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On Some New Stochastic Orders of Interest in Reliability Theory

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Abstract. The purpose of this paper is to study new notions of stochastic comparisons and ageing classes based on the total time on test transform order. We give relationships to other stochastic orders and aging classes given previously. Several preservation properties under the reliability operations of random minima and series system are given.

Key Words : *Stochastic order, total time on test transform order, increasing concave order, mean inactivity time order.*

ACRONYMS

NBUE	new better than used in expectation
HNBUE	harmonic new better than used in expectation
TTT	total time on test
DRLT	decreasing residual life in TTT
IMIT	increasing mean inactivity time
NBUT	new better than used in TTT order
NBU(2)	new better than used in second order dominace

1. INTRODUCTION

The stochastic comparisons of distributions have been an important area of research in many diverse areas of statistics and probability. Many different types of stochastic orders have been studied in the literature; a comprehensive discussion of them is available in *Shaked* and *Shanthikumar* (1994) and *Muller* and *Stoyan* (2002).

It is often easy to make value judgements when such orderings exist. For example, if X and Y are two random variables with distribution functions F and G satisfying $F(x) \geq G(x)$ for every x, then we say that X is stochastically smaller than Y, denoted by $X \leq_{ST} Y$. Stochastic ordering between two probability distributions is more informative than simply comparing their means or medians.

If $X \leq_{ST} Y$, then every quantile of the distribution of X is smaller than the corresponding quantile of the distribution Y, and any reasonable measure of location will be smaller for X than for Y.

It is well known that if one wishes to compare the dispersion between two distributions, the simplest way would be to compare their standard deviation or some such other measure of dispersion. However, such a comparison is based only on two single numbers, and therefore may not exist or they may not be the appropriate quantities to compare in some situations. A more informative way will be to compare their interquantile difference of all order at the same time. Let F^{-1} be the right continuous inverse of F defined by $F^{-1}(p) = \inf\{x : F(x) \geq p\}$, $0 < p < 1$. We use a similar notation for all other distribution functions. If

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha),$$

holds for every $0 < \alpha \leq \beta < 1$, then we say that X is less dispersive than Y and denoted by $X \leq_{DISP} Y$.

An important ordering which can be used for the joint comparison of both the location and the dispersion of distributions is the increasing concave order: X is said to be smaller than Y in the increasing concave order, denoted by $X \leq_{ICV} Y$, if

$$\int_0^x \overline{F}(u)du \leq \int_0^x \overline{G}(u)du, \quad \text{for all } x.$$

In this paper, we study a stochastic order called the total time on test (TTT) transform order that is introduced in *Section* 2 below. In that section some new properties of this order are provide. In *Section* 3 we study the applications of the TTT-transform order to the ordering of residual lives. We will give a new order and new ageing classes, based on the TTT-transform order and study some of their reliability properties.

2. THE TTT-TRANSFORM ORDER

Let F be a distribution function of a non-negative random variable X, which is strictly increasing on its interval support. Let $p \in (0,1)$, and $t \geq 0$ be two values related by $t = F^{-1}(p)$. Denote by

$$A_F \equiv \left\{(x,u) : u \in (0,p),\ x \in (0, F^{-1}(u))\right\},$$

and

$$B_F \equiv \left\{(x,u) : u \in (p,1),\ x \in (0, F^{-1}(u))\right\}.$$

The above areas of the regions have various intuitive meanings in different applications. For example, if F is the distribution function of the lifetime of a machine, then

$$T_X(p) \equiv \|A_F(p) \cup B_F(p)\|, \quad \text{for all } p \in (0,1),$$

corresponds to the total time on test (TTT) transform associated with this distribution. The TTT-transform function of X is defined by

$$\begin{aligned} T_X(p) &= E\left[\left(F^{-1}(p) - X\right)^+\right] \\ &= \int_0^{F^{-1}(p)} \overline{F}(u)du, \end{aligned}$$

where $\overline{F} = 1 - F$ denotes the survival function of X.

Recently, the TTT-transform order has also come to use in reliability and life testing *Kochar et al.* (2002). We say that a random variable X is smaller than a random variable Y in the TTT-transform order (denoted by $X \leq_{TTT} Y$ or $F \leq_{TTT} G$) if, and only if,

$$\int_0^{F^{-1}(p)} \overline{F}(x)\ dx \leq \int_0^{G^{-1}(p)} \overline{G}(x)\ dx, \ \text{ for all } p \in (0,1). \tag{2.1}$$

For the previous orders we have the following relationships (see, *Kochar et al*, 2002):

$$X \leq_{ST} Y \Rightarrow\ X \leq_{TTT} Y\ \Rightarrow\ X \leq_{ICV} Y.$$

Applications, properties, and interpretations of the TTT-transform order in the statistical theory of reliability, and in economics can be found in *Kochar et al.* (2002), *Li* and *Zuo* (2004) and *Ahmad* and *Kayid* (2007).

On the other hand, for any random variable X, let

$$X_{(t)} = [t - X \mid X < t], \quad t \in \{x : F(x) < 1\},$$

denote a random variable whose distribution is the same as the conditional distribution of $t - X$ given that $X < t$, and known in literature as inactivity time (*Chandra* and *Roy* (2001), *Kayid* and *Ahmad* (2004), *Ahmad* and *Kayid* (2005), *Ahmad et al.* (2005) and *Li* and *Xu* (2006)). When X is the lifetime of a device, $X_{(t)}$ denote the time elapsed after failure till time t, given that the unit has already failed by time t.

An important ordering which relates the inactivity times is the mean inactivity time order: X is mean inactivity time smaller than Y (denoted by $X \leq_{MIT} Y$) if, and only if,

$$\mu_X(t) \equiv \int_0^t \frac{F(x)dx}{F(t)} \geq \int_0^t \frac{G(x)dx}{G(t)} \equiv \mu_Y(t). \tag{2.2}$$

Some basic properties of the TTT-transform order are now described. First note that because $F^{-1}(p) = \overline{F}^{-1}(1-p)$ and $G^{-1}(p) = \overline{G}^{-1}(1-p)$, we see that

$$X \leq_{TTT} Y \Leftrightarrow \int_0^{\overline{F}^{-1}(p)} \overline{F}(x)dx \leq \int_0^{\overline{G}^{-1}(p)} \overline{G}(x)dx, \quad p \in (0,1).$$

It is easy to verify that if $F^{-1}(p) \leq G^{-1}(p)$, then

$$X \leq_{TTT} Y \Leftrightarrow \mu_X(F^{-1}(p)) \geq \mu_Y(G^{-1}(p)), \text{ for all } p \in (0,1). \tag{2.3}$$

In the next theorem we study the relationships between the TTT-transform order and the MIT order. Recall that a random variable is said to be increasing mean inactivity time ($IMIT$) if its mean inactivity time is increasing.

Theorem 2.1.

Let X and Y be two non-negative random variables with common left end point 0 such that $X \leq_{ST} Y$. If $Y \leq_{MIT} X$, and if either X or Y or both are $IMIT$, then $Y \leq_{TTT} X$.

Proof.

First, note that the assumption $X \leq_{ST} Y$ implies that $F^{-1}(p) \leq G^{-1}(p)$ for all $p \in (0,1)$. Now, let Y is $IMIT$, we have

$$\mu_X(F^{-1}(p)) \leq \mu_Y(F^{-1}(p)) \leq \mu_Y(G^{-1}(p)),$$

where the first inequality follows from $Y \leq_{MIT} X$, and the second from the assumption that Y is $IMIT$. Then

$$\mu_X(F^{-1}(p)) \leq \mu_Y(G^{-1}(p)), \quad \text{for all } p \in (0,1). \tag{2.4}$$

Appealing to (2.3) the result follows. If X (rather than Y) is $IMIT$ then (2.4) follow from

$$\mu_X(F^{-1}(p)) \leq \mu_X(G^{-1}(p)) \leq \mu_Y(G^{-1}(p)),$$

where the first inequality follows from the assumption that X is $IMIT$ and the second from $Y \leq_{MIT} X$. ∎

Let us now introduce and study two new orders of dispersion-type variability order which compare residual lives at quantiles. The new orders can be used to compare the variability of the underlying random variables, among which are the usual dispersive order and the TTT-transform order. Such a study is meaningful because it throw an important light on the understanding of properties of the dispersive and the TTT-transform order and of relationships among these two orders and other orders. For a similar type of results one may refer to *Hu et al.* (2002) and *Belzunce et al.*(2003).

Definition 2.1.

Let X and Y be two random variables with distribution functions F and G, respectively. Then X is said to be smaller than Y in the $*$-*order* of the dispersion-type (denoted by $X \leq_{DISP-*} Y$) if

$$\left(X - F^{-1}(p)\right)_+ \leq_* \left(Y - G^{-1}(p)\right)_+, \quad \text{for all } p \in (0,1),$$

where $* = MIT$ and TTT and $z_+ = \max(z, 0)$.

Let X be a random variable with distribution function F, and let

$$X_t = [X - t \mid X > t], \quad t \in \{x : F(x) < 1\},$$

denote a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$. When X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time t, given that the device has survived up to time t. Note that if F and G in *Definition* 2.1 are continuous then we have

$$X \leq_{DISP-*} Y \Leftrightarrow X_{F^{-1}(p)} \leq_* Y_{G^{-1}(p)}, \text{ for all } p \in (0,1),$$

where $* = MIT$ and TTT. The difference between $\left(X - F^{-1}(p)\right)_+$ and $X_{F^{-1}(p)}$ is that the former has probability mass p on the point 0 while the latter has no mass on the point 0. The dispersion-type orders compare residual lives at quantiles. Consider a system which produces units with random lifetime X. Let the units be tested until the 100 p% of the units fail, and eliminate early failures. Then the additional residual lifetime of the remaining units is distributed as $X_{F^{-1}(p)}$. Therefore the dispersion-type orders can be used to compare two lifetimes under two different systems and under the same policy to eliminate early failures (see, *Belzunce et al.* (2003)).

The relationship between the $\leq_{DISP-TTT}$ and the $\leq_{ST}$ is given in the following theorem.

Theorem 2.2.

Let X and Y be two non-negative random variables with continuous distribution functions F and G, respectively and with common left end point 0. Then

$$X \leq_{DISP-TTT} Y \Rightarrow X \leq_{ST} Y.$$

Proof.

First let l_X be the left endpoints of support of X; that is, $l_X = \inf\{x : F(x) > 0\}$. Similarly, define l_Y. Suppose that $X \leq_{DISP-TTT} Y$ and that $l_X = l_Y = 0$. Then, for all $p \in (0, 1)$,

$$\int_{F^{-1}(p)}^{2F^{-1}(p)} \overline{F}(u)du \leq \int_{G^{-1}(p)}^{2G^{-1}(p)} \overline{G}(u)du. \tag{2.5}$$

Two cases arise.

(i) If F and G are not identical, and do not cross each other, then equation (2.5) it is seen that $\overline{F}(x) \leq \overline{G}(x)$ at a right neighborhood of 0, and therefore $\overline{F}(x) \leq \overline{G}(x)$ for all x; that is, $X \leq_{ST} Y$.

(ii) If $\overline{F}(x_0) > \overline{G}(x_0)$ for some $x_0 > 0$, then there exist $0 \leq t_0 < t_2$ such that $\overline{F}(t_0) = \overline{G}(t_0)$, $\overline{F}(u) > \overline{G}(u)$ for all $u \in (t_0, t_2)$, and

$$\overline{F}(x) \leq \overline{G}(x) \text{ for all } u \leq t_0. \tag{2.6}$$

If F is strictly increasing in the right neighborhood of t_0, then taking $p = F(t_0)$ and $F^{-1}(p) = t_2 - t_0$, we have that $p \in [0, 1)$, $F^{-1}(p) \leq G^{-1}(p)$ and $t_2 = 2F^{-1}(p)$. Therefore

$$\int_{F^{-1}(p)}^{2F^{-1}(p)} \overline{F}(u)du > \int_{F^{-1}(p)}^{2F^{-1}(p)} \overline{G}(u)du \geq \int_{G^{-1}(p)}^{2G^{-1}(p)} \overline{G}(u)du,$$

which contradicts equation (2.5). Now, define

$$t_1 = \inf\{x : F(x) = F(t_0), x \in R\}, \quad t_1' = \inf\{x : G(x) = G(t_0\}, x \in R\}.$$

If F is not strictly increasing in the right neighborhood of t_0, then $t_1 \leq t_1'$ (otherwise, there exists one point $x_0 < t_0$ such that $\overline{F}(x_0) > \overline{G}(x_0)$, contradicting equation (2.6)). Choose $0 < p < F(t_0)$ such that $G^{-1}(p) \geq F^{-1}(p)$ when $t_1 < t_1'$, and $|G^{-1}(p) - F^{-1}(p)| < \delta/3$ when $t_1 = t_1'$, and also $t_1' - G^{-1}(p) < \delta/3$, where

$$\delta = \int_{t_0}^{t_2} \left[\overline{F}(u) - \overline{G}(u)\right] du > 0.$$

Setting $t_2 = 2F^{-1}(p)$, we have

$$\begin{aligned}
&\int_{F^{-1}(p)}^{2F^{-1}(p)} \overline{F}(u)du - \int_{G^{-1}(p)}^{2G^{-1}(p)} \overline{G}(u)du \\
&\qquad = \int_{F^{-1}(p)}^{G^{-1}(p)} \overline{F}(u)du + \int_{G^{-1}(p)}^{t_1'} \left[\overline{F}(u) - \overline{G}(u)\right] du \\
&\qquad + \int_{t_0}^{t_2} \left[\overline{F}(u) - \overline{G}(u)\right] du - \int_{t_2}^{x+G^{-1}(p)} \overline{G}(u)du \\
&\qquad > 0,
\end{aligned}$$

contradicting equation (2.5). Therefore, F and G do not cross each other, and thus $X \leq_{ST} Y$.

Next, if $l_Y \geq l_X > -\infty$, then $X - l_X \leq_{DISP-TTT} Y - l_Y$ and, hence, $X - l_X \leq_{ST} Y - l_Y$. therefore, $x \leq_{ST} Y - (l_Y - l_X) \leq_{ST} Y$. This completes the proof. ∎

Theorem 2.3.

Let X and Y be two random variables with continuous distribution functions with common left end point 0. Then

$$X \leq_{DISP-TTT} Y \;\Leftrightarrow\; X \leq_{DISP} Y.$$

Proof.

The implication that $X \leq_{DISP} Y \Rightarrow X \leq_{DISP-TTT} Y$ is trivial. Suppose now that $X \leq_{DISP-TTT} Y$. Then by *Theorem* 3.1 of *Belzunce et al.* (2003), we have

$$\left(X - F^{-1}(p)\right)_+ \leq_{DISP-TTT} \left(Y - G^{-1}(p)\right)_+, \quad \text{for all } p \in (0,1),$$

and hence,

$$X_{F^{-1}(p)} \leq_{DISP-TTT} Y_{G^{-1}(p)}, \quad \text{for all } p \in (0,1).$$

Note that $X_{F^{-1}(p)}$ and $Y_{G^{-1}(p)}$ have continuous distribution functions with the origin as a common left endpoint of their supports. By *Theorem* 2.2, we have

$$X_{F^{-1}(p)} \leq_{ST} Y_{G^{-1}(p)}, \quad \text{for all } p \in (0,1),$$

which is equivalent to $X \leq_{DISP} Y$. ∎

Theorem 2.4.

Let X and Y be two non-negative random variables with common left end point 0 such that $X \leq_{ST} Y$. If $Y \leq_{MIT} X$, and if either X or Y or both are $IMIT$, then $Y \leq_{DISP-MIT} X$.

Proof.

To prove $Y \leq_{DISP-MIT} X$, it suffices to check that

$$\frac{\int_0^t F\left(u + F^{-1}(p)\right) du}{\int_0^t G\left(u + G^{-1}(p)\right) du} \quad \text{is increasing in } t \text{ for each } p \in (0,1).$$

Let us observe that

$$\frac{\int_0^t F\left(u + F^{-1}(p)\right) du}{\int_0^t G\left(u + G^{-1}(p)\right) du} = \frac{\int_0^t F\left(u + F^{-1}(p)\right) du}{\int_0^t G(u + F^{-1}(p)) du} \cdot \frac{\int_0^t G(u + F^{-1}(p)) du}{\int_0^t G\left(u + G^{-1}(p)\right) du}. \tag{2.7}$$

The assumption that $Y \leq_{MIT} X$ implies that the first term in the right hand side of equation (2.7) is increasing in $t \geq 0$ and that $X \leq_{ST} Y$ implies that $F^{-1}(p) \leq G^{-1}(p)$ for each p, while $X \in IMIT$ implies that the second term in the right hand side of equation (2.7) is increasing in $t \geq 0$. ∎

3. THE TTT-TRANSFORM ORDER OF RESIDUAL LIFE

In this section we study the applications of the TTT-transform order to the residual lives. Let X and Y be two non-negative random variables. The random variable X is said to be smaller than Y in the:

(i) hazard rate order (denoted by $X \leq_{HR} Y$) if

$$X_t \leq_{ST} Y_t, \quad \text{for all } t \in (0, l_X) \cap (0, l_Y);$$

(ii) mean residual life order (denoted by $X \leq_{MRL} Y$) if

$$E(X_t) \leq E(Y_t), \quad \text{for all } t \in (0, l_X) \cap (0, l_Y).$$

Let us observe that the orders $\leq_{HR}$ and $\leq_{MRL}$ are more informative than the orders $\leq_{ST}$ and $\leq_{ICV}$, since they compare the underlying systems at any time t in contrast to the global comparison offered by the orders $\leq_{ST}$ and $\leq_{ICV}$. Note also that, the total time on test transform order can be interpreted in several ways when the random variable represents the lifetime of a system or a unit, which yields several applications of this order (see,Kochar et al. 2002). However, if a system or a unit it is known to have a survival age t, it is important to take into account the age, when we compare the remaining lifetimes. Following this idea next we introduce a new partial order based on the TTT transform order of residual lives (for other orderings in terms of residual lives see *Belzunce et al.* (1997), *Belzunce et al.* (1999), *Gao et al.* (2002) and *Ahmed* and *Kayid* (2004)).

Definition 3.1.

Let X and Y be two non-negative random variables. The random variable X is said to be smaller than Y in the TTT transform order of residual lives (denoted by $X \leq_{TTT-RL} Y$) if

$$X_t \leq_{TTT} Y_t, \quad \text{for all } t \in (0, l_X) \cap (0, l_Y).$$

The following implications among some of the previous orders are easy to prove as indicated below:

$$X \leq_{TTT-RL} Y \;\Rightarrow\; X \leq_{TTT} Y \;\Rightarrow\; X \leq_{ICV} Y.$$

The implication $X \leq_{TTT-RL} Y \Rightarrow X \leq_{TTT} Y$ follows by taking $t = 0$ for non-negative random variables with $\overline{F}_X(0) = 1 = \overline{F}_Y(0)$. And the implication $X \leq_{TTT} Y$ $\Rightarrow X \leq_{ICV} Y$ follows from *Corollary* 3.1 of *Kochar et al.* (2002).

It is well-known that useful properties of stochastic orders are their closures with respect to typical reliability operations like random minima, series systems, convolution, etc. (see, e.g., *Shaked* and *Shanthikumar,* (1994)). Next, we present some results of this kind for the $\leq_{TTT-RL}$ order.

Random minima: In life-testing, if a random censoring is adopted, then the completely observed data constitute a sample of random size, say $X_1, X_2, ..., X_N$, where $N > 0$ is a random variable of integer value. In actuarial science, the claims received by an insurer in a certain time interval should also be a sample of random size, and, $\max\{X_1, ..., X_N\}$, to evaluate the one with the largest claim amount is of positive interest there. And $\min\{X_1, ..., X_N\}$ arises naturally in survival analysis as the minimal survival time of a transplant operation, where N of them are defective and hence may cause a death, respectively. Another example in transportation theory may be found in *Shaked* and *Wong* (1997). Therefore, it is of special interest to conduct stochastic comparison between random maximums or random minimums in practical situations. For more details, one may refer to *Shaked* (1975), *Bartozewicz* (2001), *Li* and *Zuo* (2004), *Ahmad* and *Kayid* (2007).

Theorem 3.1.

Let $X_1, X_2, ...$ and $Y_1, Y_2, ...$ each be a sequence of *i.i.d.* random variables, and N is independent of X_i*'s* and Y_i*'s*. If X_i*'s* and Y_i*'s* are both non-negative and with common left end point 0, then $X_i \leq_{TTT-RL} Y_i$ for $i = 1, 2, ...$, implies

$$\min\{X_1, ..., X_N\} \leq_{TTT-RL} \min\{Y_1, ..., Y_N\}.$$

Proof.

Denote $T_N = \min\{X_1, X_2, ..., X_N\}$ and $W_N = \min\{Y_1, Y_2, ..., Y_N\}$. Note that, for the minimum for all $t \geq 0$, it is holds (*Li* and *Zuo*, 2004)

$$[T_N - t \mid T_N > t] \stackrel{st}{=} \min\{[X_i - t \mid X_i > t], \quad i = 1, ..., N\},$$

and

$$[W_N - t \mid W_N > t] \stackrel{st}{=} \min\{[Y_i - t \mid Y_i > t], \quad i = 1, ..., N\},$$

where "$\stackrel{st}{=}$" denotes equality in law. Fix $t \geq 0$. By the assumption $X_i \leq_{TTT-RL} Y_i$, we have

$$[X_i - t \mid X_i > t] \leq_{TTT} [Y_i - t \mid Y_i > t],$$

then, by *Theorem* 2 (*ii*) of *Li* and *Zuo* (2004), we have

$$\min\{[X_i - t \mid X_i > t], \ i = 1, ..., N\} \leq_{TTT} \min\{[Y_i - t \mid Y_i > t], \ i = 1, ..., N\}.$$

Thus,

$$[T_N - t \mid T_N > t] \leq_{TTT} [W_N - t \mid W_N > t] \qquad \text{for all } t \geq 0,$$

implying that $T_N \leq_{TTT-RL} W_N$. This completes the proof. ∎

Series systems: Preservation properties of an order under series and/or parallel systems are of importance in reliability theory. The next theorem below shows that if $X \leq_{TTT-RL} Y$ then a series system of n components having independent lifetimes which are copies of Y has a larger lifetime, in the sense of $\leq_{TTT-RL}$, than a similar system of n components having independent lifetimes which are copies of X. According to *Theorem* 3.1, *Corollary* 3.1 can be deduced as below.

Corollary 3.1.

Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ be two sets of independent and identically non-negative distributed random variables. If $X_i \leq_{TTT-RL} Y_i$, then

$$\min\{X_1, X_2, ..., X_n\} \leq_{TTT-RL} \min\{Y_1, Y_2, ..., Y_n\}, \qquad n \geq 1.$$

Proof.

Denote $T_n = \min\{X_1, X_2, ..., X_n\}$ and $W_n = \min\{Y_1, Y_2, ..., Y_n\}$. Note also that, for the minimum for all $t \geq 0$, it holds

$$[T_n - t \mid T_n > t] \stackrel{st}{=} \min\left\{[X_i - t \mid X_i > t],\ \ i = 1, ..., n\right\},$$

and

$$[W_n - t \mid W_n > t] \stackrel{st}{=} \min\left\{[Y_i - t \mid Y_i > t],\ \ i = 1, ..., n\right\},$$

Fix $t \geq 0$. By the assumption $X_i \leq_{TTT-RL} Y_i$, we have

$$[X_i - t \mid X_i > t] \leq_{TTT} [Y_i - t \mid Y_i > t],$$

then, by *Theorem* 5.1 (a) of *Kochar et al.* (2002), we have

$$\min\left\{[X_i - t \mid X_i > t],\ i = 1, ..., n\right\} \leq_{TTT} \min\left\{[Y_i - t \mid Y_i > t],\ i = 1, ..., n\right\}.$$

Thus,

$$[T_n - t \mid T_n > t] \leq_{TTT} [W_n - t \mid W_n > t] \qquad \text{for all } t \geq 0,$$

implying that $T_n \leq_{TTT-RL} W_n$. This completes the proof. ▮

New ageing classes: If A denotes some ageing property, a general procedure to give definition and characterizations of A (as has been pointed out in *Pellerey* and *Shaked* (1997)) is by means of stochastic orders of residual lifetimes of the form

$$X \in A \Leftrightarrow X_t \geq_{st-ord} X_s \qquad \text{whenever } t < s, \quad t, s \in (0, l_X),$$

and

$$X \in A \Leftrightarrow X \geq_{st-ord} X_t \qquad \text{whenever} \quad t \in (0, l_X),$$

where $\geq_{st-ord}$ denotes some stochastic order and $X \in A$ denote that X has the ageing property A. For example the $DMRL$ (decreasing mean residual life) class can be characterized as follows (*Shaked* and *Shanthikumar* (1994)):

$$X \in DMRL \Leftrightarrow X_t \geq_{MRL} X_s \text{ whenever } t < s,\ t, s \in (0, l_X)$$

$$\Leftrightarrow X \geq_{MRL} X_t \qquad \text{whenever, } t \in (0, l_X),$$

and the $NBU(2)$ (new better than used in the increasing concave order) class can be defined as follow (*Deshpande* et al, 1986)

$$X \in NBU(2) \quad \Leftrightarrow \quad X \geq_{ICV} X_t \quad \text{whenever, } t \in (0, l_X)$$

Other results on characterizations of this kind can be found in *Deshpand et al.*(1986), *Belzunce et al.* (1999) and *Pellerey* and *Shaked* (1997).

Other characterizations and definitions are of the form

$$X \in A \;\Leftrightarrow\; Y \geq_{st-ord} X_t, \tag{3.1}$$

where Y is an exponential random variable with mean $E[X]$. The comparison of the exponential random variable Y with a random variable X, in the form (3.1) is an indication of an ageing property associated with X. We see below some of these classes (see *Klefsjo*, 1981 and *Shaked* and *Shanthikumar,* 1994):

$$X \in NBUE \Leftrightarrow Y \geq_{MRL} X;$$

$$X \in HNBUE \Leftrightarrow Y \geq_{ICX} X.$$

For more on the previous ageing notions, please see *Barlow* and *Proschan* (1981), *Deshpande* et al. (1986), *Bryson* and *Siddiqui* (1969) and the references therein.

Recently, *Ahmad et al.* (2005) proposed the following ageing class: Let X be a nonnegative random variable with distribution function F. We say that X is new better than used in the TTT transform order (denoted by $X \in NBUT$) if

$$X_t \leq_{TTT} X \quad \text{for all } t \in (0, l_X).$$

Next, we propose new ageing classes following the previous procedures for the TTT transform order and the new one introduced in this section.

Definition 3.2.

X is decreasing residual lives in the TTT transform order (denoted by $X \in DRLT$) if

$$X_s \leq_{TTT} X_t \qquad \text{for all } t < s \in (0, l_X).$$

Observe that

$$X \in DRLT \quad \Leftrightarrow \quad X_s \leq_{TTT-RL} X_t \qquad \text{for all } t < s \in (0, l_X)$$

$$\Leftrightarrow X \geq_{TTT-RL} X_t \qquad \text{for all } t \in (0, l_X).$$

For some of the previous ageing classes the following implications are either well known or easy to prove (by *Theorem* 2.1 of *Ahmad et al*, 2004 and the definitions of the new ageing classes):

$$DRLT \Rightarrow NBUT \Rightarrow NBU(2).$$

Recently, *Ahmad et al.* (2005) have proved that the ageing notions $NBUT$ is preserved under the operation of random minima. This preservation result also holds for ageing notion $DRLT$, as is shown in the following theorem.

Theorem 3.2.

Let $X_1, X_2, ...$ be a sequence of *i.i.d.* random lives, and N is independent of X_i's. If $X_1, X_2, ..., X_N$ are all $DRLT$, then $T_N = \min\{X_1, ..., X_N\}$ is also of $DRLT$ property.

Proof.

According to *Theorem* 3.1, we have

$$\begin{aligned} T_N &= \min\{X_1, ..., X_N\} \\ &\geq_{TTT-RL} \min\{[X_1 - t|X_1 > t], ..., [X_N - t|X_N > t]\} \\ &\overset{st}{=} [T_N - t|T_N > t] \end{aligned}$$

for all $t \geq 0$. This means that $T_N \in DRLT$.

On the other hand, it is a well known fact that some ageing notions are preserved under formation of parallel or series systems (see *Barlow* and *Proschan* (1981), *Abouammoh* and *El-Neweihi* (1986), *Hendi et al.* (1993), *Li* and *Kochar* (2001) and *Pellerey* and *Petakos* (2002)). Recently, *Ahmad et al.* (2005) have proved that the ageing property $NBUT$ is preserved under the operation of series systems. Again, this preservation result also holds for ageing notions $DRLT$, as is shown in the following result.

Corollary 3.2.

Let $X_1, X_2, ..., X_n$, be non-negative random variables, where $n \geq 2$. if of independent and identically non-negative distributed random variables. If $X_1, X_2, ..., X_n$ are all $DRLT$, then $T_n = \min\{X_1, X_2, ..., X_n\}$ is $DRLT$.

Proof.

According to *Theorem* 3.2, we have

$$\begin{aligned} T_n &= \min\{X_1, ..., X_n\} \\ &\geq_{TTT-RL} \min\{[X_1 - t \mid X_1 > t], ..., [X_n - t \mid X_n > t]\} \\ &\overset{st}{=} [T_n - t \mid T_n > t] \end{aligned}$$

for all $t \geq 0$. This means that $T_n \in DRLT$.

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