, M)-topology on X.*

(2) *If $\mathcal{Q} = \{\mathcal{Q}_{x_t} \mid x_t \in Pt(X)\}$ is an (L, M)-neighborhood system on X, then $\mathcal{Q}_{x_t}^{\mathcal{T}^{\mathcal{Q}}} = \mathcal{Q}_{x_t}$, for all $x_t \in Pt(X)$.*

(3) *If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are (L, M)-neighborhood systems on X such that $\mathcal{T}^{\mathcal{Q}_1} = \mathcal{T}^{\mathcal{Q}_2}$, then $\mathcal{Q}_1 = \mathcal{Q}_2$.*

Proof. (1) (LO1) is trivial.

(LO2) For $\lambda, \mu \in L^X$, we have

$$\begin{aligned}
&\mathcal{T}^{Q}(\lambda \wedge \mu) \\
&= \bigwedge\{\mathcal{Q}_{x_t}(\lambda \wedge \mu) \mid x_t q(\lambda \wedge \mu)\} \\
&\geq \bigwedge\{\mathcal{Q}_{x_t}(\lambda) \odot \mathcal{Q}_{x_t}(\mu) \mid x_t \; q \; (\lambda \wedge \mu)\} \\
&\geq \Big(\bigwedge\{\mathcal{Q}_{x_t}(\lambda) \mid x_t q(\lambda \wedge \mu)\}\Big) \odot \Big(\bigwedge\{\mathcal{Q}_{x_t}(\mu) \mid x_t \; q \; (\lambda \wedge \mu)\}\Big) \\
&\geq \Big(\bigwedge\{\mathcal{Q}_{x_t}(\lambda) \mid x_t q \lambda\}\Big) \odot \Big(\bigwedge\{\mathcal{Q}_{x_t}(\mu) \mid x_t \; q \; \mu\}\Big) \\
&= \mathcal{T}_{x_t}^{\mathcal{Q}}(\lambda) \odot \mathcal{T}_{x_t}^{\mathcal{Q}}(\mu).
\end{aligned}$$

(LO3) Suppose $\mathcal{T}^{Q}(\bigvee_{j \in J} \mu_j) \not\geq \bigwedge_{j \in J} \mathcal{T}^{\mathcal{Q}}(\mu_j)$. Then there exists a family $\{\mu_j \mid x_t q(\bigvee_{j \in J} \mu_j)\}$ such that

$$\mathcal{Q}_{x_t}(\bigvee_{j \in J} \mu_j) \not\geq \bigwedge_{j \in J} \mathcal{Q}_{x_t}(\mu_j).$$

Since $x_t q(\bigvee_{j \in J} \mu_j)$, there exists $j \in J$ such that $x_t q \mu_j$ such that

$$\mathcal{Q}_{x_t}(\bigvee_{j \in J} \mu_j) \not\geq \mathcal{Q}_{x_t}(\mu_j).$$

It is a contradiction for a filter $\mathcal{Q}_{x_t}$. Hence the result holds.

(2)

$$\begin{aligned}\mathcal{Q}^{\mathcal{T}^{\mathcal{Q}}}_{x_t}(\lambda) &= \bigvee\{\mathcal{T}^{\mathcal{Q}}(\mu) \mid x_t q\mu \le \lambda\}\\ &= \bigvee\Big\{\bigwedge\{\mathcal{Q}_{y_s}(\mu) \mid y_s\ q\ \mu\} \mid x_t q\mu \le \lambda\Big\}\\ &= \mathcal{Q}_{x_t}(\lambda) \quad \text{(by (LN 3))}.\end{aligned}$$

(3) Since $\mathcal{T}^{\mathcal{Q}_1} = \mathcal{T}^{\mathcal{Q}_2}$, for $\lambda \in L^X$ and $x_t \in Pt(X)$, we have

$$\begin{aligned}(\mathcal{Q}_1)_{x_t}(\lambda) &= \bigvee\Big\{\bigwedge\{(\mathcal{Q}_1)_{y_s}(\mu) \mid y_s\ q\ \mu\} \mid x_t q\mu \le \lambda\Big\}\\ &= \bigvee\Big\{\mathcal{T}^{\mathcal{Q}_1}(\mu) \mid x_t q\mu \le \lambda\Big\}\\ &= \bigvee\Big\{\mathcal{T}^{\mathcal{Q}_2}(\mu) \mid x_t q\mu \le \lambda\Big\}\\ &= \bigvee\Big\{\bigwedge\{(\mathcal{Q}_2)_{y_s}(\mu) \mid y_s\ q\ \mu\} \mid x_t q\mu \le \lambda\Big\}\\ &= (\mathcal{Q}_2)_{x_t}(\lambda).\end{aligned}$$

Hence $\mathcal{Q}_1 = \mathcal{Q}_2$. □

Corollary 3.7. *Let $\mathcal{N}_x : P(X) \to M$ be a map satisfying (N1) and (N2) for all $x \in X$. We define a map $\tau_{\mathcal{N}} : P(X) \to M$ by*

$$\tau_{\mathcal{N}}(A) = \bigwedge_{x\in A} \mathcal{N}_x(A).$$

Then:

(1) $(X, \tau_{\mathcal{N}})$ is a $(2, M)$-fuzzifying topological space,

(2) if $\mathcal{N}$ is a $(2, M)$-fuzzifying neighborhood system, then $(\mathcal{N}_{\tau_{\mathcal{N}}})_x = \mathcal{N}_x$.

(3) if $\mathcal{N}_1$ and $\mathcal{N}_2$ are $(2, M)$-fuzzifying neighborhood systems on X such that $\tau_{\mathcal{N}_1} = \tau_{\mathcal{N}_2}$, then $\mathcal{N}_1 = \mathcal{N}_2$.

The following lemma is easily proved.

Lemma 3.8. *If $x_t q\lambda$, then there exists $\mu_{x_t} \in L^X$ such that $x_t q\mu_{x_t} \le \lambda$. Thus $\lambda = \bigvee_{x_t q\lambda} \mu_{x_t}$.*

THEOREM 3.9. *Let $(X, \mathcal{T})$ be an (L, M)-topological space and $\mathcal{Q}^{\mathcal{T}}$ an (L, M)-neighborhood system in $(X, \mathcal{T})$. Then $\mathcal{T} = \mathcal{T}^{\mathcal{Q}^{\mathcal{T}}}$.*

Proof. Since $\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda) = \bigvee\{\mathcal{T}(\mu) \mid x_t\ q\ \mu \leq \lambda\} \geq \mathcal{T}(\lambda)$ for all $x_t\ q\ \lambda$, we have $\bigwedge\{\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda) \mid x_t\ q\ \ \lambda\} \geq \mathcal{T}(\lambda)$. So, $\mathcal{T}^{\mathcal{Q}^{\mathcal{T}}} \geq \mathcal{T}$.

Conversely, there exists $\lambda \in L^X$ such that $\mathcal{T}^{\mathcal{Q}^{\mathcal{T}}}(\lambda) \not\leq \mathcal{T}(\lambda)$. For each $x_t \in P(X)$ with $x_t\ q\ \lambda$, if $x_t\ q\ \mu_{x_t} \leq \lambda$, then by Lemma 3.8, we get $\lambda = \bigvee_{x_t\ q\ \lambda} \mu_{x_t}$. So,

$$\mathcal{T}(\lambda) = \mathcal{T}(\bigvee \mu_{x_t}) \geq \bigwedge \mathcal{T}(\mu_{x_t}).$$

Thus, $\bigwedge \mathcal{T}(\mu_{x_t}) \not\geq \mathcal{T}^{\mathcal{Q}^{\mathcal{T}}}(\lambda) = \bigwedge\{\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda) \mid x_t\ q\ \ \lambda\}$. There exists μ_{x_t} with $x_t q \mu_{x_t} \leq \lambda$ such that

$$\mathcal{T}(\mu_{x_t}) \not\geq \bigwedge\{\mathcal{Q}^{\mathcal{T}}_{x_t}(\lambda) \mid x_t\ q\ \ \lambda\}$$

It is a contradiction. Thus, $\mathcal{T} \geq \mathcal{T}^{\mathcal{Q}^{\mathcal{T}}}$. □

COROLLARY 3.10. *Let (X, τ) be a $(2, M)$-fuzzifying topological space and $\mathcal{N}_\tau$ a $(2, M)$-fuzzifying neighborhood system in $(X, \mathcal{T})$. Then $\tau = \tau_{\mathcal{N}_\tau}$.*

THEOREM 3.11. *Let $(X, \mathcal{Q}_1)$, $(Y, \mathcal{Q}_2)$ be (L, M)-neighborhood spaces. A mapping $\phi : (X, \mathcal{Q}_1) \to (Y, \mathcal{Q}_2)$ is an LN-map iff $\phi : (X, \mathcal{T}^{\mathcal{Q}_1}) \to (Y, \mathcal{T}^{\mathcal{Q}_2})$ is LF-continuous.*

Proof. Since $\forall\lambda \in L^Y$,$\forall x_t \in P(X), x_t q \phi^{\leftarrow}(\lambda)$ if and only if $(\phi^{\rightarrow}(x_t) = \phi(x)_t) q\lambda$ and

$$\{y_t \in Pt(Y) \mid y_t q\lambda\} \supset \{\phi(x)_t \in Pt(Y) \mid x_t \in Pt(X),\ \phi(x)_t q\lambda\},$$

we have

$$\begin{aligned} \mathcal{T}^{\mathcal{Q}_2}(\lambda) &= \bigwedge\{(\mathcal{Q}_2)_{y_t}(\lambda) \mid y_t\ q\ \lambda\} \\ &\leq \bigwedge\{(\mathcal{Q}_2)_{\phi^{\rightarrow}(x_t)}(\lambda) \mid \phi^{\rightarrow}(x_t)\ q\ \lambda\} \\ &\leq \bigwedge\{(\mathcal{Q}_1)_{x_t}(\phi^{\leftarrow}(\lambda)) \mid x_t\ q\ \phi^{\leftarrow}(\lambda)\} \\ &= \mathcal{T}^{\mathcal{Q}_1}(\phi^{\leftarrow}(\lambda)). \end{aligned}$$

Thus, $\phi : (X, \mathcal{T}^{\mathcal{Q}_1}) \to (Y, \mathcal{T}^{\mathcal{Q}_2})$ is LF-continuous.

Conversely, since $\forall \lambda \in L^Y, \mathcal{T}^{\mathcal{Q}_2}(\lambda) \leq \mathcal{T}^{\mathcal{Q}_1}(\phi^{\leftarrow}(\lambda))$, $\mathcal{Q}_1 = \mathcal{Q}^{\mathcal{T}^{\mathcal{Q}_1}}$ and $\mathcal{Q}_2 = \mathcal{Q}^{\mathcal{T}^{\mathcal{Q}_2}}$, we have

$$\begin{aligned}
(\mathcal{Q}_2)_{\phi^{\to}(x_t)}(\lambda) &= \bigvee \{\mathcal{T}^{\mathcal{Q}_2}(\mu) \mid \phi^{\to}(x_t)\ q\ \mu \leq \lambda\} \\
&\leq \bigvee \{\mathcal{T}^{\mathcal{Q}_2}(\mu) \mid x_t\ q\ \phi^{\leftarrow}(\mu) \leq \phi^{\leftarrow}(\lambda)\} \\
&\leq \bigvee \{\mathcal{T}^{\mathcal{Q}_1}(\phi^{\leftarrow}(\mu)) \mid x_t\ q\ \phi^{\leftarrow}(\mu) \leq \phi^{\leftarrow}(\lambda)\} \\
&\leq (\mathcal{Q}_1)_{x_t}(\phi^{\leftarrow}(\lambda)).
\end{aligned}$$

Hence the proof is complete . □

Corollary 3.12. *Let $(X, \mathcal{N}_1)$ and $(Y, \mathcal{N}_2)$ be $(2, M)$-fuzzifying neighborhood spaces. A map $f : (X, \mathcal{N}_1) \to (Y, \mathcal{N}_2)$ is a N-fuzzifying map iff $f : (X, \tau_{\mathcal{N}_1}) \to (Y, \tau_{\mathcal{N}_2})$ is fuzzifying continuous.*

From Theorems 3.9 and 3.11 we obtain the following corollaries.

Corollary 3.13. *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be (L, M)-topological spaces. A mapping $\phi : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is LF-continuous if and only if $\phi : (X, \mathcal{Q}^{\mathcal{T}_1}) \to (Y, \mathcal{Q}^{\mathcal{T}_2})$ is an LN-map.*

Corollary 3.14. *Let (X, τ_1) and (Y, τ_2) be $(2, M)$-fuzzifying topological spaces. A map $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzifying continuous iff $f : (X, \mathcal{N}_{\tau_1}) \to (Y, \mathcal{N}_{\tau_2})$ is an N-fuzzifying map.*

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