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Higher Order Moments of Record Values From the Inverse Weibull Lifetime Model and Edgeworth Approximate Inference

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Abstract. In this paper, we derive exact explicit expressions for the triple and quadruple moments of the lower record values from inverse the Weibull (IW) distribution. Next, we present and calculate the coefficients of the best linear unbiased estimates of the location and scale parameters of IW distribution (BLUEs) for different choices of the shape parameter and records size. We then use the higher order moments and the calculated BLUEs to compute the mean, variance, and the coefficients of skewness and kurtosis of certain linear functions of lower record values. By using the coefficients of the skewness and kurtosis, we develop approximate confidence intervals for the location and scale parameters of the IW distribution using Edgeworth approximate values and then compare them with the corresponding intervals constructed through Monte Carlo simulations. Finally, we apply the findings of the paper to some simulated data.

Key Words : *Lower record values; exact moments; single moments; double moments; triple moments; quadruple moments; Edgeworth approximation; coefficients of skewness and kurtosis; approximate confidence interval; pivotal quantity; best linear unbiased estimates; probability coverage; average width; Mote Carlo and simulations.*

1. INTRODUCTION

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Record values arise naturally in many real life applications involving data relating to weather, sport, economics and life testing studies. Many authors have studied record values and the associated statistics; see, for example, Chandler (1952), Nevzorov (1988), Nagaraja (1988), Ahsanullah (1980, 1988, 1990, 1995) and Arnold, Balakrishnan and Nagaraja (1992, 1998). Balakrishnan, Ahsanullah and Chan (1992) have established some recurrence relations for the moments of record values from the Gumbel distribution. Similar work has been carried out by Balakrishnan, Chan and Ahsanullah (1993) and Balakrishnan and Ahsanullah (1994a, 1994b, 1995) for the generalized extreme value, generalized Pareto, Lomax and exponential distributions, respectively. Ahsanullah (1980, 1990), Balakrishnan and Chan (1993), and Balakrishnan, Ahsanullah and Chan (1995) have also discussed some inferential methods based on record values from exponential, Gumbel, Weibull and logistic distributions, respectively. Sultan and Balakrishnan (1999) and Sultan and Moshref (2000) have discussed inferential techniques based on Weibull and generalized Pareto distributions, respectively.

Let $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ be the first n lower record values from the IW density function (*pdf*)

$$f(x) = cx^{-c-1}e^{-x^{-c}}, \quad c > 0,\ x \geq 0, \tag{1.1}$$

and cumulative distribution function (*cdf*)

$$F(x) = e^{-x^{-c}}, \quad x \geq 0. \tag{1.2}$$

¿From (1.1) and (1.2), it is easy to see that

$$f(x) = \frac{c}{x}\{-\log F(x)\}F(x). \tag{1.3}$$

The location-scale IW distribution has its density function given by

$$f(y) = \frac{c}{\sigma}\left(\frac{\sigma}{y-\theta}\right)^{c+1} \exp\{-(\frac{\sigma}{y-\theta})^{c}\},\ y \geq \theta, \sigma > 0,\ \theta \geq 0. \tag{1.4}$$

Drapella (1993) calls the IW distribution as the complementary Weibull distribution, while Mudholker and Kollia (1994) call it the reciprocal Weibull distribution. Jiag, Murthy and Ji (2001) have discussed some useful measures for the IW distribution. Nigm and Khalil (2006) used the relation in (1.3) to establish some recurrence relations for the single and the product moments of lower record values from IW distribution in (1.1).

The IW distribution plays an important role in many applications, including the dynamic components of diesel engines and several data set such as the times to breakdown of an insulating fluid subject to the action of a constant tension; see Nelson (1982). Calabria and Pulcini (1990) provide an interpretation of the IW distribution in the context of the load-strength relationship for a component.

Recently, Maswadah (2003) has the fitted IW distribution to the flood data reported from Dumonceaux and Antle (1973). For more details on the IW distribution see for example Johnson, Kotz and Balakrishnan (1995) and Murthy, Xie and Jiang, (2004).

In the following section, we derive the exact explicit expressions of the triple and quadruple moments of record values from the IW distribution. The higher order moments are then used together with the BLUEs in Section 3 to determine the coefficients of skewness and kurtosis of some pivotal quantities which depend on liner functions of lower records values from the IW. We then propose Edgeworth approximations for the distributions of these pivotal quantities and show that this method provides close approximations to the percentage points of the pivotal quantities determined by Monte Carlo simulations.Finally, examples to illustrate the methods of inference developed in this paper are discussed in Section 4.

2. HIGHER ORDER MOMENTS

In this section, we derive exact expressions for the triple and quadruple moments of the lower record values from the IW distribution in (1.1).

The joint density function of the first n lower record values $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ is given by [Arnold, Balakrishnan and Nagaraja (1998)]

$$f_{1,2,\ldots,n}(x_{L(1)}, x_{L(2)}, \ldots, x_{L(n)}) = f(x_{L(n)}) \prod_{i=1}^{n-1} \frac{f(x_{L(i)})}{F(x_{L(i)})}. \tag{2.1}$$

¿From (2.1), the *pdf* of $X_{L(m)}$ can be obtained as

$$f_m(x) = \frac{1}{\Gamma(m)} \{-\log[F(x)]\}^{m-1} f(x) \; x \geq 0, \; m = 1, 2, \ldots, \tag{2.2}$$

where $f(.)$ and $F(.)$ are given in (1.1) and (1.2), respectively.

The the joint *pdf* of $X_{L(m)}$ and $X_{L(n)}$ is given by

$$\begin{aligned} f_{m,n}(x,y) &= \frac{1}{\Gamma(m)\Gamma(n-m)} \{-\log[F(x)]\}^{m-1} \{-\log[F(y)] \\ &+ \log[F(x)]\}^{n-m-1} \frac{f(x)}{F(x)} f(y) \; 0 \leq y < x < \infty, \; m, n = 1, 2, \ldots, m < n, \end{aligned} \tag{2.3}$$

where $f(.)$ and $F(.)$ are given in (1.1) and (1.2), respectively.

By using (2.2) and (2.3) Nigm and Khalil (2006) have derived the single and double moments as

$$\mu_m^{(i)} = \frac{\Gamma(m - \frac{i}{c})}{\Gamma(m)}, \; i < mc, \tag{2.4}$$

and

$$\mu_{m,n}^{(i,j)} = \frac{\Gamma(m-\frac{i}{c})\Gamma(n-\frac{i+j}{c})}{\Gamma(m)\Gamma(n-\frac{i}{c})}, \quad i+j < nc, \tag{2.5}$$

where $\Gamma(\cdot)$ is the gamma function. Then they used the single and double moments of the lower record values to compute the BLUEs when $n = 5$.

2.1 Triple moments

From (2.1), the joint *pdf* of $X_{L(m)}$, $X_{L(n)}$ and $X_{L(p)}$, $(m < n < p)$, is obtained to be

$$\begin{aligned} f_{m,n,p}(x,y,z) &= \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(p-n)} \{-\log[F(x)]\}^{m-1} \{-\log[F(y)] \\ &+ \log[F(x)]\}^{n-m-1} \{-\log[F(z)] + \log[F(y)]\}^{p-n-1} \\ &\times \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} f(z), \ -\infty < z < y < x < \infty, \ m,n = 1,2,\ldots, m < n < p, \end{aligned} \tag{2.6}$$

where $f(.)$ and $F(.)$ are as given in (1.1) and (1.2), respectively.

¿From (2.6), we derive the triple moments of the m-th, n-th and p-th lower record values form IW distribution as

$$\mu_{m,n,p}^{(i,j,k)} = E(X_{L(m)}^i X_{L(n)}^j X_{L(p)}^k) = \frac{\Gamma(m-\frac{i}{c})\Gamma(n-\frac{i+j}{c})\Gamma(p-\frac{i+j+k}{c})}{\Gamma(m)\Gamma(n-\frac{i}{c})\Gamma(p-\frac{i+j}{c})}, \quad i+j+k < pc. \tag{2.7}$$

The required triple moments of record values to develop the Edgeworth approximation are $\mu_{m,n,p}^{(1,1,1)}$, $\mu_{m,n,p}^{(1,1,2)}$, $\mu_{m,n,p}^{(1,2,1)}$ and $\mu_{m,n,p}^{(2,1,1)}$, where $\mu_{m,n,p}^{(i,j,k)}$ is given by (2.7).

2.2 Quadruple moments

From (2.1), the joint *pdf* of $X_{L(m)}$, $X_{L(n)}$, $X_{L(p)}$ and $X_{L(q)}$, $(m < n < p < q)$, is given by

$$\begin{aligned} f_{m,n,p,q}(x,y,z,w) &= \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(p-n)\Gamma(q-p)} \{-\log[F(x)]\}^{m-1} \\ &\times \{-\log[F(y)] + \log[F(x)]\}^{n-m-1} \{-\log[F(z)] \\ &+ \log[F(y)]\}^{p-n-1} \{-\log[F(w)] + \log[F(z)]\}^{q-p-1} \\ &\times \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} \frac{f(z)}{F(z)} f(w), \end{aligned} \tag{2.8}$$

where $-\infty < w < z < y < x < \infty$, $m, n, p, q = 1, 2, \ldots, m < n < p < q$, and $f(.)$ and $F(.)$ are as given in (1.1) and (1.2), respectively.

From (2.8), we derive the quadruple moment generating function of the m-th, n-th, p-th and q-th lower record values as

$$\begin{aligned}\mu_{m,n,p,q}^{(i,j,k,l)} &= E(X_{L(m)}^{i} X_{L(n)}^{j} X_{L(p)}^{k} X_{L(q)}^{l}) \\ &= \frac{\Gamma(m-\frac{i}{c})\Gamma(n-\frac{i+j}{c})\Gamma(p-\frac{i+j+k}{c})\Gamma(q-\frac{i+j+k+l}{c})}{\Gamma(m)\Gamma(n-\frac{i}{c})\Gamma(p-\frac{i+j}{c})\Gamma(q-\frac{i+j+k}{c})},\end{aligned} \tag{2.9}$$

where $i + j + k + l < qc$.

The required quadruple moment of lower record values to develop the Edgeworth approximation is $\mu_{m,n,p,q}^{(1,1,1,1)}$, where $\mu_{m,n,p,q}^{(i,j,k,l)}$ is given by (2.9).

3. INFERENCE

In this section, we use the single and double moments of record values derived by Nigm and Khalil (2006) to calculate the coefficient of BLUEs for records of size 4,5,6 and 7. Then we use these BLUEs together with our new forms of the triple and quadruple moments of the lower record values to develop Edgeworth approximate inference for the location and scale parameters of IW distribution. In addition, we compare the confidence intervals based on Edgeworth approximation to the corresponding intervals constructed using Monte Carlo simulation.

3.1 BLUE's of θ and σ

Let $Y_{L(1)} \geq Y_{L(2)} \geq \ldots \geq Y_{L(n)}$ be the lower record values from the IW distribution given in (1.4), and let $X_{L(i)} = \left(Y_{L(i)} - \theta\right)/\sigma$, $i = 1, \ldots, n$, be the corresponding lower record values from the one parameter the IW distribution given in (1.1). Let us denote $E(X_{L(i)})$ by μ_i, $Var(X_{L(i)})$ by $\sigma_{i,i}$, and $Cov(X_{L(i)}, X_{L(j)})$ by $\sigma_{i,j}$. Further, let

$$\begin{aligned} \boldsymbol{Y} &= \left(Y_{L(1)}, Y_{L(2)}, \ldots, Y_{L(n)}\right)^T \\ \boldsymbol{\mu} &= (\mu_1, \mu_2, \ldots, \mu_n)^T \\ \mathbf{1} &= \underbrace{(1, 1, \ldots, 1)^T}_{n} \\ \text{and} \quad \Sigma &= ((\sigma_{i,j})), \; 1 \leq i, \; j \leq n. \end{aligned}$$

Then, the BLUEs of θ and σ are given by [see Balakrishnan and Cohen (1991)]

$$\theta^* = \left\{ \frac{\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1} - \boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^2} \right\} \boldsymbol{Y} = \sum_{i=1}^{n} A_i Y_{L(i)}, \tag{3.1}$$

and

$$\sigma^* = \left\{ \frac{\mathbf{1}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1}\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1} - \mathbf{1}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\mathbf{1}^{\mathbf{T}}\mathbf{\Sigma}^{-1}}{(\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1})^2} \right\} \boldsymbol{Y} = \sum_{i=1}^{n} B_i Y_{L(i)}. \tag{3.2}$$

Furthermore, the variances and covariance of these BLUEs are given by [see Balakrishnan and Cohen (1991)]

$$Var(\theta^*) = \sigma^2 \left\{ \frac{\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\boldsymbol{\mu}}{(\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1})^2} \right\} = \sigma^2 V_1, \tag{3.3}$$

$$Var(\sigma^*) = \sigma^2 \left\{ \frac{\mathbf{1}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1}}{(\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1})^2} \right\} = \sigma^2 V_2, \tag{3.4}$$

and

$$Cov(\theta^*, \sigma^*) = \sigma^2 \left\{ \frac{-\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1}}{(\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1})^2} \right\} = \sigma^2 V_3. \tag{3.5}$$

Table 3.1 The Coefficients of the BLUEs

	$c = 3$		$c = 4$		$c = 5$	
n	A_i	B_i	A_i	B_i	A_i	B_i
4	-0.2353	0.3519	-0.5294	0.7182	-0.8421	1.0764
	-1.4118	2.1112	-1.7647	2.3939	-2.1053	2.6909
	-2.1176	3.1668	-2.3529	3.1919	-2.6316	3.3636
	4.7647	-5.6299	5.6471	-6.3040	6.5789	-7.1309
5	-0.1400	0.2284	-0.3383	0.4896	-0.5591	0.7523
	-0.8400	1.3704	-1.1278	1.6319	-1.3978	1.8807
	-1.2600	2.0555	-1.5038	2.1759	-1.7473	2.3509
	-1.6200	2.6428	-1.8045	2.6111	-2.0161	2.7126
	4.8600	-6.2972	5.7744	-6.9085	6.7204	-7.6966
6	-0.0942	0.1647	-0.2395	0.3649	-0.4071	0.5706
	-0.5653	0.9881	-0.7985	1.2162	-1.0178	1.4265
	-0.8479	1.4821	-1.0646	1.6216	-1.2723	1.7831
	-1.0902	1.9055	-1.2776	1.9459	-1.4680	2.0575
	-1.3082	2.2866	-1.4601	2.2239	-1.6311	2.2861
	4.9058	-6.8270	5.8403	-7.3725	6.7964	-8.1238
7	-0.0684	0.1266	-0.1809	0.2875	-0.3143	0.4556
	-0.4104	0.7595	-0.6029	0.9582	-0.7857	1.1391
	-0.6156	1.1393	-0.8038	1.2776	-0.9821	1.4239
	-0.7915	1.4648	-0.9646	1.5331	-1.1332	1.6429
	-0.9498	1.7578	-1.1024	1.7521	-1.2591	1.8255
	-1.0959	2.0282	-1.2249	1.9468	-1.3686	1.9842
	4.9316	-7.2763	5.8794	-7.7552	6.8429	-8.4712

Table 3.2 The variances and covariances of the BLUEs

c	n	$Var(\theta^*)$	$Var(\sigma^*)$	$Cov(\theta^*,\sigma^*)$
3	4	0.3152	0.6834	-0.4713
3	5	0.1875	0.4901	-0.3059
3	6	0.1262	0.3803	-0.2206
3	7	0.0916	0.3103	-0.1696
4	4	0.3128	0.5646	-0.4243
4	5	0.1999	0.4138	-0.2893
4	6	0.1415	0.3255	-0.2156
4	7	0.1069	0.2680	-0.1698
5	4	0.3135	0.5055	-0.4007
5	5	0.2082	0.3739	-0.2801
5	6	0.1516	0.2960	-0.2124
5	7	0.1170	0.2447	-0.1696

For details, refer to Balakrishnan and Cohen (1991), and Arnold, Balakrishnan and Nagaraja (1992).

Table 3.1 represents the coefficients of the BLUEs A_i and B_i for records of sizes 4, 5, 6, 7 and the shape parameter c =3,4 and 5. As a check, the entries of Table 3.1 stratify the identities

$$\sum_{i=1}^{n} A_i = 1 \text{ and } \sum_{i=1}^{n} B_i = 0.$$

The variance and covariances of the BLUEs given in Table 3.2 have been calculated by setting $\sigma = 1$.

3.2 Edgeworth Approximate Inference

In this section, we use the higher moments of record values derived in Section 2 to develop confidence intervals for the location and scale parameters θ and σ of the IW distribution based on the following pivotal quantities:

$$R_1 = \frac{\theta^* - \theta}{\sigma\sqrt{V_1}}, \quad R_2 = \frac{\sigma^* - \sigma}{\sigma\sqrt{V_2}} \text{ and } R_3 = \frac{\theta^* - \theta}{\sigma^*\sqrt{V_1}}, \tag{3.6}$$

where θ^* and σ^* are the BLUEs of θ and σ with variances $\sigma^2 V_1$ and $\sigma^2 V_2$, respectively. R_1 can be used to draw inferences on θ when σ is known, while R_3 can be used to draw inference on θ when σ is unknown. Similarly, R_2 can be used to draw inference for σ when θ is unknown.

Notice that R_1 and R_2 in (3.6) can be rewritten as

$$R_1 = \frac{1}{\sqrt{V_1}}\left(\sum_{i=1}^{n} A_i X_{L(i)}\right) = \frac{R_1^*}{\sqrt{V_1}} \text{ and } R_2 = \frac{1}{\sqrt{V_2}}\left(\sum_{i=1}^{n} B_i X_{L(i)} - 1\right) = \frac{R_2^* - 1}{\sqrt{V_2}}, \tag{3.7}$$

where $X_{L(i)} = (Y_{L(i)} - \theta)/\sigma$, $i = 1, 2, \ldots, n$.

Thus, they are linear functions of record values arising from the one parameter IW distribution in (1.1). Since the distribution of a linear function of record values will in general not be known, we consider finding the approximate distribution by using Edgeworth approximation for a statistic T (with mean 0 and variance 1) given by [see Johnson, Kotz and Balakrishnan (1994)]

$$G(t) \approx \Phi(t) - \phi(t)\left\{\frac{\sqrt{\beta_1}}{6}(t^2-1) + \frac{\beta_2 - 3}{24}(t^3 - 3t) + \frac{\beta_1}{72}(t^5 - 10t^3 + 15t)\right\}, \quad (3.8)$$

where $\sqrt{\beta_1}$ and β_2 are the coefficients of skewness and kurtosis of T, respectively, and $\Phi(t)$ is the *cdf* of the standard normal distribution with corresponding *pdf* $\phi(t)$.

By making use of the exact expressions of moments presented in Section 2, and the BLUEs A_i and B_i, we determined the values of the mean, variance and the coefficients of skewness and kurtosis ($\sqrt{\beta_1}$ and β_2) of R_1^* and R_2^*, for $n = 4(1)7$ and $c = 5$. Notice that Edgeworth approximate is valid only when $c > 4$, that is because of the conditions on the quadruple moments.

The coefficients of skewness and kurtosis of R_1^* are given in Lemma 2.1.

Lemma 3.1

$$\sqrt{\beta_1}(R_1^*) = \frac{L_3 - 3L_2L_1 - 2L_1^2}{(L_2 - L_1^2)^{3/2}}, \quad (3.9)$$

and

$$\beta_2(R_1^*) = \frac{L_4 - 3L_1^4 + 6L_2L_1^2 - 4L_1L_3}{(L_2 - L_1^2)^2}, \quad (3.10)$$

where

$$L_1 = E(R_1^*) = E\left(\sum_{i=1}^{n} A_i Z_{i:n}\right) = \sum_{i=1}^{n} A_i \mu_{i:n}^{(1)}, \quad (3.11)$$

$$\begin{aligned} L_2 &= E(R_1^*)^2 = E\left(\sum_{i=1}^{n} A_i Z_{i:n}\right)^2 \\ &= \sum_{i=1}^{n} A_i^2 \mu_{i:n}^{(2)} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} A_i A_j \mu_{i,j:n}^{(1,1)}, \end{aligned} \quad (3.12)$$

$$\begin{aligned} L_3 &= E(R_1^*)^3 = E\left(\sum_{i=1}^{n} A_i Z_{i:n}\right)^3 \\ &= \sum_{i=1}^{n} A_i^3 \mu_{i:n}^{(3)} + 3\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} A_i^2 A_j \mu_{i,j:n}^{(2,1)} \\ &+ 3\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} A_i A_j^2 \mu_{i,j:n}^{(1,2)} + 6\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} A_i A_j A_k \mu_{i,j,k:n}^{(1,1,1)}, \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} L_4 &= E(R_1^*)^4 = E\left(\sum_{i=1}^{n} A_i Z_{i:n}\right)^4 \\ &= \sum_{i=1}^{n} A_i^4 \mu_{i:n}^{(4)} + 4\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} A_i^3 A_j \mu_{i,j:n}^{(3,1)} \\ &+ 4\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} A_i A_j^3 \mu_{i,j:n}^{(1,3)} + 6\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} A_i A_j \mu_{i,j:n}^{(2,2)}, \\ &+ 12\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} A_i^2 A_j A_k \mu_{i,j,k:n}^{(2,1,1)} + 12\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} A_i A_j^2 A_k \mu_{i,j,k:n}^{(1,2,1)} \\ &+ 12\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} A_i A_j A_k^2 \mu_{i,j,k:n}^{(1,1,2)} + 24\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{l=k+1}^{n} A_i A_j A_k A_l \mu_{i,j,k,l:n}^{(1,1,1,1)}. \end{aligned} \tag{3.14}$$

The coefficients of skewness and kurtosis for $R_2^* = \sum_{i=1}^{n} B_i Z_{i:n}$ can be obtained following steps similar to those in R_1^* and replacing A_i by B_i. Table 3.3 displays the values of the mean, variance and the coefficients of skewness and kurtosis ($\sqrt{\beta_1}$ and β_2) of R_1^* and R_2^*.

Table 3.3 Mean, variance and coefficients of skewness and kurtosis of R_1^* and R_2^* when c, θ and σ are 5.0, 0.0 and 1.0

	R_1^*				R_2^*			
n	Mean	V_1	$\sqrt{\beta_1}$	β_2	Mean	V_2	$\sqrt{\beta_1}$	β_2
4	.000	.314	-2.081	13.305	1.000	.505	2.115	13.305
5	.000	.208	-1.720	9.543	1.000	.374	1.751	9.575
6	.000	.152	-1.494	7.716	1.000	.296	1.524	7.756
7	.000	.117	-1.337	6.662	1.000	.245	1.365	6.702

An examination of the $(\sqrt{\beta_1}, \beta_2)$ values in Table 3.3 reveals that the distribution of R_2^* (and hence of R_2) is positively skewed, while the distribution of R_1^* (and hence of R_1) is negatively skewed. In addition, $\sqrt{\beta_1}$ for R_1 increases as n decreases while $\sqrt{\beta_1}$ for R_2 decreases as n increases. Also, the coefficient of kurtosis β_2 of both R_1^* and R_2^* decrease as n increases. Further, β_2 of both R_1^* and R_2^* are almost equal.

By making use of the entries in Table 3.3, we determined the lower and upper 1%, 2.5%, 5% and 10% points of R_1 and R_2 through the Edgeworth approximation in (3.8). These values, for $n = 4(1)7$ and $c = 5$ are presented in Tables 3.4 and 3.5. For the purpose of comparison, these percentage points were also determined by Monte Carlo simulations (based on 10001 runs) and they are presented along with the Edgeworth percentage points in Tables 3.4 and 3.5 when $c = 3, 4, 5$. From Tables 3.4 and 3.5, we see that the Edgeworth approximation of the distribution

Table 3.4 [Edgeworth] Approximate values and simulated values of percentage points of R_1 when θ and σ are 0.0 and 1.0

c	n	1%	2.5%	5%	10%	90%	95%	97.5%	99%
3	4	-3.548	-2.511	-1.861	-1.196	.862	.988	1.105	1.233
	5	-3.537	-2.489	-1.803	-1.174	.929	1.070	1.189	1.309
	6	-3.397	-2.386	-1.797	-1.112	.990	1.143	1.272	1.419
	7	-3.243	-2.341	-1.714	-1.109	1.053	1.209	1.338	1.485
4	4	-3.479	-2.592	-1.842	-1.261	.930	1.053	1.158	1.282
	5	-3.275	-2.461	-1.832	-1.213	.997	1.143	1.261	1.409
	6	-3.264	-2.436	-1.808	-1.166	1.034	1.192	1.315	1.436
	7	-3.167	-2.277	-1.710	-1.147	1.099	1.268	1.388	1.539
5	4	[-3.421]	[-3.190]	[-2.762]	[-1.647]	[.871]	[.974]	[1.714]	[2.461]
		-3.473	-2.524	-1.856	-1.239	.955	1.084	1.177	1.287
	5	[-3.320]	[-3.060]	[-2.691]	[-.906]	[.967]	[1.096]	[2.173]	[2.639]
		-3.319	-2.410	-1.830	-1.238	1.010	1.156	1.284	1.403
	6	[-3.297]	[-3.041]	[-2.315]	[-.862]	[1.022]	[1.171]	[2.625]	[2.889]
		-3.297	-2.488	-1.775	-1.214	1.072	1.238	1.364	1.484
	7	[-3.198]	[-2.951]	[-2.055]	[-.750]	[1.058]	[1.222]	[3.329]	[3.409]
		-3.109	-2.245	-1.736	-1.170	1.123	1.303	1.423	1.569

Table 3.5 [Edgeworth] Approximate values and (vimulated values) of percentage points of R_2 when θ and σ are 0.0 and 1.0

c	n	1%	2.5%	5%	10%	90%	95%	97.5%	99%
3	4	-1.096	-1.046	-.985	-.897	1.291	1.857	2.652	3.779
	5	-1.231	-1.161	-1.078	-.958	1.220	1.774	2.652	3.732
	6	-1.354	-1.270	-1.177	-1.045	1.175	1.625	2.629	3.546
	7	-1.448	-1.353	-1.248	-1.103	1.071	1.518	2.434	3.348
4	4	-1.192	-1.127	-1.056	-.953	1.296	1.881	2.648	3.703
	5	-1.322	-1.230	-1.147	-1.012	1.247	1.779	2.566	3.395
	6	-1.424	-1.324	-1.220	-1.065	1.203	1.740	2.532	3.272
	7	-1.528	-1.417	-1.301	-1.140	1.182	1.676	2.345	3.247
5	4	[-2.876]	[-1.731]	[-.980]	[-.879]	[1.961]	[2.773]	[3.660]	[3.887]
		-1.541	-1.665	-1.084	-.965	1.261	1.904	2.570	3.547
	5	[-2.652]	[-1.174]	[-1.098]	[-.972]	[1.317]	[2.699]	[3.251]	[3.728]
		-1.503	-1.471	-1.167	-1.027	1.258	1.866	2.501	3.445
	6	[-2.593]	[-1.161]	[-1.171]	[-1.025]	[1.169]	[2.318]	[3.101]	[3.607]
		-1.373	-1.372	-1.263	-1.098	1.206	1.811	2.423	3.362
	7	[-1.402]	[-1.324]	[-1.221]	[-1.059]	[1.154]	[2.052]	[2.962]	[3.509]
		-1.278	-1.247	-1.332	-1.158	1.187	1.760	2.301	3.172

of R_1 and R_2 are in close agreement with the simulated percentage points in most cases.

It should be mentioned here that, though the Edgeworth approximation has been shown to be quite satisfactory for the choices of n and ν considered here, it will be necessary to check the validity of its use for any other choice of n and c; for details concerning the validity of the Edgeworth approximation, one may refer to Johnson, Kotz and Balakrishnan (1994, p.29).

In conclusion, we observe that the Edgeworth approximations of the distributions of R_1 and R_2 both work quite satisfactorily; this is also clear from the probability coverages and the average width of the confidence intervals based on R_1 and R_2 which are presented in Tables 3.6 and 3.7, respectively.

Table 3.6 Probability coverages of C.I.'s based on R_1 and R_2 using Edgeworth percentage points when θ and σ are 0.0 and 1.0

	R_1		R_2		R_1 (using $\sigma = \sigma^*$)	
n	95%	90%	95%	90%	95%	90%
4	.8940	.8526	.9452	.8487	.8843	.6850
5	.9421	.9133	.9514	.9053	.7421	.7276
6	.9217	.9012	.9358	.8927	.7759	.7345
7	.9445	.8969	.9345	.8864	.7534	.7321

Table 3.7 Average width of the simulated and [Edgeworth] C.I.'s based on R_1 and R_2 when $\theta = 0.0$ and $\sigma = 1.0$

		R_1 (Simulated)		R_2 (Simulated)		R_1 (using $\sigma = \sigma^*$) (simulated using R_3)	
c	n	95%	90%	95%	90%	95%	90%
3	4	1.543	2.030	4.974	7.065	3.153	4.430
	5	1.244	1.592	3.624	4.967	2.106	2.867
	6	1.080	1.370	3.123	4.151	1.680	2.266
	7	0.885	1.114	2.642	3.443	1.350	1.751
4	4	1.619	2.098	4.389	6.142	3.192	4.438
	5	1.330	1.664	3.298	4.328	2.209	2.914
	6	1.166	1.449	2.772	3.633	1.754	2.301
	7	0.974	1.198	2.412	3.132	1.454	1.896
5	4	[1.531]	[2.186]	[2.865]	[1.614]	[1.536]	[2.192]
		1.646	2.072	3.948	5.484	3.071	4.233
	5	[1.428]	[2.023]	[2.660]	[3.532]	[1.724]	[2.017]
		1.363	1.686	3.016	4.081	2.211	2.936
	6	[1.357]	[1.932]	[2.262]	[2.754]	[1.327]	[2.003]
		1.212	1.500	2.648	3.445	1.855	2.372
	7	[1.121]	[1.464]	[1.928]	[2.372]	[1.066]	[1.392]
		1.040	1.255	2.279	2.898	1.517	1.953

It should also be pointed out here that a similar Edgeworth approximation can not be developed for the percentage points of the pivotal quantity R_3 since it is not a linear function of record values. However, as displayed in Tables 3.6 and 3.7, we do not recommend drawing approximate inference based on R_1 with σ replaced by σ^* since it does not provide close results to those based on R_3. For this purpose, we have presented in Table 3.8 some selected percentage points of R_3 determined by Monte Carlo simulations (based on 10001 runs).

Table 3.8 Simulated percentage points of R_3 when θ and σ are 0.0 and 1.0

c	n	1%	2.5%	5%	10%	90%	95%	97.5%	99%
3.0	4	-.890	-.804	-.712	-.589	3.193	4.925	7.116	10.882
	5	-.997	-.899	-.792	-.642	2.742	4.092	5.752	8.283
	6	-1.087	-.987	-.868	-.697	2.742	3.965	5.530	7.620
	7	-1.142	-1.007	-.877	-.698	2.645	3.827	5.096	6.869
4.0	4	-.956	-.868	-.771	-.625	3.245	4.985	7.134	10.385
	5	-1.051	-.951	-.839	-.682	2.846	4.192	5.688	8.401
	6	-1.148	-1.034	-.911	-.738	2.592	3.822	5.175	7.104
	7	-1.183	-1.044	-.911	-.742	2.521	3.777	5.067	7.037
5.0	4	-.997	-.901	-.795	-.661	3.044	4.675	6.640	9.937
	5	-1.087	-.974	-.860	-.704	2.674	3.997	5.478	8.165
	6	-1.172	-1.054	-.925	-.750	2.631	3.945	5.174	7.121
	7	-1.217	-1.065	-.934	-.762	2.586	3.729	4.940	6.824

4. NUMERICAL ILLUSTRATIONS

In order to illustrate the usefulness of the inference procedures discussed in the previous sections, we consider here simulated data sets of size $n = 4, 5, 6$ and 7 (with $\theta = 0.0$, $\sigma = 1.0$). The BLUEs were calculated by making use of the entries in Table 3.1. The observed record values and the estimates obtained are presented in Table 4.1.

With these estimates and the use of Tables 3.2 and 3.4, we can determine the confidence intervals for θ (when σ is known to be 1.0) based on the Edgeworth approximation as well as using the simulated percentage points, based on the pivotal quantity R_1 through the formula

$$P\left(\theta^* - \sigma\sqrt{V_1}\,(R_1)_{1-\alpha/2} \le \theta \le \theta^* - \sigma\sqrt{V_1}\,(R_1)_{\alpha/2}\right) = 1 - \alpha.$$

For example, when $n = 7$ and $c = 5$, we have 90% C.I's of θ as

Edgeworth	Simulated
(-0.420 , 0.701)	(-0.448 , 0.592)

Table 4.1 The observed record values and the estimates of θ and σ

c	n	Records	θ^*	σ^*
3	4	1.105, 1.015, .705, .669	.000838	.999279
	5	.846, .844, .777, .718, .611	-.002344	.999917
	6	1.482, .853, .712, .688, .648, .576	004462	.999863
	7	.808, .776, .767, .692, .630, .610, .539	-.000584	.999462
4	4	1.232, .913, .802, .735	-.001431	.998474
	5	.995, .927, .900, .709, .696	006045	.998997
	6	1.067, 1.029, .896, .673, .667, .663	004123	1.00284
	7	1.734, .891, .755, .734, .676, .639, .628	-.000120	.998634
5	4	1.176, .975, .794, .780	.000854	.996102
	5	1.257, .983, .798, .754, .743	002342	.996655
	6	.984, .882, .862, .850, .739, .714	001434	.998260
	7	1.012, .971, .848, .830, .709, .702, .688	-.002229	1.000068

It is clear that the confidence intervals based on the Edgeworth approximation and those determined by simulation are quite close to those determined through the exact probabilities.

Similarly, with the use of Tables 3.2 and 3.5, we determined the confidence intervals for σ, through the formula

$$P\left(\frac{\sigma^*}{1+\sqrt{V_2}\,(R_2)_{1-\alpha/2}} \leq \sigma \leq \frac{\sigma^*}{1+\sqrt{V_2}\,(R_2)_{\alpha/2}}\right) = 1-\alpha.$$

For example, when $n = 7$ and $c = 5$, we have 90% C.I's of σ as

Edgeworth	Simulated
(0.496 , 2.528)	(0.534, 2.528)

Once again, we observe that the confidence intervals based on the Edgeworth approximation are somewhat close to those based on the exact results except for $m = 3$.

In the case when σ is unknown, the Edgeworth approximation method can not be used to draw inference for θ using R_3. So, we computed the confidence intervals for θ based on the simulated percentage points of the pivotal quantity R_3 (given in Table 3.6) through the formula

$$P\left(\theta^* - \sigma^*\sqrt{V_1}\,(R_3)_{1-\alpha/2} \leq \theta \leq \theta^* - \sigma^*\sqrt{V_1}\,(R_3)_{\alpha/2}\right) = 1-\alpha,$$

For example, when $n = 7$ and $c = 5$, we have 90% C.I's of θ when $\sigma^* = 1.000068$ as

$$(-0.448, 1.040)$$

As we can see from all the above tables, all confidence intervals become narrower as n increases.

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