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(weak) R-mingle: toward a fuzzy-relevance logic* †

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This paper investigates the relevance system **R**-mingle (**RM**) as a fuzzy-relevance logic. It shows that **RM** is *fuzzy* in Cintula's sense, i.e., **RM** is complete with respect to linearly ordered **L**-matrices (or **L**-algebras). More exactly, we first introduce **RM** and its weak versions **wwRM** and **wRM**. We next provide algebraic and matrix completeness results for them.

【Key Words】 (**w**)**RM**, fuzzy-relevance logic, algebraic and matrix completeness.

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1. Introduction

The aim of this paper is to extend the world of fuzzy logic to the realm of relevance logics. For this purpose, recall first some recent historical facts in fuzzy logic. In the last decade Hájek [6] introduced the basic fuzzy logic **BL** and investigated the well-known infinite-valued systems **Ł** (Łukasiewicz logic), **G** (Gödel-Dummett logic), and **Π** (Product logic) as its extensions. **BL** is the residuated fuzzy logic capturing the tautologies of *continuous* t-norms and their residua. Esteva and Godo [4] introduced the monoidal t-norm logic **MTL**, which copes with the logic of *left-continuous* t-norms and their residua, as a weakening of **BL** (and a strengthening of Affine multiplicative additive intuitionistic linear logic **AMAILL** introduced by Höhle [7]).

The system **R** of Relevance with "mingle" (**RM**) has been treated as a relevance logic, and Dunn [3] algebraically investigated **RM** capturing the tautologies on *denumerable* infinite sets of truth values. Then **RM** seems both fuzzy and relevant. However, there are some difficulties in regarding it as a fuzzy-relevance logic. Because while all the above fuzzy logics are logics of t-norms in the sense that their algebraic counterparts satisfy all of the conditions of a t-norm, **RM** is not.

The intergral condition (Int) that the greatest element 1 is the unit element $(1 * x = x$ for all $x \in [0, 1])$ does not hold in Sugihara-algebras, whose class (more exactly the class of Sugihara-matrices) characterizes **RM** (see [3]). **RM** drops the divisibility axiom (D) $(\phi \& (\phi \to \psi)) \leftrightarrow (\phi \wedge \psi)$ for **BL** related with the *continuity* of a t-norm, its weakening (wD) $(\phi \& (\phi \to \psi)) \to (\phi \wedge \psi)$ for **MTL**, and the &-absurdity (&-AB) $(\phi \& {\sim}\phi) \to \psi$ for **AMAILL**. It also omits the &-elimination (&-E) $(\phi \& \psi) \to \phi$, a common axiom of all the above (t-norm) logics, from which the weakening (W) $\phi \to (\psi \to \phi)$ can be proved using the "residuation" below and vice versa. Note that (&-E) concerns the above integral condition of a t-norm in the sense that (Int) is the algebraic counterpart of (&-E). Unfortunately, (W) (and so (&-E)) is(are) rejected not merely in **RM**, but in relevance systems such as **R** because a logic **L** with (W) allows *irrelevance* between ϕ and ψ in case $\phi \to \psi$ is a theorem.

It will be interesting to state that Sugihara algebras instead satisfy all the conditions of a uninorm, which is a generalization of t-norm where the identity can lie in anywhere in [0, 1]. This means that using the properties of it, we may provide algebraic semantics for **RM**.

In this paper we shall first introduce **wwRM** and **wRM**., briefly (w)**wRM**. (w)**wRM** is a weak version of **RM** in the sense that it has weak negation in place of strong negation of **RM**. (w)**wRM** is *relevant* in the *weak* sense that it satisfies the *weak* relevance principle (WRP) in [3] that ϕ

$\to \psi$ is a theorem only if either (i) ϕ and ψ share a sentential variable or (ii) both $\sim\phi$ and ψ are theorems; wRM is *fuzzy* in the sense that it satisfies the fuzzy condition (of a logic) of Cintula in [2] that the logic L is complete with respect to (w.r.t.) linearly ordered L-matrices (or L-algebras). (w)wRM is a system belonging to the class of weakly implicative (fuzzy) logics (**WI(F)L**) investigated by Cintula [2] in the sense that the former satisfies the conditions for a logic to be called **WI(F)L**. We verify this by investigating the class of (w)wRM and its extensions such as RM (**(w)wRM**) as the subclass of **WI(F)L**. We shall provide algebraic and matrix completeness results for a (w)wRM L.

While wRM is fuzzy-relevant, it is not necessary that all of the schematic extensions of wRM are fuzzy-relevant. Thus the completeness of **wRM** does not ensure that any system in that class is relevant. We shall consider the relevant and irrelevant subclasses of **wRM** **R-wRM** and **P-wRM**.

For convenience, we shall adopt the notation and terminology similar to those in [2], [4], [5], and [6], and assume being familiar with them (together with results found in them).

2. Syntax

We base logics on a countable propositional language with formulas *FOR* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, &, $\wedge$, $\vee$, and constants **F**, f, **t**. Further definable connectives are:

df1. $\sim\phi := \phi \rightarrow \mathrm{f}$,
df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We moreover define **T** as $\sim$**F**, and ϕ_t as $\phi \wedge \mathbf{t}$. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiom schemes and rules for **wwRM**.

Definition 2.1 **wwRM** consists of the following axiom schemes and rules:

A1. $\phi \rightarrow \phi$ (self-implication, SI)
A2. $(\phi \wedge \psi) \rightarrow \phi$, $(\phi \wedge \psi) \rightarrow \psi$ ($\wedge$-elimination, $\wedge$-E)
A3. $((\phi\rightarrow\psi)\wedge(\phi\rightarrow\chi)) \rightarrow (\phi\rightarrow(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)
A4. $\phi \rightarrow (\phi \vee \psi)$, $\psi \rightarrow (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)
A5. $((\phi\rightarrow\chi)\wedge(\psi\rightarrow\chi)) \rightarrow ((\phi\vee\psi)\rightarrow\chi)$ ($\vee$-elimination, $\vee$-E)
A6. $(\phi\wedge(\psi\vee\chi))\rightarrow((\phi\wedge\psi)\vee(\phi\wedge\chi))$ ($\wedge\vee$-distributivity, $\wedge\vee$-D)
A7. $\mathbf{F} \rightarrow \phi$ (ex falso quadlibet, EF)
A8. $(\phi \,\&\, (\psi \,\&\, \chi)) \leftrightarrow ((\phi \,\&\, \psi) \,\&\, \chi)$ (&-associativity, AS)

A9. $(\phi \& \psi) \rightarrow (\psi \& \phi)$ (&-commutativity, &-C)
A10. $(\phi \& t) \leftrightarrow \phi$ (push and pop, PP)
A11. $(\psi \rightarrow \chi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$ (prefixing, PF)
A12. $(\phi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow ((\phi \& \psi) \rightarrow \chi)$ (residuation, RE)
A13. $(\phi \& \phi) \leftrightarrow \phi$ (idempotence, ID)
A14. $((\phi\&\psi)\rightarrow(\phi\wedge\psi)) \vee ((\phi\vee\psi)\rightarrow(\phi\&\psi))$ (weak RM, wRM)
$\phi \rightarrow \psi, \phi \vdash \psi$ (modus ponens, mp)
$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj)

Definition 2.2 (i) **wRM** is **wwRM** plus
A15. $(\phi \rightarrow \psi)_t \vee (\psi \rightarrow \phi)_t$ (t-prelinearity, PL_t).
(ii) **RM** is **wRM** plus
A16. $\sim\sim\phi \rightarrow \phi$ (double negation elimination, DNE),
where (df3) $\phi \& \psi$ is defined as $\sim(\phi \rightarrow \sim\psi)$.

Definition 2.3 ((w)wRMs) A logic is a schematic extension of L if and only if (iff) it results from L by (finitely or infinitely many) axioms. L is a wRM iff L is a schematic extension of (w)wRM.

The following proposition ensures that **(w)wRM** is a subclass of **WI(F)L**.

Proposition 2.4 (i) (**wwRM** $\subset$ **WIL**) A wwRM L is a weakly implicative logic.
(ii) (**wRM** $\subset$ **WIFL**) A finitary wRM L is a weakly implicative fuzzy logic.

Proof: (i) We first note that a weakly implicative logic (WIL) is a logic having (SI), (mp), transitivity ($\phi \to \psi, \psi \to \chi \vdash \phi \to \chi$), and congruence w.r.t. connectives. Since **L** has A1, (mp), and (SF) below, it suffices to check that $\leftrightarrow$ is a congruence w.r.t. $\wedge$, $\vee$, &, and $\to$: we check one direction. Let $\vdash \phi \to \psi$. W.r.t. $\wedge$, by A2 and transitivity, $(\phi \wedge \chi) \to \psi$, and thus $(\phi \wedge \chi) \to (\psi \wedge \chi)$ by A2, A3, (adj), and (mp); w.r.t. $\vee$, analogously to $\wedge$; w.r.t. &, $(\phi \& \chi) \to (\psi \& \chi)$ and $(\chi \& \phi) \to (\chi \& \psi)$ by (IT) below and A9; w.r.t. $\to$, $(\psi \to \chi) \to (\phi \to \chi)$ and $(\chi \to \phi) \to (\chi \to \psi)$ by (SF) below and A11.

(ii) By (i) and Proposition 5.4 below. □

As we mentioned above, **wRM** includes non-relevant systems such as the classical propositional system **CL**. To restrict **wRM** to the subclass of fuzzy-relevance ones, we note one important historical fact in relevance logic: the most famous relevance systems **R**, **E** of Entailment, **T** of Ticket Entailment were introduced as logics *free from* paradoxes, i.e., *positive* paradoxes such as (W) above and ($\to$-triviality, $\to$-TR) $\phi \to (\psi \to \psi)$ (so called the *paradoxes of implication*), and negative paradoxes such as ($\wedge$-absurdity, $\wedge$-AB) $(\phi \wedge \sim\phi) \to \psi$ and ($\vee$-TR) $\phi \to (\psi \vee \sim\psi)$ (so called the *paradoxes of material implication*).[1)]

2) Here *paradox* means that while $\phi \to \psi$ is a theorem, the antecedent does not have any meaningful (or intensional) relation with the consequent (in other words, they do not share any sentential

These systems satisfy the *strong* relevance principle (SRP) in [1] that $\phi \rightarrow \psi$ is a theorem only if ϕ and ψ share a sentential variable, and systems satisfying SRP are free from the paradoxes (W), ($\rightarrow$-TR), ($\wedge$-AB), and ($\vee$-TR). But systems satisfying WRP such as **(w)RM** are not necessarily free from paradoxes because such systems prove some paradoxical sentences such as $\sim(\phi \rightarrow \phi) \rightarrow (\psi \rightarrow \psi)$. We call systems free from the paradoxes above *strong paradoxes-free* systems, and systems either (i) free from the paradoxes or (ii) satisfying the (ii) of WRP *weak paradoxes-free* systems. And, in a non-constructive way, we define a *relevant* wRM as follows:

Definition 2.5 (Relevant wRMs, R-wRMs) L is an R-wRM iff (i) L is a schematic extension of **wRM** and (ii) it is weakly paradoxes-free.

We call any weak paradoxes-free wRM *R-wRM*. We also call any wRM accepting at least one of the paradoxes paradoxical wRM (P-wRM), more exactly, a wRM accepting at least one of the positive (negative resp) paradoxes *positive (negative resp) P-wRM*. (Note that some negative P-wRM L may be a weak paradoxes-free R-wRM.)

Remark 2.6 In case a wRM L has (df4) t = T and so

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variable).

(df5) f = F, it is not relevant any more because it proves $(\phi \,\&\, T) \to \phi$ and thus (W). Since it proves (W), it is instead a positive P-wRM.

By ***R-wRM*** (***P-wRM*** resp), let us express the class of R-wRMs (P-wRMs resp). Since **R-wRM** (**P-wRM** resp) is a subclass of **wRM**, it is immediate that

Corollary 2.7 (i) (**R-wRM** $\subset$ **WIFL**) A finitary R-wRM L is a weakly implicative fuzzy logic.

(ii) (**P-wRM** $\subset$ **WIFL**) A finitary P-wRM L is a weakly implicative fuzzy logic.

In a wRM L f can be defined as $\sim$t and vice versa; in a wRM with (DNE), $\wedge$ defined using $\sim$ and $\vee$, and $\to$ instead defined as $\phi \to \psi := \sim(\phi \,\&\, \sim\psi)$.

Proposition 2.8 (i) **wwRM** proves:

(1) $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$ (SF)

(2) $(\phi \to \psi) \to ((\phi \,\&\, \chi) \to (\psi \,\&\, \chi))$ (IT)

(3) $(\phi \to (\psi \to \chi)) \to (\psi \to (\phi \to \chi))$ (PM)

(4) $(\phi \to (\phi \to \psi)) \to (\phi \to \psi)$ (CT)

(5) $(\phi \to \psi) \to \sim\psi \to \sim\phi$ (CP)

(6) $\phi \to \sim\sim\phi$ (double negation introduction, DNI)

(7) $\sim(\phi \vee \psi) \leftrightarrow \sim\phi \wedge \sim\psi$ (de MorganI, DM1)

(8) $\sim\phi \vee \sim\psi \leftrightarrow \sim(\phi \wedge \psi)$ (de MorganII, DM2)

(9) $(\phi\&(\psi\vee\chi))\leftrightarrow((\phi\&\psi)\vee(\phi\&\chi))$ (&$\vee$-distributivity, &$\vee$-D)

(ii) wRM additionally proves:

(1) $(\phi\&(\psi\wedge\chi))\leftrightarrow((\phi\&\psi)\wedge(\phi\&\chi))$ ($\&\wedge$-D)

(2) $(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$ (PL)

(iii) RM additionally proves: A8 to A10, A14, A15, and

(1) $\phi \leftrightarrow \sim\sim\phi$ (double negation, DN)

Proof: We prove (i-2) as an example:

1. $((\phi \rightarrow \psi) \& \phi) \rightarrow \psi$ (A1, A12, MP)
2. $\psi \rightarrow (\chi \rightarrow (\psi \& \chi))$ (A1, A12, MP)
3. $((\phi \rightarrow \psi) \& \phi) \rightarrow (\chi \rightarrow (\psi \& \chi))$ (1, 2, transitivity)
4. $((\phi \rightarrow \psi) \& (\phi \& \chi)) \rightarrow (\psi \& \chi)$ (3, A8, A12)
5. $(\phi \rightarrow \psi) \rightarrow ((\phi \& \chi) \rightarrow (\psi \& \chi))$ (4, A12)

Proof of the rest is left to the interested reader. □

A *theory* over L is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of L or a member of T or follows from some preceding members of the sequence using the rules (mp) and (adj). $T \vdash \phi$, more exactly $T \vdash_L \phi$, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T. The relevant deduction theorem (RDT) for L is as follows:

Proposition 2.9 Let L be a wwRM, T a theory, and ϕ, ψ formulas. $T \cup \{\phi\} \vdash_L \psi$ iff $T \vdash_L \phi_t \rightarrow \psi$.

Proof: It is just Enthymematic Deduction Theorem (see [8]). □

A theory T is *inconsistent* if T $\vdash$ F; otherwise it is *consistent*.

For convenience, "$\sim$", "$\wedge$", "$\vee$", and "$\rightarrow$" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

Remark 2.10 Let **L** be a wRM having either f $\rightarrow$ t or (FP) f $\leftrightarrow$ t. It proves such formulas as $\sim(\phi \vee \sim\phi) \rightarrow (\psi \vee \sim\psi)$ and $\sim(\phi \rightarrow \phi) \rightarrow (\psi \rightarrow \psi)$, and so it is not relevant in the strong sense any more. But, since these formulas still satisfy the (ii) of WRP, **L** may be instead relevant in the weak sense.

3. Semantics

Suitable algebraic structures for (w)wRMs are obtained as varieties of monoidal residuated lattices.

Definition 3.1 An *idempotent coummtative monoidal residuated lattice* (icmr-lattice) is a structure **A** = (A, $\top$, $\bot$, $\top_t$, $\bot_f$, $\wedge$, $\vee$, $*$, $\rightarrow$) such that:

(I) (A, $\top$, $\bot$, $\wedge$, $\vee$) is a bounded distributive lattice with top element $\top$ and bottom element $\bot$.

(Ⅱ) $(A, *, \top_t)$ satisfies for all $x, y, z \in A$,

(a) $x * y = y * x$ (commutativity)

(b) $\top_t * x = x$ (identity)

(c) $x \leq y$ implies $x * z \leq y * z$ (isotonicity)

(d) $x * (y * z) = (x * y) * z$ (associativity)

(e) $x * x = x$ (idempotence)

(Ⅲ) $y \leq x \to z$ iff $x * y \leq z$, for all $x, y, z \in A$ (residuation)

$(A, *, \top_t)$ satisfying (Ⅱ-b, d) is a *monoid*. Thus $(A, *, \top_t)$ satisfying (Ⅱ-a, b, d) is a commutative monoid, and $(A, *, \top_t)$ satisfying (Ⅱ-a, b, d, e) an idempotent commutative monoid. $(A, *, \top_t)$ satisfying (Ⅱ-a, b, c, d) on [0, 1] is a *uninorm* and it is a *t-norm* in case $\top_t = \top$.

To define an icmr-lattice we may take in place of (Ⅱ-c)

(IV) $x * (y \vee z) = (x * y) \vee (x * z)$.

Using $\to$ and $\bot_f$ we can define $\top_t$ as $\bot_f \to \bot_f$, and $\sim$ as in (df1). In an icmr-lattice, $\sim$ is a *weak* negation in the sense that for all x, $x \leq \sim\sim x$ holds in it.

Definition 3.2 (i) (wwRM-algebra) A *wwRM-algebra* is an icmr-lattice satisfying the condition: for all x, y,

(wrm) $\top_t \leq ((x*y)\to(x\wedge y)) \vee ((x\vee y)\to(x*y))$.

(i) (wRM-algebra) A *wRM-algebra* is a wwRM-algebra satisfying the condition: for all x, y,

(pl_t) $\top_t \leq (x \to y)_t \vee (y \to x)_t$.

Let **A** be an equationally definable wRM-algebra. **A** is *paradoxical* in case it satisfies at least one of the following conditions:

(Paradox Conditions, PC) for all x, y ∈ A,

(a) $x \leq (y \rightarrow x)$ (weakening)
(b) $x \leq (y \rightarrow y)$ ($\rightarrow$-triviality)
(c) $(x \wedge \sim x) \leq y$ ($\wedge$-absurdity)
(d) $x \leq (y \vee \sim y)$ ($\vee$-triviality)

We call it P-wRM-algebra. **A** is *strongly relevant* in case it rejects (PC), and *weakly relevant* in case it either (i) rejects (PC) or (ii) satisfies that $\top_t \leq (x \rightarrow y)$ implies $\top_t \leq \sim x$ and $\top_t \leq y$. In case **A** is weakly relevant, we call it *R-wRM-algebra*.

In an analogy to Definition 3.2, we can define an RM-algebra corresponding to the system **RM**. An algebra A is *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 3.3 (Evaluation) Let **A** be an algebra. An *A-evaluation* is a function v : FOR → **A** satisfying:

$$v(\#(\phi_1, \cdots, \phi_m)) = \#_A(v(\phi_1), \cdots, v(\phi_m)),$$

where # ∈ {&, →, ∧, ∨, t, f, **T**, **F**}, $\#_A$ ∈ {*, →, ∧, ∨, $\top_t$, $\bot_f$, ⊤, ⊥}, and m is the arity of # and $\#_A$.

Definition 3.4 Let *L* be a propositional language, L a logic in *L*, T a theory in *L*, ϕ a formula, and **K** a class of

A-algebras.

(i) (Tautology) ϕ is a *$\top_t$-tautology* in **A**, briefly an *A-tautology* (or *A-valid*), if $v(\phi) \geq \top_t$ for each A-evaluation v.

(ii) (Model) An A-evaluation v is an *A-model* of T if $v(\phi) \geq \top_t$ for each $\phi \in$ T. By *Mod(T, A)*, we denote the class of A-models of T.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. **K**, denoting by T $\vDash_K \phi$, if Mod(T, A) = Mod(T $\cup$ {ϕ}, A) for each **A** $\in$ **K**.

Definition 3.5 (L-algebra) Let **L** be a logic in *L*, T a theory in *L*, ϕ a formula, and **A** an algebra. **A** is an *L-algebra* iff whenever ϕ is L-provable in T, i.e., T $\vdash_L \phi$, it is a semantic consequence of T w.r.t. the set of **A**. By *$MOD^{(l)}(L)$*, we denote the class of (linearly ordered) L-algebras. We write T $\vDash^{(l)}_L \phi$ in place of T $\vDash_{MOD^{(l)}L} \phi$.

Since the class of ML-algebras forms a variety and a wRM-algebra is just an ML-algebra generalizing identity, it is obvious that the class of all wRM-algebra is a variety of algebras. This ensures that

Proposition 3.6 Let L be a wRM. The class of all L-algebras is a variety of algebras.

Proof: We prove that the class of all wRM-algebras is a variety. Note first that the class of (bounded) distributive lattices is a variety and each of the conditions of (II-a, b, d, e) has a form of equation. Note also that in each wRM-algebra the equations for (II-c) and (III) can be given: e.g., for the equations for (III), see Lemma 2.3.10 in [6]. Analogously the equations for (wrm) and (pl_t) can be given. Thus, since each condition for a wRM-algebra has a form of equation or can be defined in equation, it can be eusured that the class of all wRM-algebras is a variety. □

Let A be an algebra. A_M-matrix, briefly *M-matrix*, is an A-algebra with D, a subset of A. The elements of D are usually called designated elements of matrix M. Then, in an analogy to the above, we can define a (w)wRM-matrix. Furthermore, by taking $v(\phi) \in D$ in place of $v(\phi) \geq \top_t$, we can analogously define tautology, model, semantic consequence, and L-matrix on M-matrices in place of A-algebras.

Let us take $D = \{x: x = v(\phi) \geq \top_t\}$. Then it is immediate that

Corollary 3.7 A (w)wRM-algebra A is an L-algebra iff $T \vdash_L \phi$ implies $T \vDash_L \phi$ iff a (w)wRM-matrix M = (A, D) is an L-matrix.

4. Algebraic completeness

Let L be a (w)wRM, and A a (corresponding) (w)wRM-algebra. We first note that the nomenclature of the prelinearity condition is explained by the following subdirect representation theorem.

Proposition 4.1 Each wRM-algebra is a subdirect product of linearly ordered wRM-algebras.

Proof: Its proof is as usual. □

We next show that classes of provably equivalent formulas form an L-algebra. Let T be a fixed theory over L. For each formula ϕ, let $[\phi]_T$ be the set of all formulas ψ such that $T \vdash_L \phi \leftrightarrow \psi$ (formulas T-provably equivalent to ϕ). A_T is the set of all the classes $[\phi]_T$. We define that $[\phi]_T \rightarrow [\psi]_T = [\phi \rightarrow \psi]_T$, $[\phi]_T * [\psi]_T = [\phi \,\&\, \psi]_T$, $[\phi]_T \wedge [\psi]_T = [\phi \wedge \psi]_T$, $[\phi]_T \vee [\psi]_T = [\phi \vee \psi]_T$, $\bot = [F]_T$, $\top = [T]_T$, $\top_t = [t]_T$, and $\bot_f = [f]_T$. By $\mathcal{A}_T$, we denote this algebra.

Proposition 4.2 For T a theory over L, A_T is an L-algebra.

Proof: Note that A2 to A6 ensure that $\wedge$, $\vee$, and $\rightarrow$

satisfy (I) in Definition 3.1; that A8 to A10, A13, and (IT) ensure that & satisfies (II) (a) - (e); that A12 ensures that (III) holds; that A14 ensures that (wrm) holds; (and that A15 ensures that (pl_t) holds.) It is obvious that $[\phi]_T \leq [\psi]_T$ iff $T \vdash_L \phi \leftrightarrow (\phi \wedge \psi)$ iff $T \vdash_L \phi \rightarrow \psi$. Finally recall that A_T is an L-algebra iff $T \vdash_L \psi$ implies $T \models_L \psi$, and observe that for ϕ in T, since $T \vdash_L t \rightarrow \phi$, it follows that $[t]_T \leq [\phi]_T$. Thus it is an L-algebra. □

Theorem 4.3 (Strong completeness) Let T be a theory and ϕ a formula.

(i) Let L be a wwRM. $T \vdash_L \phi$ iff $T \models_L \phi$.

(ii) Let L be a wRM. $T \vdash_L \phi$ iff $T \models_L \phi$ iff $T \models^1_L \phi$.

Proof: (i) Left to right follows from definition. Right to left is as follows: from Proposition 4.2, we obtain $A_T \in$ MOD(L), and for A_T-evaluation v defined as $v(\psi) = [\psi]_T$, it holds that $v \in Mod(T, A_T)$. Thus, since from $T \models_L \phi$ we obtain that $[\phi]_T = v(\phi) \geq \top_t$, $T \vdash_L t \rightarrow \phi$. Then, since $T \vdash_L t$, by (mp) $T \vdash_L \phi$, as required.

(ii) That $T \vdash_L \phi$ iff $T \models_L \phi$ is analogous to (i). That $T \models_L \phi$ iff $T \models^1_L \phi$ follows from Proposition 4.1. □

Corollary 4.4 (Weak completeness) For each formula ϕ, ϕ is a theorem iff for each (linearly ordered) L-algebra A, ϕ is an A-tautology, i.e., $\vdash_L \phi$ iff $\models^{(1)}_L \phi$.

Given an equationally definable R-wRM-algebra (P-wRM-algebra resp) A, we can provide a corresponding R-wRM (P-wRM resp) L. Then, since an R-wRM (P-wRM resp) L is also a wRM, Theorem 4.3 ensures that

Corollary 4.5 Let T be a theory and ϕ a formula.

(i) Let **A** be a (linearly ordered) R-wRM-algebra (equationally definable) and L a corresponding R-wRM. T $\vdash_L \phi$ iff T $\models^{(l)}{}_L \phi$.

(ii) Let **A** be a (linearly ordered) P-wRM-algebra (equationally definable) and L a corresponding P-wRM. T $\vdash_L \phi$ iff T $\models^{(l)}{}_L \phi$.

5. Matrix completeness

Following Cintula [2], let a wwRM L be *fuzzy* in case it is complete w.r.t. linearly ordered L-matrices, i.e., L = $\models^1{}_L$. We shall show that a logic L is so in case it is a wRM, i.e., a wwRM satisfying A15. Note that even though Cintula [2] does not investigate any system exactly corresponding to our L, his results are useful. Following his idea we can provide easy (strong) completeness for L.

To achieve completeness for L, following [2] we add more definitions on a theory T to the definitions above.

Definition 5.1 Let L be a wwRM.

(i) T is *linear* if T is consistent and for each pair ϕ, ψ of formulas, $T \vdash \phi \rightarrow \psi$ or $T \vdash \psi \rightarrow \phi$.

(ii) T is *prime* if for each pair ϕ, ψ of formulas such that $T \vdash \phi \vee \psi$, $T \vdash \phi$ or $T \vdash \psi$.

(iii) L has the *Linear Extension Property* (LEP) if for each theory T and formula ϕ such that $T \nvdash \phi$, there is a linear theory T' such that $T \subseteq T'$ and $T' \nvdash \phi$.

(iv) L has the *Prelinearity Property* (PP) if for each theory T, we get $T \vdash \chi$ whenever $T, \phi \rightarrow \psi \vdash \chi$ and $T, \psi \rightarrow \phi \vdash \chi$.

(v) L has the *Subdirect Decomposition Property* (SDP) if each ordered L-matrix is a subdirect product of linearly ordered L-matrices.

(vi) L has the *Prime Extension Property* (PEP) if for each theory T and formula ϕ such that $T \nvdash \phi$, there is a prime theory T' such that $T \subseteq T'$ and $T' \nvdash \phi$.

(vii) L has the *Proof by Cases Property* (PCP) if for each theory T, we get $T, \phi \vee \psi \vdash \chi$ whenever $T, \phi \vdash \chi$ and $T, \psi \vdash \chi$.

We consider L as a *finitary* logic in the sense that for each theory T and formula ϕ we have that if $T \vdash \phi$ there is a *finite* theory $T' \subseteq T$ such that $T' \vdash \phi$. Then, since a wwRM L is a WIL, by Lemma 17 in [2], we can obtain that

Proposition 5.2 Let L be a wwRM and T a theory.
(i) T is linear iff the L-matrix M_T is linearly ordered;
(ii) In case L is a wRM, T is linear iff T is prime;
(iii) L has PP iff L has PCP and (PL), i.e., $(\phi \to \psi) \vee (\psi \to \phi)$.

Note that Cintula showed that as a finitary WIL L is fuzzy iff L has LEP iff L has PP iff L has SDP iff L has PCP and (PL) (see Theorem 3 and Lemma 17 in [2]). Since a finitary wRM L is a finitary WIL proving (PL), it is immediate that

Corollary 5.3 For a finitary wRM L,
L is fuzzy iff L has PCP iff L has PEP.

Let us consider L with deduction theorem. In an analogy to Lemma 22 in [2], we can show that

Proposition 5.4 Let L be a finitary logic with RDT. Then L is a fuzzy logic iff it holds: $\vdash_L (\phi \to \psi)_t \vee (\psi \to \phi)_t$, i.e., A15.

Proof: Left to right is obvious. For right to left, we just show that L has PP. Let T, $\phi \to \psi \vdash_L \chi$ and T, $\psi \to \phi \vdash_L \chi$. By RDT, T $\vdash_L (\phi \to \psi)_t \to \chi$ and T $\vdash_L (\psi \to \phi)_t \to \chi$. Then by A5 (together with (adj) and (mp)), T $\vdash_L ((\phi \to \psi)_t \vee (\psi \to \phi)_t) \to \chi$. Thus, by A15 and (mp), T $\vdash_L \chi$, as desired. □

Then using Proposition 5.4 (and soundness as usual), we can easily show that

Theorem 5.5 (Completeness) Let T be a theory over a finitary wRM L and ϕ a formula. Then $T \vdash_L \phi$ iff $T \models^1_L \phi$.

Since an R-wRM (P-wRM resp) L is a WIFL (see Corollary 2.6), it is immediate that

Corollary 5.6 (i) Let T be a theory over a finitary R-wRM L and ϕ a formula. Then $T \vdash_L \phi$ iff $T \models^1_L \phi$.

(ii) Let T be a theory over a finitary P-wRM L and ϕ a formula. Then $T \vdash_L \phi$ iff $T \models^1_L \phi$.

References

Anderson, A. R.(1962), and Belnap, Jr., N. D., "The pure calculus of entailment", *The Journal of symbolic Logic*, 27, pp. 19-52.

Cintula, P.(2006), "Weakly Implicative (Fuzzy) Logics I: Basic properties", *Archive for Mathematical Logic.*

Dunn, J. M.(1970), "Algebraic Completeness for R-Mingle and Its Extension", *Journal of symbolic Logic*, 35, pp. 13.

Esteva, F., and Godo, L.(2001), "Monoidal t-norm based logic: towards a logic for left-continous t-norms", *Fuzzy Sets and Systems*, 124, pp. 271-288.

Esteva, F., Godo, L., Hájek, P.(2000), and Navara, M., "Residuated fuzzy logics with an involutive logics", *Archive for Mathematical Logic*, 39, pp. 103-124.

Hájek, P.(1998), **Metamathematics of Fuzzy Logic**, Amsterdam, Kluwer.

Höhle, U.(1995), "Commutative, residuated l-monoids", U. Höhle and E. P. Klement (eds.), **Non-Classical Logics and Their Applications to Fuzzy Subsets**, Amsterdam, Kluwer, pp. 53-106.

Meyer, R. K., Dunn, J. M., and Leblanc, H.(1976), "Completeness of relevant quantification theories", *Notre Dame Journal of Formal Logic*, 15, pp. 97-121.

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