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LINEAR OPERATORS PRESERVING MAXIMAL COLUMN RANKS OF NONNEGATIVE REAL MATRICES

KYUNG-TAE KANG, DUK-SUN KIM, SANG-GU LEE*, and HAN-GUK SEOL

Abstract. For an m by n nonnegative real matrix A, the maximal column rank of A is the maximal number of the columns of A which are linearly independent. In this paper, we analyze relationships between ranks and maximal column ranks of matrices over nonnegative reals. We also characterize the linear operators which preserve the maximal column rank of matrices over nonnegative reals.

1. Introduction

Much attention has been paid to the study of linear operators preserving the rank or maximal column rank of matrices over several semirings([1]-[13]). Nonnegative matrices also have been the interesting subject of research by many authors([3]-[5], [9]-[11]). In 1985, Beasley et al. [3] obtained a characterization of the linear operators preserving the rank of matrices over $\mathbb{R}_+$, the set of all nonnegative elements in the reals $\mathbb{R}$.

In 1994, Hwang et al. [6] defined a maximal column rank of a matrix over a semiring and compared it with rank. And they characterized linear operators that preserve maximal column ranks of matrices over Boolean algebra. But the analysis of maximal column ranks over $\mathbb{R}_+$ remains open until now. As a partial result, in 1998, Song [10] characterized the linear operators that preserve maximal column ranks of matrices over $\mathbb{Z}_+$, the set of all nonnegative integers.

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*Corresponding author.

In this paper, we study the maximal column rank of matrices over $\mathbb{R}_+$. Consequently, we analyze the relationship between ranks and maximal column ranks of matrices over nonnegative reals. We also characterize the linear operators preserving the maximal column rank of matrices over $\mathbb{R}_+$.

2. Comparison of rank and maximal column rank of matrices over the nonnegative reals

For nonnegative reals $\mathbb{R}_+$, we denote $\mathcal{M}_{m,n}(\mathbb{R}_+)$ as the set of $m \times n$ matrices with entries in $\mathbb{R}_+$. Addition, scalar multiplication, and the product of matrices are defined as if $\mathbb{R}_+$ were a field. Let A be a nonzero element in $\mathcal{M}_{m,n}(\mathbb{R}_+)$. Then the *rank* [3] or *factor rank*, $r(A)$, of A is defined as the least integer k such that there exist $m \times k$ and $k \times n$ matrices B and C with $A = BC$. The real rank of A will be denoted by $\rho(A)$ [3]. The rank of a zero matrix is zero. Also we can easily obtain that

$$0 \leq \rho(A) \leq r(A) \leq \min(m, n), \tag{2.1}$$

$$r(AB) \leq \min(r(A), r(B)) \tag{2.2}$$

for all matrices $A \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ and $B \in \mathcal{M}_{n,p}(\mathbb{R}_+)$.

Let $\mathbb{S}$ be a nonempty subset of $(\mathbb{R}_+)^n \equiv \mathcal{M}_{n,1}(\mathbb{R}_+)$. Then $\mathbb{S}$ is *linearly dependent* if there exists $\boldsymbol{x} \in \mathbb{S}$ such that $\boldsymbol{x}$ is a linear combination of elements in $\mathbb{S} \backslash \{\boldsymbol{x}\}$. Otherwise $\mathbb{S}$ is *linearly independent.* Thus an independent set cannot contain a zero vector.

The *maximal column rank,* $mc(A)$, of $A \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ is the maximal number of the columns of A which are linearly independent. The maximal column rank of a zero matrix is zero. It follows that

$$0 \leq r(A) \leq mc(A) \leq n \tag{2.3}$$

for all matrices $A \in \mathcal{M}_{m,n}(\mathbb{R}_+)$.

The maximal column rank of a matrix may actually exceed its rank. For an example, we consider a matrix

$$A = \begin{bmatrix} a & b & 0 & 0 \\ 0 & c & d & 0 \\ 0 & 0 & e & f \end{bmatrix} \in \mathcal{M}_{3,4}(\mathbb{R}_+), \tag{2.4}$$

where a, b, c, d, e and f are nonzero elements in $\mathbb{R}_+$. Then example 2.5 (below) implies that $r(A) = 3$, but $mc(A) = 4$.

LEMMA 2.1. *Let A be a matrix in $\mathcal{M}_{m,n}(\mathbb{R}_+)$ with $\min(m,n) \geq 1$. Then we have that $r(A) = 1$ if and only if $mc(A) = 1$.*

Proof. If $r(A) = 1$, then A can be factored as

$$A = \boldsymbol{b}\,[\,c_1 \; c_2 \; \cdots \; c_n\,] = [\,c_1\boldsymbol{b}, c_2\boldsymbol{b}, \cdots, c_n\boldsymbol{b}\,],$$

where $\boldsymbol{b}$ is an $m \times 1$ matrix and $[\,c_1 \; c_2 \; \cdots \; c_n\,]$ is a $1 \times n$ matrix. Since $r(A) = 1$, $\boldsymbol{b}$ is not a zero vector. Then it is obvious that any two columns of A are linearly dependent. So we have $mc(A) = 1$. The converse is obvious from (2.3). □

Let $\beta(\mathbb{R}_+, m, n)$ be the largest integer k such that for all $A \in \mathcal{M}_{m,n}(\mathbb{R}_+)$, $r(A) = mc(A)$ if $r(A) \leq k$ and there is at least one $m \times n$ matrix A with $r(A) = k$. The matrix A in (2.4) shows that $\beta(\mathbb{R}_+, 3, 4) < 3$. In general, $0 \leq \beta(\mathbb{R}_+, m, n) \leq n$. We also obtain that

$$r\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = r(A) \quad \text{and} \quad mc\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = mc(A) \tag{2.5}$$

for all matrices $A \in \mathcal{M}_{m,n}(\mathbb{R}_+)$.

LEMMA 2.2. *If A is a matrix in $\mathcal{M}_{p,q}(\mathbb{R}_+)$ such that $mc(A) > r(A)$, then we have that $\beta(\mathbb{R}_+, m, n) < r(A)$ for all $m \geq p$ and $n \geq q$.*

Proof. Since $mc(A) > r(A)$ for some $A \in \mathcal{M}_{p,q}(\mathbb{R}_+)$, we have $\beta(\mathbb{R}_+, p, q) < r(A)$ from the definition of β. Let $B = A \oplus 0_{m-p,n-q}$ be the matrix in $\mathcal{M}_{m,n}(\mathbb{R}_+)$, where $m \geq p$ and $n \geq q$. Then (2.5) implies that

$$r(B) = r(A) < mc(A) = mc(B).$$

So, we have $\beta(\mathbb{R}_+, m, n) < r(A)$ for all $m \geq p$ and $n \geq q$. □

LEMMA 2.3. *For any matrix A in $\mathcal{M}_{2,n}(\mathbb{R}_+)$ with $n \geq 2$, we have that $r(A) = 2$ if and only if $mc(A) = 2$.*

Proof. Suppose that $r(A) = 2$. If $n = 2$, then (2.3) implies that $mc(A) = 2$. So we can assume that $n \geq 3$. Let

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \quad \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$$

be any three columns of A. We claim those three columns are linearly dependent. This implies $mc(A) < 3$, and hence $mc(A) = 2$ by lemma 2.1. Let $X = \{a_i, b_i \,|\, i = 1, 2, 3\}$.

First, assume that X has at least two zero elements. Then we can easily show those three columns are linearly dependent.

Next, assume that X has only one zero element. Without loss of generality, we may take $a_1 = 0$ or $b_1 = 0$. If $a_1 = 0$, let $\min\{\frac{a_2}{b_2}, \frac{a_3}{b_3}\} = \frac{a_3}{b_3}$. Then we have

$$\frac{a_2b_3 - a_3b_2}{a_2b_1}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \frac{a_3}{a_2}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}.$$

If $b_1 = 0$, let $\max\{\frac{a_2}{b_2}, \frac{a_3}{b_3}\} = \frac{a_3}{b_3}$. Then we now have

$$\frac{a_3b_2 - a_2b_3}{a_1b_2}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \frac{b_3}{b_2}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}.$$

So those three columns are linearly dependent.

Finally, assume that X has no zero element. If $\frac{a_i}{b_i} = \frac{a_j}{b_j}$ for some $i \neq j$, then it is obvious that the three vectors are linearly dependent. Thus, without loss of generality, we may assume that $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \frac{a_3}{b_3}$. Then we obtain that

$$\frac{a_2b_3 - a_3b_2}{a_1b_3 - a_3b_1}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \frac{a_1b_2 - a_2b_1}{a_1b_3 - a_3b_1}\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

Hence those three columns are linearly dependent.

The converse follows from (2.3) and lemma 2.1. □

THEOREM 2.4. *Let A be a matrix in $\mathcal{M}_{m,n}(\mathbb{R}_+)$ with $\min(m, n) \geq 2$. Then we have that $r(A) = 2$ if and only if $mc(A) = 2$.*

Proof. Let $r(A) = 2$. Then A can be factored as $A = BC$ for some $m \times 2$ matrix $B = [\,\boldsymbol{x}\ \boldsymbol{y}\,]$ and $2 \times n$ matrix C with $r(B) = r(C) = 2$. If $n = 2$, then (2.3) implies that $mc(A) = 2$. So we can assume that $n \geq 3$. Then any column of A has the form $a\boldsymbol{x} + b\boldsymbol{y}$ with a column $\begin{pmatrix} a \\ b \end{pmatrix}$ of C. Let $a_1\boldsymbol{x} + b_1\boldsymbol{y}$, $a_2\boldsymbol{x} + b_2\boldsymbol{y}$ and $a_3\boldsymbol{x} + b_3\boldsymbol{y}$ be arbitrary chosen three columns of A. Then

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \text{ and } \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$$

are columns of C and hence they are linearly dependent by lemma 2.3. Without loss of generality, we may write $\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \alpha\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \beta\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ for some $\alpha, \beta \in \mathbb{R}_+$. Then we obtain that

$$a_3\boldsymbol{x} + b_3\boldsymbol{y} = \alpha(a_1\boldsymbol{x} + b_1\boldsymbol{y}) + \beta(a_2\boldsymbol{x} + b_2\boldsymbol{y}).$$

Therefore any three columns of A are linearly dependent, and hence $mc(A) \le 2$. By lemma 2.1, we have $mc(A) = 2$. The converse is obvious from (2.3) and lemma 2.1. $\square$

EXAMPLE 2.5. Consider a matrix

$$A = \begin{bmatrix} a & b & 0 & 0 \\ 0 & c & d & 0 \\ 0 & 0 & e & f \end{bmatrix} \in \mathcal{M}_{3,4}(\mathbb{R}_+),$$

where a, b, c, d, e and f are positive reals. Since all columns of A are linearly independent over $\mathbb{R}_+$, we have $mc(A) = 4$. Also $2 \le r(A) \le 3 = \min(3, 4)$ by lemma 2.1 and (2.1). It follows from theorem 2.4 that $r(A) \ne 2$. Therefore $r(A) = 3$.

THEOREM 2.6. *For the nonnegative reals* $\mathbb{R}_+$, *we have the value of* β *as follows*:

$$\beta(\mathbb{R}_+, m, n) = \begin{cases} 1 & \text{if } \min(m,n) = 1; \\ 3 & \text{if } m \ge 3, \text{ and } n = 3; \\ 2 & \text{otherwise.} \end{cases}$$

Proof. If $\min(m, n) = 1$, then we have $\beta(\mathbb{R}_+, m, n) = 1$ from lemma 2.1. Consider the matrix $A \in \mathcal{M}_{3,4}(\mathbb{R}_+)$ in example 2.5. Then $r(A) = 3$ and $mc(A) = 4$. Thus we have $\beta(\mathbb{R}_+, m, n) \le 2$ for all $m \ge 3$ and $n \ge 4$ by lemma 2.2. Suppose $m \ge 2$ and $n \ge 2$. Then we have $\beta(\mathbb{R}_+, m, n) \ge 2$ for all $m \ge 2$ and $n \ge 2$ by theorem 2.4. Finally, consider the case with $m \ge 3$ and $n = 3$. Then we have $\beta(\mathbb{R}_+, m, n) = 3$ by lemma 2.1 and theorem 2.4. Therefore we have values of β as required. $\square$

3. Maximal column rank preservers of matrices over $\mathbb{R}_+$

In this section we have characterizations of the linear operators that preserve the maximal column rank of matrices over $\mathbb{R}_+$.

For a linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_+)$, it is said to *preserve the maximal column rank* if $mc(T(X)) = mc(X)$ for all $X \in \mathcal{M}_{m,n}(\mathbb{R}_+)$. It *preserves maximal column rank* r if $mc(T(X)) = r$ whenever $mc(X) = r$. For the terms *rank preserver* and *rank* r *preserver* on $\mathcal{M}_{m,n}(\mathbb{R}_+)$, they can be defined similarly (see [3]).

Let $\mathbb{S}$ be any set. Then an $n \times n$ square matrix A over $\mathbb{S}$ is called $\mathbb{S}$-*invertible* if there exists a matrix $B \in \mathcal{M}_{n,n}(\mathbb{S})$ such that $AB = BA = I$. It is well known [3] that a square matrix A over $\mathbb{R}_+$ is $\mathbb{R}_+$-invertible if and only if some permutation of its rows makes it a diagonal matrix all of whose diagonal entries are nonzero in $\mathbb{R}_+$. Also, we say that A is $\mathbb{S}$-*singular* if there exist some nonzero vectors $\boldsymbol{x} \in \mathcal{M}_{1,n}(\mathbb{S})$ and $\boldsymbol{y} \in \mathcal{M}_{n,1}(\mathbb{S})$ such that $\boldsymbol{x}A = 0$ and $A\boldsymbol{y} = 0$. Otherwise A is $\mathbb{S}$-*nonsingular*. If $\mathbb{S}$ is a field, nonsingularity and invertibility are equivalent.

LEMMA 3.1. *Let X be any matrix in $\mathcal{M}_{m,n}(\mathbb{R}_+)$. Suppose that Q is a $\mathbb{R}$-invertible matrix in $\mathcal{M}_{m,m}(\mathbb{R}_+)$, and P a $\mathbb{R}_+$-invertible matrix in $\mathcal{M}_{n,n}(\mathbb{R}_+)$. Then we have that*

$$mc(QX) = mc(X) = mc(XP).$$

Proof. First, we show that $mc(X) = mc(QX)$. Suppose that $mc(X) = r$ and $mc(QX) = s$. Since $mc(X) = r$, there exist r linearly independent columns $\boldsymbol{x}_{i(1)}, \cdots, \boldsymbol{x}_{i(r)}$ in X which are maximal. Then $Q\boldsymbol{x}_{i(1)}, \cdots, Q\boldsymbol{x}_{i(r)}$ are linearly independent columns in QX because Q is a $\mathbb{R}$-invertible matrix. Thus we have $s \geq r$. Also, since $mc(QX) = s$, there exist s linearly independent columns $\boldsymbol{y}_{i(1)}, \cdots, \boldsymbol{y}_{i(s)}$ in QX which are maximal. Then $Q^{-1}\boldsymbol{y}_{i(1)}, \cdots, Q^{-1}\boldsymbol{y}_{i(s)}$ are linearly independent columns in $X(= Q^{-1}QX)$. Hence $r \geq s$. Therefore we have $mc(X) = mc(QX)$.

Next, we show that $mc(X) = mc(XP)$. Since P is $\mathbb{R}_+$-invertible, each column of P has only one nonzero entry. Let p_i be the nonzero entry of the ith column of P. Then we have

$$XP = [\boldsymbol{x}_1 \cdots \boldsymbol{x}_n]P = [p_1\boldsymbol{x}_{i(1)} \cdots p_n\boldsymbol{x}_{i(n)}],$$

where $\{i(1), \cdots, i(n)\}$ is a permutation of $\{1, \cdots, n\}$. If $\boldsymbol{x}_a, \boldsymbol{x}_b, \cdots, \boldsymbol{x}_c$ are linearly independent columns of X, then $p_a\boldsymbol{x}_{i(a)}, p_b\boldsymbol{x}_{i(b)}, \cdots, p_c\boldsymbol{x}_{i(c)}$ are linearly independent columns of XP, and conversely. Hence we have $mc(X) = mc(XP)$. □

For a given matrix P in $\mathcal{M}_{n,n}(\mathbb{R}_+)$, we define a linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ by $T(X) = XP$. If P is $\mathbb{R}_+$-invertible, lemma 3.1 implies that T preserves the maximal column rank. But the following example shows that if P is $\mathbb{R}$-invertible, T may not preserve the maximal column rank.

EXAMPLE 3.2. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{4,4}(\mathbb{R}_+) \quad \text{and} \quad P = A \oplus I_{n-4,n-4},$$

where $n \geq 4$. Then we can easily show that P is $\mathbb{R}$-invertible in $\mathcal{M}_{n,n}(\mathbb{R}_+)$, but not $\mathbb{R}_+$-invertible. Let T be a linear operator defined by $T(X) = XP$ on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ with $\min(m,n) \geq 4$. Consider the following matrices

$$Y = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{4,4}(\mathbb{R}_+) \quad \text{and} \quad X = Y \oplus 0_{m-4,n-4}.$$

Since all columns of Y are linearly independent, we have $mc(Y) = 4$, and hence $mc(X) = 4$ by (2.5). But

$$YA = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

has $mc(YA) < 4$ because the second column is the sum of the last two columns of YA. Notice that $XP = YA \oplus 0_{m-4,n-4}$, and hence $mc(XP) = mc(YA) < 4$ by (2.5). Therefore T does not preserve maximal column rank 4 on $\mathcal{M}_{m,n}(\mathbb{R}_+)$.

We say that a linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ is a (U,V)-*operator* if there exist $\mathbb{R}_+$-invertible matrices U and V in $\mathcal{M}_{m,m}(\mathbb{R}_+)$ and $\mathcal{M}_{n,n}(\mathbb{R}_+)$, respectively such that either $T(A) = UAV$ or $m = n$, $T(A) = UA^tV$ for all A in $\mathcal{M}_{m,n}(\mathbb{R}_+)$. Lemma 3.1 shows that if T is a (U,V)-operator on $\mathcal{M}_{m,n}(\mathbb{R}_+)$, then T preserves all maximal column ranks.

Beasley et al. [3] obtained the following two theorems:

THEOREM 3.3. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ with $\min(m,n) \geq 2$. Then the following are equivalent:*

(a) *T preserves ranks 1 and 2;*
(b) *T is injective, and there exist two matrices U and V over $\mathbb{R}_+$ such that either*
  (1) *$T(X) = UXV$ for all X in $\mathcal{M}_{m,n}(\mathbb{R}_+)$, or*
  (2) *$T(X) = UX^tV$ for all X in $\mathcal{M}_{m,n}(\mathbb{R}_+)$, possibly $m \neq n$.*

[Here, T need not to be a (U,V)-operator because U or V need not be invertible.]

THEOREM 3.4. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ with $\min(m,n) \geq 4$. Then the following are equivalent:*

(a) *T preserves ranks* $1, 2$, *and* 4;
(b) *T is a (U, V)-operator on* $\mathcal{M}_{m,n}(\mathbb{R}_+)$;
(c) *T preserves all ranks.*

The next sequence of lemmas will be used to prove the main theorem.

LEMMA 3.5. *Let Q be a given matrix in $\mathcal{M}_{m,m}(\mathbb{R}_+)$ and T a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ defined by $T(X) = QX$. If T preserves all maximal column ranks, then Q is $\mathbb{R}$-nonsingular, and conversely.*

Proof. We use the law of contraposition. Assume that Q is $\mathbb{R}$-singular. If $m = 1$, then we have $Q = 0$. Thus T does not preserve maximal column rank 1. Let $m \geq 2$. Since Q is $\mathbb{R}$-singular, $Q\boldsymbol{y}= 0$ for some nonzero vector $\boldsymbol{y}= [y_1, \cdots, y_m]^t$ in $\mathcal{M}_{m,1}(\mathbb{R})$. We choose a positive real $\alpha = \max\{|\ y_i|\ \mid i = 1, \cdots, m\}$ such that $\boldsymbol{z} = \boldsymbol{j}+\boldsymbol{y} \in \mathcal{M}_{m,1}(\mathbb{R}_+)$, where $\boldsymbol{j}$ is the vector in $\mathcal{M}_{m,1}(\mathbb{R}_+)$ with all entries α. Then $Q\boldsymbol{z} = Q(\boldsymbol{j}+\boldsymbol{y}) = Q\boldsymbol{j}$. Consider the vector $\boldsymbol{e_1}= [1, 0, \cdots, 0]^t \in \mathcal{M}_{n,1}(\mathbb{R}_+)$. Then $\boldsymbol{z}\boldsymbol{e_1^t}$ and $\boldsymbol{j}\boldsymbol{e_1^t}$ are distinct elements of $\mathcal{M}_{m,n}(\mathbb{R}_+)$ such that

$$T(\boldsymbol{z}\boldsymbol{e_1^t}) = Q\boldsymbol{z}\boldsymbol{e_1^t} = Q\boldsymbol{j}\boldsymbol{e_1^t} = T(\boldsymbol{j}\boldsymbol{e_1^t}).$$

This shows that T is not injective. By theorem 3.3, T does not preserve rank (and hence maximal column rank by theorem 2.6) 1 or 2. The converse follows from lemma 3.1. □

LEMMA 3.6. *For a matrix A in $\mathcal{M}_{n,n}(\mathbb{R}_+)$, define a linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ by $T(X) = XA$. If A is $\mathbb{R}$-singular, then T does not preserve maximal column rank* 1 *or* 2.

Proof. If $n = 1$, then $A = 0$, and the result is obvious. Let $n \geq 2$. Since A is $\mathbb{R}$-singular, there exists a nonzero vector $\boldsymbol{x}= [x_1, \cdots, x_n]$ in $\mathcal{M}_{1,n}(\mathbb{R})$ such that $\boldsymbol{x}A = 0$. Choose a positive real $\beta = \max\{|\ x_i|\ \mid i = 1, \cdots, n\}$ such that $\boldsymbol{w} = \boldsymbol{l} + \boldsymbol{x} \in \mathcal{M}_{1,n}(\mathbb{R}_+)$, where $\boldsymbol{l}$ is the vector in $\mathcal{M}_{1,n}(\mathbb{R}_+)$ with all entries β. Then $\boldsymbol{w}A = (\boldsymbol{l} + \boldsymbol{x})A = \boldsymbol{l}A$. Consider the vector $\boldsymbol{e_1}= [1, 0, \cdots, 0]^t \in \mathcal{M}_{m,1}(\mathbb{R}_+)$. Then $\boldsymbol{e_1}\boldsymbol{w}$ and $\boldsymbol{e_1}\boldsymbol{l}$ are distinct elements of $\mathcal{M}_{m,n}(\mathbb{R}_+)$ such that

$$T(\boldsymbol{e_1}\boldsymbol{w}) = \boldsymbol{e_1}\boldsymbol{w}A = \boldsymbol{e_1}\boldsymbol{l}A = T(\boldsymbol{e_1}\boldsymbol{l}).$$

This shows that T is not injective and the result follows from theorems 3.3 and 2.6. □

For any two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathcal{M}_{m,n}(\mathbb{R}_+)$, we say *$A$ dominates B* (written $B \leq A$ or $A \geq B$) if $a_{ij} = 0$ implies $b_{ij} = 0$.

Let A be a fixed matrix in $\mathcal{M}_{n,n}(\mathbb{R}_+)$. Then lemma 3.6 shows that if $T(X) = XA$ preserves the maximal column rank on $\mathcal{M}_{m,n}(\mathbb{R}_+)$, then A dominates a permutation matrix P.

LEMMA 3.7. *For any positive reals $\alpha_1, \alpha_2, \cdots, \alpha_n$, there is a set of linearly independent vectors $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \cdots, \boldsymbol{x_n}\}$ in $\mathbb{R}_+^n$ such that*

$$\alpha_1\boldsymbol{x_1} + \alpha_2\boldsymbol{x_2} = \alpha_3\boldsymbol{x_3} + \cdots + \alpha_n\boldsymbol{x_n}.$$

Proof. Choose $\boldsymbol{x_1} = \alpha_1^{-1}[1, 1, 0, \cdots, 0]^t$, $\boldsymbol{x_2} = \alpha_2^{-1}[0, 0, 1, \cdots, 1]^t$, $\boldsymbol{x_3} = \alpha_3^{-1}[1, 0, 1, 0, \cdots, 0]$, $\boldsymbol{x_4} = \alpha_4^{-1}[0, 1, 0, 1, 0, \cdots, 0]^t$, and for $i \geq 5$ $\boldsymbol{x_i} = \alpha_i^{-1}[0, \cdots, 0, 1, 0, \cdots, 0]^t$. Then these are vectors satisfying the required property. □

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices in $\mathcal{M}_{m,n}(\mathbb{R}_+)$ with $A \geq B$. Then we say A *strictly dominates* B (written $A > B$) if for some (i, j)-entry, $a_{ij} \neq 0$, but $b_{ij} = 0$.

LEMMA 3.8. *Let $n \geq 4$. If A is a $\mathbb{R}$-nonsingular matrix in $\mathcal{M}_{n,n}(\mathbb{R}_+)$ with $A > P$ for some permutation matrix P, then $T(X) = XA$ does not preserve maximal column rank n on $\mathcal{M}_{n,n}(\mathbb{R}_+)$.*

Proof. Let $A = [a_{ij}] \in \mathcal{M}_{n,n}(\mathbb{R}_+)$ be a $\mathbb{R}$-nonsingular matrix which strictly dominates a permutation matrix P. Since $A > P$, one of the columns of A has more than one positive entry. Without loss of generality we may assume that the first column has at least two nonzero entries. We like to show that there is a linearly independent set $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \cdots, \boldsymbol{x_n}\}$ in $\mathbb{R}_+^n$ such that for some nonnegative reals $\beta_2, \cdots, \beta_n$,

$$\boldsymbol{y_1} = \beta_2\boldsymbol{y_2} + \beta_3\boldsymbol{y_3} + \cdots + \beta_n\boldsymbol{y_n}, \tag{3.1}$$

where $T(X) = XA = [\boldsymbol{y_1}\ \ \boldsymbol{y_2}\ \ \cdots\ \ \boldsymbol{y_n}]$, $X = [\boldsymbol{x_1}\ \ \boldsymbol{x_2}\ \ \cdots\ \ \boldsymbol{x_n}] \in \mathcal{M}_{n,n}(\mathbb{R}_+)$ and $\boldsymbol{y_i} = a_{1i}\boldsymbol{x_1} + a_{2i}\boldsymbol{x_2} + \cdots + a_{ni}\boldsymbol{x_n}$ for all $i = 1, 2, \cdots, n$. Now, (3.1) is equivalent to

$$\lambda_1\boldsymbol{x_1} + \lambda_2\boldsymbol{x_2} + \cdots + \lambda_n\boldsymbol{x_n} = 0, \tag{3.2}$$

where for all $i = 1, 2, \cdots, n$

$$\lambda_i = a_{i1} - a_{i2}\beta_2 - a_{i3}\beta_3 - \cdots - a_{in}\beta_n. \tag{3.3}$$

Claim: *There are nonnegative reals $\beta_2, \beta_3, \cdots, \beta_n$ such that exactly two of λ_k are positive and the other $n-2$ are negative. More precisely, there exist positive reals $\beta_2, \beta_3, \cdots, \beta_n$ and a permutation σ of $\{1, 2, \cdots, n\}$ such that*

$$\lambda_{\sigma(i)} \ \ is \ \begin{cases} positive & if\ i \leq 2, \\ negative & if\ i \geq 3 \end{cases}. \tag{3.4}$$

Assume the claim holds. Let $\alpha_i = \lambda_{\sigma(i)}$ for $i \leq 2$, and $\alpha_i = -\lambda_{\sigma(i)}$ for $i \geq 3$. Then the equation (3.2) becomes,

$$\alpha_1 \boldsymbol{x_{\sigma(1)}} + \alpha_2 \boldsymbol{x_{\sigma(2)}} = \alpha_3 \boldsymbol{x_{\sigma(3)}} + \alpha_4 \boldsymbol{x_{\sigma(4)}} + \cdots + \alpha_n \boldsymbol{x_{\sigma(n)}}. \tag{3.5}$$

Now, by lemma 3.7, the equation (3.5) has a linearly independent solution vectors $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \cdots \boldsymbol{x_n}\}$. For $X = [\boldsymbol{x_1}\ \boldsymbol{x_2}\ \cdots\ \boldsymbol{x_n}] \in \mathcal{M}_{n,n}(\mathbb{R}_+)$, the corresponding $\boldsymbol{y_i}$'s satisfy (3.1). This shows we have found $X \in \mathcal{M}_{n,n}(\mathbb{R}_+)$ such that $mc(X) = n$ but $mc(XA) \leq n-1$. So we only need to show the claim.

Proof of the claim. Consider the relations

$$\begin{array}{ccccccccc} a_{12}z_2 & + & a_{13}z_3 & + & \cdots & + & a_{1n}z_n & = & a_{11} \\ a_{22}z_2 & + & a_{23}z_3 & + & \cdots & + & a_{2n}z_n & = & a_{21} \\ \vdots & & & & \vdots & & & & \vdots \\ a_{n2}z_2 & + & a_{n3}z_3 & + & \cdots & + & a_{nn}z_n & = & a_{n1}. \end{array} \tag{3.6}$$

First we consider the case when each of the relations in (3.6) is an equation in $z_2, z_3, \cdots, z_n$. (This is the case when coefficients of $z_2, z_3, \cdots, z_n$ in any of the relations do not vanish simultaneously.) Then (3.6) represents equations of n planes in $\mathbb{R}^n$. Since A is $\mathbb{R}$-nonsingular, the planes are distinct. Moreover, each of them have nonempty intersection with $\mathbb{R}^n$. For any positive vector $(b_2, b_3, \cdots, b_n) \in \mathbb{R}_+^{n-1}$, let

$$\alpha_k = \frac{a_{k1}}{a_{k2}b_2 + a_{k3}b_3 + \cdots + a_{kn}b_n} \qquad \text{for} \quad k = 1, 2, \cdots, n.$$

Then the positive ray passing through the origin and $(b_2, b_3, \cdots, b_n)$ meets the k-th plane in (3.6) at the point $P_k = (\alpha_k b_2, \alpha_k b_3, \cdots, \alpha_k b_n)$. By our choice of the first column of the matrix A, at least two of α_k are nonzero. Thus any positive ray from the origin meets at least two of the planes. Now, we choose the vector $(b_2, b_3, \cdots, b_n)$ in such a way that those points P_k which are not the origin are all distinct. Let σ be a permutation of $\{1, 2, \cdots, n\}$ such that $P_{\sigma(1)}$, $P_{\sigma(2)}, \cdots$, $P_{\sigma(n)}$ are arranged so that their distances from the origin are in descending order. Then $P_{\sigma(2)}$ and $P_{\sigma(3)}$ are distinct points in $\mathbb{R}_+^{n-1}$. Let Q be the mid-point of the line joining $P_{\sigma(2)}$ and $P_{\sigma(3)}$. If Q has coordinate $(\beta_2', \beta_3', \cdots, \beta_n')$ then for these values of $(\beta_2, \beta_3, \cdots, \beta_n)$, (3.4) holds and we are done.

Next, consider the case when some of the relations in (3.6) is not an equation of z_i. Then $a_{i2} = a_{i3} = \cdots = a_{in} = 0$ for some $i = 1, 2, \cdots, n$. Since A is $\mathbb{R}$-nonsingular, $a_{i1} > 0$ and, consequently, $\lambda_i > 0$. Moreover, there can not be two such i, because in that case the rows of A would be linearly dependent. So exactly $n-1$ relations in (3.6) represent equations of $z_2, z_3, \cdots, z_n$ in $\mathbb{R}^{n-1}$. Using the argument of the previous case we can find a point Q with positive coordinates $(\beta_2', \beta_3', \cdots, \beta_n')$ such that for these values of $(\beta_2, \beta_3, \cdots, \beta_n)$ exactly one value of λ_k other than λ_i is positive and the other $n-2$ are negative. This completes the proof of the claim. $\square$

Lemma 3.9. *Let $m \geq n \geq 4$ and A be a matrix in $\mathcal{M}_{n,n}(\mathbb{R}_+)$. If $T(X) = XA$ preserves maximal column rank $1, 2$ and n on $\mathcal{M}_{m,n}(\mathbb{R}_+)$, then A is $\mathbb{R}_+$-invertible.*

Proof. By lemma 3.6, we have A is $\mathbb{R}$-nonsingular and therefore $A \geq P$ for some permutation matrix P. Suppose that $A > P$. Then, by lemma 3.8, there is a matrix X in $\mathcal{M}_{n,n}(\mathbb{R}_+)$ such that $mc(X) = n$ and $mc(XA) \leq n-1$. If O is the zero matrix in $\mathcal{M}_{m-n,n}(\mathbb{R}_+)$, then $Y = \begin{bmatrix} X \\ O \end{bmatrix} \in \mathcal{M}_{m,n}(\mathbb{R}_+)$. By the property (2.5), we have that $mc(Y) = n$ and $mc(YA) = mc\left(\begin{bmatrix} XA \\ O \end{bmatrix}\right) \leq n-1$. Thus T does not preserve maximal column rank n, a contradiction. This implies A is $\mathbb{R}_+$-invertible. $\square$

Let T be a linear operator on $\mathcal{M}_{n,n}(\mathbb{R}_+)$ defined by $T(X) = X^t$, a transpose of $X \in \mathcal{M}_{n,n}(\mathbb{R}_+)$. Then T preserves all ranks since it is a (U, V)-operator. But the following example shows that the transposition operator does not preserve all maximal column ranks.

Example 3.10. Let

$$B = \begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ 0 & d & e & 0 \\ 0 & 0 & f & 0 \end{bmatrix}$$

be a matrix in $\mathcal{M}_{4,4}(\mathbb{R}_+)$, where a, b, c, d, e and f are all positive reals. Then we have $mc(B) = 3$ since the first three columns are linearly independent. But the maximal column rank of

$$B^t = \begin{bmatrix} a & b & 0 & 0 \\ 0 & c & d & 0 \\ 0 & 0 & e & f \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is 4 by (2.5) and example 2.5. Thus the transposition operator does not preserve maximal column rank 3 on $\mathcal{M}_{4,4}(\mathbb{R}_+)$.

LEMMA 3.11. *If T is a transposition operator on $\mathcal{M}_{n,n}(\mathbb{R}_+)$ with $n \geq 4$, then T does not preserve maximal column rank r for $r \geq 3$, but preserves all ranks.*

Proof. Let B be the matrix in example 3.10. Consider $C = B \oplus 0_{n-4} \in \mathcal{M}_{n,n}(\mathbb{R}_+)$. Then $mc(C) = 3$ by (2.5), but $T(C) = C^t$ has maximal column rank 4 by (2.5). Let

$$D = B \oplus I_k \oplus 0_{n-k-4} \in \mathcal{M}_{n,n}(\mathbb{R}_+),$$

where I_k is the identity matrix of order k. Then $mc(D) = 3 + k$ but $T(D) = D^t$ has maximal column rank $4 + k$. Therefore T does not preserve maximal column rank r for $r \geq 3$, but it is obvious that T preserves all ranks since the transposition operator is a (U, V)-operator. □

LEMMA 3.12. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_+)$, and $m \geq n \geq 4$. If T preserves maximal column ranks $1, 2$ and n, then there exist $Q \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $P \in \mathcal{M}_{n,n}(\mathbb{R}_+)$ such that $T(X) = QXP$ for all $X \in \mathcal{M}_{m,n}(\mathbb{R}_+)$, where Q is $\mathbb{R}$-invertible and P is $\mathbb{R}_+$-invertible.*

Proof. Since T preserves maximal column ranks 1 and 2, it preserves ranks 1 and 2 by theorem 2.6. Thus T is injective and has the form (1) $T(X) = QXP$ or (2) $T(X) = QX^tP$ given in theorem 3.3. Suppose (1) holds. First, we show that Q is $\mathbb{R}$-invertible, equivalently, $\mathbb{R}$-nonsingular. If Q is $\mathbb{R}$-singular, then, as in the proof of lemma 3.5, we can choose $X_1, X_2 \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ such that $X_1 \neq X_2$ and $QX_1 = QX_2$. We then have $T(X_1) = T(X_2)$, contradicting the fact that T is injective. Thus, Q is $\mathbb{R}$-nonsingular. Next, we show that P is $\mathbb{R}_+$-invertible. It follows from lemma 3.6 that P is $\mathbb{R}$-nonsingular. If P is not $\mathbb{R}_+$-invertible, then, by lemma 3.9, there exists $Y \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ of maximal column rank $1, 2$ or n such that $mc(Y) \neq mc(YP)$. We then have, $mc(T(Y)) = mc(QYP) = mc(YP) \neq mc(Y)$, a contradiction. Thus P must be $\mathbb{R}_+$-invertible.

Now, suppose (2) holds. Then Q and P are all $m \times n$ matrices, and $T^2(X) = WXZ$, where $W = QP^t$ and $Z = Q^tP$. Since T preserves maximal column ranks $1, 2$, and n, so does T^2. By (1), W is $\mathbb{R}$-invertible in $\mathcal{M}_{m,m}(\mathbb{R}_+)$ and Z is $\mathbb{R}_+$-invertible in $\mathcal{M}_{n,n}(\mathbb{R}_+)$. If $m > n$, then by the properties (2.1) and (2.2), we obtain that

$$\rho(W) = \rho(QP^t) \leq \min(\rho(Q), \rho(P^t)) \leq \rho(Q) \leq \min(m, n) = n < m,$$

a contradiction. Thus we have $m = n$, and so Q and P are $\mathbb{R}_+$-invertible matrices. But $T(X) = QX^tP$ does not preserve maximal column rank n by lemma 3.11. □

THEOREM 3.13. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ with $m \geq n \geq 4$. Then the following are equivalent:*

(1) *T preserves maximal column ranks $1, 2$, and n;*
(2) *There exist $Q \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $P \in \mathcal{M}_{n,n}(\mathbb{R}_+)$ such that $T(X) = QXP$ for all $X \in$*
$\mathcal{M}_{m,n}(\mathbb{R}_+)$, where Q is $\mathbb{R}$-invertible and P is $\mathbb{R}_+$-invertible;
(3) *T preserves all maximal column ranks.*

Proof. The proof follows lemma 3.12. □

If $n \leq 3$, then the linear operators that preserve maximal column rank on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ are the same as the rank preservers, which were characterized in [3].

Thus we have characterizations of the linear operators that preserve the maximal column rank of matrices over nonnegative reals.

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KYUNG-TAE KANG
Combinatorial and Computational Mathematics Center, POSTECH
Pohang 790-784, Republic of Korea

DUK-SUN KIM
Department of Mathematics, SungKyunKwan University,
Suwon 440-746, Republic of Korea

SANG-GU LEE
Department of Mathematics, SungKyunKwan University,
Suwon 440-746, Republic of Korea
E-mail: `sglee@skku.edu`

HAN-GUK SEOL
Department of Mathematics, Daejin University,
Pocheon, 487-711, Republic of Korea