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A Note on a Recent Approach to Some Life Testing Problems

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Abstract. In this note, some comments concerning the testing procedures for the problems of stochastic comparison, testing IFR and testing NBU ageing properties are presented showing that the tests proposed in Belzunce, Candel and Ruiz(1998) are cumbersome to calculate and do not have as good asymptotic Pittman efficacies as simpler others already in the literature. This is done via new presentations of the measures on which Belzunce et al. base their tests.

Key Words: *Stochastic order, IFR (DFR) and NBU (NWU) ageing life distributions.*

1. INTRODUCTION

In a recent article, Belzunce, Candel and Ruiz(1998) present a new approach to the following testing problems: stochastic order and testing exponentiality versus either the increasing failure rate (IFR) or new better than used (NBU) alternatives. They proceed to obtain the exact null distributions for the IFR and NBU tests, the asymptotic distributions of all three tests and proceed to defend the IFR and NBU tests by comparing them to other tests in the literature via the concept of Pitman's asymptotic relative efficiencies (PARE). By new representations of the measures on which Belzunce et al.(1998) base their tests, we show in the current note that these tests are truly cumbersome to calculate and have rather poor PARE relative to other procedures in print that are as is a simpler to implement.

Let X (and Y) be a nonnegative rv with cdf F and sf $\bar{F}$ $\left(G \text{ and } \bar{G}\right)$. Let X_t, $t \geq 0$ denote the random residual life at age t (RRL(t)). Thus the sf of X_t is given by:

$$\bar{F}_t(x) = \bar{F}(x+t)/\bar{F}(t), \qquad x, t \geq 0. \tag{1.1}$$

The stochastic order between X and Y is defined as: $X \underset{st}{\leq} Y$ if and only if $\bar{F}(x) \leq \bar{G}(x)$, for all $x \geq 0$. Using this definition we see that, cf. Barlow and Proschan(1981), X is IFR if and only if $X_t \underset{st}{\leq} X_s, 0 \leq s \leq t$, i.e., $\bar{F}_t(x)$ is decreasing in

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$t \geq 0$ for all $x \geq 0$. Also, X in NBU if and only if $X_t \underset{st}{\leq} X, 0 \leq t$, i.e., $\bar{F}_t(x) \leq \bar{F}(x)$ for all $x, t \geq 0$.

The following proposition whose proof is trivial is crucial for the development in this note.

Proposition 1.1. Let $X_1, \ldots, X_n$ denote k independent copies of X. Then

$$\delta_k(X) = \int_0^\infty \bar{F}^k(x)dx = E\{\min(X_1, \ldots, X_n)\}. \tag{1.2}$$

In Section 2, we comment on the two-sample stochastic order problem, while in Sections 3 and 4, respectively, we comment on the IFR and NBU tests.

2. TESTING STOCHASTIC ORDER

Here we center attention on the two-sample case leaving the much easier one-sample case to interested readers. The problem is to test the null by hypothesis $H_0 : X \underset{st}{=} Y$ (i.e. X and Y have common distribution) against the alternative $H_1 : X \underset{st}{<} Y$ (i.e. $\bar{F}(x) \leq \bar{G}(x)$ for all $x \geq 0$ with at least one strict inequality). Belzunce et al.(1998) propose as a measure of departure for H_0 in favor of H_1 :

$$\Delta_{st}(X,Y) = \delta_2(Y) - \delta_2(X) = E\{\min(Y_1, Y_2)\} - E\{\min(X_1, X_2)\}. \tag{2.1}$$

They proceed to estimate $\Delta_{st}(X,Y)$ by its empirical form

$$\hat{\Delta}_{st}(X,Y) = \hat{\delta}_2(Y) - \delta_2(X) = \int_0^\infty \bar{G}_n^2(x)dx - \int_0^\infty \bar{F}_m^2(x)dx \tag{2.2}$$

Another version of (2.2) is given by:

$$\hat{\Delta}_{st}(X,Y) = \frac{1}{n^2}\sum_i \sum_j \min(Y_i, Y_j) - \frac{1}{m^2}\sum_k \sum_l \min(X_k, X_l). \tag{2.3}$$

Hence $\hat{\Delta}_{st}(X,Y)$ may be viewed as a difference of two independent U-statistics of order 2 each, cf. Lee [5].

The classical test for this problem is the Mann-Whitney-Wilcoxon (MWW) statistic based on the measure

$$\theta_{st}(X,Y) = \int_0^\infty F(x)dG(x) \tag{2.4}$$

which is estimated by

$$\hat{\theta}_{st}(X,Y) = \int_0^\infty F_m(x)dG_n(x) = \frac{1}{mn}\sum_i \sum_j I(X_i \leq Y_j). \tag{2.5}$$

Clearly (2.5) is simpler than (2.3). Furthermore, (2.5) is distribution free cf. Randles and Wolfe(1979) while (2.5) is not and its asymptotic null variance needs to be estimated while the asymptotic null variance of (2.5) is $\frac{1}{12\lambda(1-\lambda)}$. The null variance of (2.3) may be calculated through the U-statistics theory to be:

$$\sigma_0^2 = \frac{4}{\lambda(1-\lambda)} Var\left\{ \int_0^X x dF(x) + X\bar{F}(X) \right\} = \frac{4\sigma_{*0}^2}{\lambda(1-\lambda)}, \text{ say,} \tag{2.6}$$

where $\lambda = \lim \frac{n}{n+m}$. Let us now asses the APE for the two procedures. For the MWW statistic, we have APE for the location and scale problem given by, cf. Randles and Wolffe(1979)

Location Case: $G_\theta(x) = F(x-\theta)$

$$\text{APE}= \sqrt{12\lambda(1-\lambda)} \int_0^\infty f^2(x)\,dx$$

Scale Case: $G_\theta(x) = F(\theta x)$

$$\text{APE}= \sqrt{12\lambda(1-\lambda)} \int_0^\infty x f^2(x)\,dx .$$

Now, for the test in Bezunce et al.(1998) we have

Location Case:

$$\text{APE}= 2\sigma_0^{-1} \int_0^\infty f(x)\bar{F}(x) dx$$

Scale Case:

$$\text{APE}= 2\sigma_0^{-1} \int_0^\infty x f(x)\bar{F}(x) dx$$

Hnce the APRE's are:

Location: $\sqrt{3\sigma_{*0}} \int_0^\infty f^2(x)\,dx \Big/ \int_0^\infty f(x)\bar{F}(x) dx$

Scale: $\sqrt{3\sigma_{*0}} \int_0^\infty x f^2(x)\,dx \Big/ \int_0^\infty x f|x|\bar{F}(x)$

Let us consider the two cases. The Uniform [0,1] distribution and the exponential distribution. In those cases we get that σ_{*0}^2 is equal to $\frac{4}{45}$ and $\frac{1}{3}$ respectively. The PARE's are $\frac{3}{4}\sqrt{\frac{5}{3}} = 0.968$ and $\frac{1}{2}\sqrt{\frac{5}{3}} = 0.646$ for the uniform distribution and those values are equal to one in both cases for the exponential. Hence the MWW is better as well as simpler.

3. TESTING IFR ALTERNATIVES

In this section, we wish to test $H_0 : \bar{F}(x) = e^{-x/\mu}, \mu$ is the unknown mean against $H_1 : \bar{F}$ is IFR and not exponential. Belzunce et al.(1998) propose the measure of departure from H_0 given by:

$$\Delta_{IFR}(X) = \iint_{s<t} \bar{F}^2(s)\bar{F}^2(t)\Delta_{st}(X_t, X_s)\,dF(s)\,dF(t) \quad (3.1)$$

They base their test on an empirical version of (9). To appreciate the complexity of this procedure we have the following proposition.

Proposition 3.1. Let $\delta_k(X) = \int_0^\infty \bar{F}^k(x)\,dx$. Then

$$\Delta_{IFR}(X) = \frac{1}{6}\{\delta_2(X) - 2\delta_3(X) + \delta_6(X)\}$$

$$= \frac{1}{6}\left[E\{\min(X_1, X_2)\} - 2E\{\min(X_1, X_2, X_3)\} + E\{\min(X_1, \ldots, X_6)\}\right] \quad (3.2)$$

Proof. Note that it is easy to see that

$$\Delta_{IFR}(X) = \iint_{s<t}\left[\bar{F}^2(s)\int_t^\infty \bar{F}^2(u)\,du - \bar{F}^2(t)\int_s^\infty \bar{F}^2(v)\,dv\right]dF(s)\,dF(t)$$

= I-II, say.

But

$$I = \int_0^\infty\left(\int_t^\infty \bar{F}^2(u)\,du\right)\int_0^t \bar{F}^2(s)\,dF(s)\,dF(t)$$

$$= \frac{1}{3}\int_0^\infty\int_t^\infty \bar{F}^2(u)\,du\left(1 - \bar{F}^3(t)\right)dF(t)$$

$$= \frac{1}{3}\left\{\int_0^\infty \bar{F}^2(t)\left(1 - \bar{F}(t)\right)dt - \int_0^\infty \bar{F}^3(t)\int_t^\infty \bar{F}^2(u)\,du\,dF(t)\right\}. \quad (3.3)$$

Similarly,

$$II = \int_0^\infty\left(\int_s^\infty \bar{F}^2(v)\,dv\right)\int_s^\infty \bar{F}^2(t)\,dF(t)\,dF(s)$$

$$= \frac{1}{3}\int_0^\infty \bar{F}^3(s)\int_s^\infty \bar{F}^2(v)\,dv\,dF(s). \quad (3.4)$$

Hence it follows form (3.3) and (3.4) that

$$\Delta_{IFR}(X)=\frac{1}{3}\left[\delta_2(X)-\delta_3(X)\right]-\frac{2}{3}\int_0^\infty \bar{F}^3(t)\int_t^\infty \bar{F}^2(u)\,du\,dF(t) \tag{3.5}$$

Note that

$$\int_0^\infty \bar{F}^3(t)\int_t^\infty \bar{F}^2(u)\,du\,dF(t)=-\int_0^\infty \bar{F}^2(u)\int_0^u \bar{F}^3(t)\,dF(t)\,du=\frac{1}{4}\left(\delta_2(X)-\delta_6(X)\right) \tag{3.6}$$

The result follows from (3.5) and (3.6).

It follows from Proposition 3.1 that any estimate of $\Delta_{IFR}(X)$ will be a form of a U-statistics with kernel of six degree order. Thus the equivalent estimate to that of Belzunce et al.(1998) is:

$$\hat{\Delta}_{IFR}(X)=\frac{1}{n^6}\sum_{i_1}\cdots\sum_{i_6}\varphi\left(X_{i_1},\ldots,X_{i_6}\right), \tag{3.7}$$

where $\varphi(X_1,\ldots,X_6)=\left\{\min(X_1,X_2)-2\min(X_1,X_2,X_3)+\min(X_1,\ldots,X_6)\right\}/6.$

Thus this test is much more complicated than others in the literature. To make this test scale-invariant, Belzunce et al.(1998) use $\breve{\Delta}_{IFR}(X)=\Delta_{IFR}(X)/\bar{X}$. The yardstick test they compare to is the test of Ahmad [1] based on the measure:

$$\theta_{IFR}(X)=\int_0^\infty\int_\theta^\infty \bar{F}^2\left(\frac{x+y}{2}\right)dF(x)\,dF(y)-\frac{1}{4}, \tag{3.8}$$

which can be estimated by

$$\hat{\theta}_{IFR}(X)=\frac{1}{n^4}\sum_{i_1}\cdots\sum_{i_4} I\left(2\min(X_{i_1},X_{i_2})>X_{i_3}+X_4\right)-\frac{1}{4}.$$

The test requires less computation than that of Belzunce et al(1998) and is scale invariant. A much simpler test than either of the two above two tests could be based on

$$\zeta_{IFR}(X)=\int_0^\infty\int_0^\infty \bar{F}\left(\frac{x+y}{2}\right)dF(x)\,dF(y)-\frac{4}{9}, \tag{3.9}$$

which is estimated by

$$\hat{\zeta}_{IFR}(X)=\frac{1}{n^3}\sum_{i_1}\sum_{i_2}\sum_{i_3} I\left(2X_{i_1}>X_{i_2}+X_{i_3}\right)-\frac{4}{9}.$$

Again, this test is scale invariant. Note that the null variances of the $\hat{\Delta}_{IFR}(X)$, $\hat{\theta}_{IFR}(X)$ and $\hat{\zeta}_{IFR}(X)$ tests are given by $\frac{23}{83610},\frac{82}{25725}$ and $\frac{248}{55125}$, respectively. Proposition 31 above makes it easy to calculate the APE of the $\hat{\breve{\Delta}}_{IFR}(X)$ test as we need to get:

$$\left|\frac{\delta}{\delta\theta}\left[\Delta_{IFR}(X)/\mu_\theta\right]_{\theta\to\theta_0}\right|=\left|\int_0^\infty\left[\frac{1}{3}e^{-x}-e^{-2x}+e^{-5x}\right]\bar{F}'_{\theta_0}(x)\,dx\right|, \tag{3.10}$$

where $\bar{F}'_{\theta_0}(X) = \frac{\delta}{\delta\theta}\bar{F}_\theta(x)\Big|_{\theta\to\theta_0}$. Then

$$APE(\Delta) = \left|\int_0^\infty \left(\frac{1}{3}e^{-x} - e^{-2x} + e^{-5x}\right)\bar{F}'_{\theta_0}(x)\,dx\right| / \sigma_0$$

Similar formulas are possible for θ and ζ. Considering the following three alternatives is common in this area:

1. Weihall family: $\bar{F}_\theta(x) = e^{-x^\theta}$
2. Linear Failure Rate family: $\bar{F}_\theta(x) = e^{-x-\frac{\theta}{2}x^2}$
3. Makeham family: $\bar{F}_\theta(x) = e^{-x-\theta(x+e^{-x}-1)}$

The numeric values of the above three families of the $APE(\Delta)$ are:

0.8742, 0.5583, 0.2393, while for ζ and θ there values are 0.8553, 0.5535, 0.2363 and 0.8553, 0.7363, 0.2485, respectively. Hence the Belzunce et al.(1998) test fairs nearly equal or sometime worse than the above two tests which are a lot easier to calculate. The efficiency calculations in Belzunce et al.(1998) are apparently in error.

4. TESTING NBU ALTERNATIVES

Here we test $H_0 : \bar{F}(x) = e^{-x/\mu}$ against $H_1 : \bar{F}$ is NBU and not exponential. Belzunce et al.(1998) proposed the following measures for departure form H_0 in favor of H_1:

$$\Delta_{NBU}(X) = \int_0^\infty \bar{F}^2(t)\Delta_{st}(X_t, X)\,dF(t). \tag{4.1}$$

Proposition 4.1. Let $\delta_k(X) = \int_0^\infty \bar{F}_k(x)\,dx$. Then

$$\Delta_{NBU}(X) = \frac{2}{3}\delta_2(X) - \delta_3(X) = \frac{2}{3}E\{\min(X_1, X_2)\} - E\{\min(X_1, X_2, X_3)\}. \tag{4.2}$$

Proof. Note that

$$\Delta_{NBU}(X) = \int_0^\infty\int_t^\infty \bar{F}^2(u)\,du\,dF(t) - \int_0^\infty \bar{F}^2(x)\,dx\int_0^\infty \bar{F}^2(t)\,dF(t)$$
$$= \int_0^\infty \bar{F}^2(x)\,dx - \int_0^\infty \bar{F}^3(x)\,dx - \frac{1}{3}\int_0^\infty \bar{F}^2(x)\,dx.$$

The result now follows.

Now, the empirical estimate of $\Delta_{NBU}(X)$ is given by:

$$\hat{\Delta}_{NBU}(X) = \frac{1}{n^3}\sum_{i_1}\sum_{i_2}\sum_{i_3}\left\{\frac{2}{3}\min(X_{i_1}, X_{i_2}) - \min(X_{i_1}, X_{i_2}, X_{i_3})\right\}.$$

The test is not scale invariant and to make it so we use $\hat{\Delta}^{*}_{NBU}(X) = \Delta_{NBU}(X)/\bar{X}$. The null variance is $\frac{2}{135}$. Again the yardstick test for NBU is that of Hollander and Proschan(1981) based on the functional

$$\theta_{NBU}(X) = \frac{1}{4} - \int_0^\infty\int_0^\infty \bar{F}(x+y)\,dF(x)\,dF(y)$$

which is estimated by:

$$\hat{\theta}_{NBU}(X) = \frac{1}{4} - \frac{1}{n^3}\sum_{i_1}\sum_{i_2}\sum_{i_3} I\left(X_{i_1} > X_{i_2} + X_{i_3}\right).$$

Hence the Belzunce et al.(1998) test is not any easier to calculate. The APE of $\Delta_{NBU}(X)$ is given by:

$$APE(\Delta) = \left|\int_0^\infty\left(\frac{4}{3}e^{-x} - 3e^{-2x}\right)\bar{F}'_{\theta_0}(x)\,dx\right| / \sigma_0 \qquad (4.3)$$

For three alternatives in Section 3 above we get the values 1.109, 0.571 and .0228. The corresponding values for the Hollander and Proschan(1981) test are, respectively, 1.16986, 0.58095 and 0.28544. Hence Belzunce et al.(1998) is not better. Again, the efficiency calculations in Belzunce et al.(1998) appear to be in error.

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