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ON THE INTEGRAL CLOSURES OF IDEALS

H. ANSARI-TOROGHY AND F. DOROSTKAR

Abstract. Let R be a commutative Noetherian ring (with a non-zero identity) and let M be an $R-$module. Further let I be an ideal of R. In this paper, by putting a suitable condition on $Ass_R(M)$, we obtain some results concerning $I^{*(M)}$ and prove that the sequence of sets

$$Ass_R(R/(I^n)^{*(M)}) \ , \ n \in \mathbf{N},$$

is increasing and ultimately constant. (Here $(I^n)^{*(M)}$ denotes the integral closure of I^n relative to M.)

1. introduction

Throughout this paper R denotes a commutative Noetherian ring (with a non-zero identity) and $\mathbf{N}$ denotes the set of positive integers.

The important ideas of reduction and integral closure for ideals in a commutative Noetherian ring were introduced by Northcott and Rees in [7], a brief and direct approach to their theory is given in [9, 1.1] and it is appropriate to summarize some of those facts.

Let I be an ideal of R. Then I is a reduction of an ideal J of R if $I \subseteq J$ and there exists an integer $n \in \mathbf{N}$ such that $IJ^n = J^{n+1}$. An element x of R is said to be integrally dependent on I if there exists $n \in \mathbf{N}$ and elements $c_1, ..., c_n \in R$ with $c_i \in I^i$ for $i = 1, ..., n$ such that

$$x^n + c_1 x^{n-1} + ... + c_{n-1}x + c_n = 0.$$

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The set of all elements of R which are integrally dependent on I is an ideal of R called integral closure of I, and denoted by $\overline{I}$. In fact $\overline{I}$ is the largest ideal of R which has I as a reduction.

In [1] H. Ansari-Toroghey and R.Y. Sharp introduced the concepts of reduction and integral closure of ideals relative to injective $R-$modules and showed that these concepts have properties which reflect those of classical concepts in [7]. Here we recall some of these facts.

Let I be an ideal of R and E be an injective $R-$module. Then I is said to be a reduction of an ideal J of R relative to E if $I \subseteq J$ and there exists a positive integer n such that

$$(0 :_E IJ^n) = (0 :_E J^{n+1}).$$

An element x of R is said to be integrally dependent on I relative to E if there exists a positive integer n such that

$$(0 :_E \sum_{i=1}^{n} x^{n-i} I^i) \subseteq (0 :_E x^n).$$

The set of ideals of R which have I as a reduction relative to E has a unique maximal member, denoted by $I^{*(E)}$ and called the integral closure of I relative to E. It is shown that (see [1, 2.6] and [6, Lemma 4]) $I^{*(E)} = \overline{I}(Ass_R(E))$, where $\overline{I}(Ass_R(E))$ is the intersection of those primary terms in a minimal primary decomposition of $\overline{I}$ which are contained in at least one member of $Ass_R(E)$. Further it is proved that (see [1, 2.7])

$$I^{*(E)} = \{x \in R : x \text{ is integrally dependent on } I \text{ relative to } E\}.$$

Now let M be an $R-$module with $Ass_R(M) \subseteq Ass(R)$ and let $Ass_R(M)$ have reduced property (see 2.2). Further suppose that I is an ideal of R. In this paper, similar to those of mentioned in the last paragraph, the concepts of reduction and integral closure of ideals relative to an arbitrary $R-$module are introduced (see 2.8) and it is shown (see 3.8, 3.9) that $I^{*(M)} = \overline{I}(Ass_R(M))$. Also it is shown (see 3.10, 3.11) that

$$I^{*(M)} = \{x \in R : x \textit{ is integrally dependent on } I \textit{ relative to } M\}.$$

Further it is proved (see 4.1, 4.3) that the sequence of sets

$$Ass_R(R/(I^n)^{*(M)})\ ,\ n \in \mathbf{N},$$

is increasing and ultimately constant. Finally it is proved (see 4.4, 4.6) that if I contains a non-zero divisor on R then the sequence of sets

$$Ass_R((I^n)^{*(M)}/I^n)\ ,\ n \in \mathbf{N},$$

is increasing and ultimately constant.

2. Auxiliary results

Lemma 2.1. Let M be an $R-$module. Then we have $Ass_R(M) \subseteq Ass(R)$ in each of the following cases.
(i) R is Artinian ring.
(ii) M is a faithful multiplication $R-$module with $Ass_R(M) \subseteq Max(R)$.
Proof. (i) Since R is an Artinian ring, we have

$$Ass_R(M) \subseteq Max(R) = Supp(R) = Ass(R).$$

(ii) Since M is a faithful multiplication $R-$module, $Z(M) = Z(R)$ by [3, 4.3]. Now if $P \in Ass_R(M)$, then $P \subseteq Z(M) = Z(R)$. Thus $P \subseteq Q$ for some $Q \in Ass(R)$. So by assumption, $P = Q$. Thus $Ass_R(M) \subseteq Ass(R)$.

Definition 2.2. A subset T of $Ass(R)$ has reduced property if for every $P \in T$, there exists an element $x \in R$ such that $P = Ann(x)$ and $x^2 \neq 0$.

Example 2.3. (i) Suppose that $R = Z_6$ and $M = R\overline{2}$. Then $Ass(R) = \{(\overline{2}), (\overline{3})\}$ and $Ass_R(M) = \{(\overline{3})\}$. Hence $Ass_R(M)$ has reduced property.
(ii) Let $R = k[x, y]$, where k be a field. Let $M = Rx + Ry$. Then

$Ass_R(M)$ has reduced property because R is an integral domain and $Ass_R(M) = Ass(R)$ by [2, p. 286, Ex. 5].
(iii) Let S be a commutative Noetherian reduced ring (with a non-zero identity) and let $Ass(S) = \{P_1, ..., P_n\}$. Set $M = \oplus_{i=1}^{n} S/P_i$. Then $Ass_S(M)$ has reduced property.

The following Lemma denotes a characterization for a commutative Noetherian reduced ring.

Lemma 2.4. Let S be a commutative Noetherian ring (with a non-zero identity). Then S is a reduced ring if and only if $Ass(S)$ has reduced property.

Proof. ($\Leftarrow$) Suppose that $Ass(S)$ has reduced property and assume that S is not a reduced ring. Set

$$\Theta = \{Ann_S(a) : a \neq 0 \ and \ a^2 = 0\}$$

Then Θ is not empty and it has a maximal element, $Ann(y)$ say, where y is a non-zero element of S whit $y^2 = 0$. Set $P' = Ann(y)$. It is clear P' is a prime ideal so that $P' \in Ass(S)$. Thus by assumption, there exists $z \in S$ such that $P' = Ann(z)$ and $z^2 \neq 0$. This implies that $z \notin Ann(z) = Ann(y)$, which is a contradiction. The reverse implication is clear and the proof is completed.

We need the following notation from [1].

Notation 2.5 (see [1, 1.1]). Let I be an ideal of R and let T be a subset of $Spec(R)$. The notation $I(T)$ will denote (I if I=R and) , if I is a proper, the intersection of those primary terms in a minimal primary decomposition of I which are contained in at least one member of T (the intersection of an empty family of ideals of R is assumed to be R itself). This definition is unambiguous and $I(\{P\})$ is denoted by $I(P)$. It is clear that $I(P) = (IR_P)^c$ is just contraction back to R of the extension of I to R_P under the natural ring homomorphism. Also we have $I(T) = \bigcap_{P \in T} I(P)$ and $(J \cap K)(T) = (J(T) \cap K(T))$ for every ideal J and K of R (see [1, 2.5]).

Remark 2.6. Let E be an injective R–module and let I be an ideal of R. Then
(i) $I^{*(E)} = \overline{I}(Ass_R(E))$ (see [1, 2.6] and [6, Lemma 4]).
(ii) $(0 :_E I) = (0 :_E I(Ass_R(E)))$ (see [1, 2.5(iii)]).

Lemma 2.7. Let M be an R–module. Let M be an R–module. Then
(i) $(0 :_M I) = (0 :_M I(Ass_R(M)))$,
(ii) $(0 :_M IJ) = (0 :_M I(Ass_R(M))J)$,
(iii) $(0 :_M I^n J) = (0 :_M (I(Ass_R(M)))^n J)$ for all $n \in \mathbf{N}$.
Proof. (i) It is clear that

$$(0 :_M I(Ass_R(M))) \subseteq (0 :_M I).$$

To see the reverse inclusion, let $m \in (0 :_M I)$. But $(0 :_M I) \subseteq (0 :_{E(M)} I)$ and by 2.6(ii), we have

$$(0 :_{E(M)} I) = (0 :_{E(M)} I(Ass_R(E(M)))) = (0 :_{E(M)} I(Ass_R(M))).$$

Hence $(I(Ass_R(M)))m = 0$ so that $m \in (0 :_M I(Ass_R(M)))$. Thus $(0 :_M I) = (0 :_M I(Ass_R(M)))$.
(ii) This is clear from (i) because $(0 :_M IJ) = ((0 :_M I) :_M J)$.
(iii) This follows from part (ii) by using induction on n.

Definition 2.8. Let I and J be ideals of R. Then I is a reduction of J relative to M if $I \subseteq J$ and there exists $n \in \mathbf{N}$ such that $(0 :_M IJ^n) = (0 :_M J^{n+1})$.
It is straightforward to see (using, for example, the argument of the proof of [10, (1.2)]) that if I is a reduction of the ideal J of R relative to M and also a reduction of the ideal K of R relative to M, then I is a reduction of $J + K$ relative to M. Thus, since R is Noetherian, the set of ideals

of R which have I as a reduction relative to M has a unique maximal member which is denoted by $I^{*(M)}$ and called the integral closure of I relative to M. This is in fact the largest ideal which has I as a reduction.

The proof of the next Lemma is easy and is omitted.

Lemma 2.9. Let I, J, K and L be ideals of R.
(i) If I is a reduction of J relative to M and J is a reduction of K relative to M, then I is a reduction of K relative to M.
(ii) If I is a reduction of J relative to M and K is a reduction of L relative to M, then IK is a reduction of JL relative to M.
(iii) If $I \subseteq J \subseteq K$ and I is a reduction of K relative to M, then J is a reduction of K relative to M.
(iv) We have $(I^{*(M)})^{*(M)} = I^{*(M)}$.

Definition 2.10. We say that the element x of R is integrally dependent on I relative to M if there exists $n \in \mathbf{N}$ such that

$$(0 :_M \sum_{i=1}^{n} x^{n-i} I^i) \subseteq (0 :_M x^n).$$

The ideas of the proof of [10, 2.2] can be used to establish the following Lemma.

Lemma 2.11. Let $x \in R$. Then x is integrally dependent on I relative to M if and only if I is a reduction of $I + Rx$ relative to M.

Remark 2.12. Let M be an $R-$module and let I be an ideal of R. Then

$$\{x \in R : x \textit{ is integrally dependent on } I \textit{ relative to } M\} \subseteq I^{*(M)}.$$

Proof. This follows from 2.11.

Remark 2.13 (see [12, 1.19]). Let M be an $R-$module, and suppose that F is a flat $R-$ module. Then

$$Ass_R(M \otimes_R F) = \{P \in Ass_R(M) : P \subseteq Q \; for \; some \; Q \in Coass_R(F)\}.$$

3. Main results

Lemma 3.1. Let M be an $R-$module and let N be a submodule of M. Let I and J be ideals of R and let I be a reduction of J relative to M. Then I is a reduction of J relative to N.

Proof. The proof is straightforward.

Corollary 3.2. Let I be an ideal of R and let M be an $R-$module. Suppose that N is a submodule of M. Then

$$I^{*(M)} \subseteq I^{*(N)}.$$

Proof. By 3.1, I is a reduction of $I^{*(M)}$ relative to N. Hence $I^{*(M)} \subseteq I^{*(N)}$.

Lemma 3.3. Let M be an $R-$module, then $\overline{I}(Ass_R(M)) \subseteq I^{*(M)}$.

Proof. By 3.2, $I^{*(E(M))} \subseteq I^{*(M)}$. Further by 2.6(i),

$$I^{*(E(M))} = \overline{I}(Ass_R(E(M))).$$

Thus we have

$$\overline{I}(Ass_R(M)) = I^{*(E(M))} \subseteq I^{*(M)}.$$

Lemma 3.4. Let M be an $R-$module. Then $0^{*(M)} = \sqrt{(0 :_R M)}$.

Proof. The proof is easy and is omitted.

The following example shows that the inclusion in 3.3 may be strict.

Example 3.5. Let (R, m) be a local ring with maximal ideal m such that $\sqrt{0} \neq m$. By 3.4, $0^{*(R/m)} = m$. Now by 3.4 and [11, page 47, Coro.2], we have $0^{*(E(R/m))} = \sqrt{0}$. Thus

$$\overline{0}(Ass_R(R/m)) = \overline{0}(Ass_R(E(R/m))) = 0^{*(E(R/m))} \neq 0^{*(R/m)}.$$

Lemma 3.6. Let $P \in Spec(R)$ and let $P = Ann(x)$, where $x \in R$ with $x^2 \neq 0$. Then for every non-zero element $y \in E(R/P)$, there exists $t \in R \setminus P$ such that $ty \in R/P$.

Proof. Let y be a non-zero element of $E(R/P)$. Since $x^2 \neq 0$, $x \notin P$. Set $I = Rx$ so that $I \neq 0$. Then it is easy to see that $I \cap P = 0$. Now we claim that $Iy \neq 0$. Otherwise, there exists $0 \neq h \in I$ such that $hy = 0$. Since multiplication by h on $E(R/P)$ is an automorphism (see [4, 3.2]), it follows that $y = 0$, which is a contradiction. Now the result follows from the fact that $Iy \cap R/P \neq 0$ and that $I \cap P = 0$.

Lemma 3.7. Let $P \in Spec(R)$ and let $P = Ann(x)$, where $x \in R$ and $x^2 \neq 0$. Then $I^{*(R/P)} = I^{*(E(R/P))}$.

Proof. By 3.2, it is enough to prove that $I^{*(R/P)} \subseteq I^{*(E(R/P))}$. To see this, chose $n \in \mathbf{N}$ such that

$$(0 :_{R/P} I(I^{*(R/P)})^n) = (0 :_{R/P} (I^{*(R/P)})^{n+1}).$$

Then we will show that

$$(0 :_{E(R/P)} I(I^{*(R/P)})^n) = (0 :_{E(R/P)} (I^{*(R/P)})^{n+1}).$$

To do this let $y \in (0 :_{E(R/P)} I(I^{*(R/P)})^n)$. Then by 3.6, there exists $t \in R \setminus P$ such that $ty \in R/P$ and $I(I^{*(R/P)})^n ty = 0$. Hence

$$ty \in (0 :_{R/P} I(I^{*(R/P)})^n) = (0 :_{R/P} (I^{*(R/P)})^{n+1}).$$

Since the multiplication by t on $E(R/P)$ is an automorphism, it follows that $y \in (0 :_{E(R/P)} (I^{*(R/P)})^{n+1})$. Hence

$$(0 :_{E(R/P)} I(I^{*(R/P)})^n) \subseteq (0 :_{E(R/P)} (I^{*(R/P)})^{n+1}).$$

The revers inclusion is clear. It follows that $I^{*(R/P)} \subseteq I^{*(E(R/P))}$ as desired.

Theorem 3.8. Let M be an R–module with $Ass_R(M) \subseteq Ass(R)$.

Suppose that $Ass_R(M)$ has reduced property. Then

$$I^{*(M)} = \overline{I}(Ass_R(M)).$$

Proof. Let $P \in Ass_R(M)$. By 3.2, $I^{*(M)} \subseteq I^{*(R/P)}$. But $I^{*(R/P)} = I^{*(E(R/P))}$ by 3.7. Also by 2.6(i), $I^{*(E(R/P))} = \overline{I}(P)$. Thus

$$I^{*(M)} \subseteq I^{*(R/P)} = \overline{I}(P).$$

Hence for every $P \in Ass_R(M)$, $I^{*(M)} \subseteq \overline{I}(P)$ so that

$$I^{*(M)} \subseteq \bigcap_{P \in Ass_R(M)} \overline{I}(P) = \overline{I}(Ass_R(M)).$$

Now by 3.3, $\overline{I}(Ass_R(M)) \subseteq I^{*(M)}$. Therefore,

$$I^{*(M)} = \overline{I}(Ass_R(M)).$$

Corollary 3.9. Let R be a reduced ring and let M be an $R-$module. Then

$$I^{*(M)} = \overline{I}(Ass_R(M))$$

in each of the following cases.
(i) R is an Artinian ring.
(ii) M is a $R-$module with $Ass_R(M) \subseteq Ass(R)$.
(iii) M is a faithful multiplication $R-$module with $Ass_R(M) \subseteq Max(R)$.
Proof. This follows from 2.1, and 3.8.

Corollary 3.10 (see [1, 2.7]). Let M be an $R-$module with $Ass_R(M) \subseteq Ass(R)$. Suppose that $Ass_R(M)$ has reduced property. Let I and J be ideals of R such that $I \subseteq J \subseteq I^{*(M)}$. Then I is a reduction of J relative to M. Consequently,

$$I^{*(M)} = \{x \in R : x \textit{ is integrally dependent on } I \textit{ relative to } M\}.$$

Proof. One can apply the same technique as in [1, 2.7] by using 3.8, 2.7(ii), 2.7(iii), 2.11, and 2.12.

Corollary 3.11. Let R be a reduced ring and let M be an $R-$module. Further let I be an ideal of R. Then

$$I^{*(M)} = \{x \in R : x \text{ is integrally dependent on } I \text{ relative to } M\}$$

in each of the following cases.
(i) R is an Artinian ring.
(ii) M is a $R-$module with $Ass_R(M) \subseteq Ass(R)$.
(iii) M is a faithful multiplication $R-$module with $Ass_R(M) \subseteq Max(R)$.
Proof. This follows from 2.1, and 3.10.

4. Asymptotic behavior

In [8, 2.4] Ratliff showed that the sequence of sets

$$Ass_R(R/\overline{I^n}),\ n \in \mathbf{N},$$

is increasing and ultimately constant. We denote the ultimate constant value of this sequence by $\overline{As}(I,R)$.

Theorem 4.1. Let M be an $R-$module with $Ass_R(M) \subseteq Ass(R)$. Suppose that $Ass_R(M)$ has reduced property and I is an ideal of R. Then the sequence of sets

$$Ass_R(R/(I^n)^{*(M)}),\ n \in \mathbf{N},$$

is increasing and ultimately constant. Further if we denote the ultimate constant value of the above sequence by $\overline{As}^*(I,R)$, then

$$\overline{As}^*(I,R) = \{P \in \overline{As}(I,R) : P \subseteq P' \text{ for some } P' \in Ass_R(M)\}.$$

Proof. Let $\overline{I} = Q_1 \cap Q_2 \cap ... \cap Q_m$ be a minimal primary decomposition of $\overline{I}$, where $Q_j (1 \leq j \leq m)$ is a P_j-primary ideal of R. By 3.8, we have

$$I^{*(M)} = \overline{I}(Ass_R(M)) = \bigcap_{Q_j \subseteq P \text{ for some } P \in Ass_R(M)} Q_j.$$

This is in fact a minimal primary decomposition of $I^{*(M)}$. Thus

$$Ass_R(R/I^{*(M)}) = \{P_j \in Ass_R(R/\overline{I}) : P_j \subseteq P \; for \; some \; P \in Ass_R(M)\}.$$

Hence

$$Ass_R(R/(I^n)^{*(M)}) = \{P_j \in Ass_R(R/\overline{I^n}) : P_j \subseteq P \; for \; some \; P \in Ass_R(M)\}.$$

Now the result follows from the fact that the sequence of sets

$$Ass_R(R/\overline{I^n}), \; n \in \mathbf{N},$$

is increasing and ultimately constant as mentioned above.

Corollary 4.2. Let M be an $R-$module with $Ass_R(M) \subseteq Ass(R)$. Suppose that $Ass_R(M)$ has reduced property and F is a flat $R-$module. Then for every ideal I of R the sequence of sets

$$Ass_R(F/(I^n)^{*(M)}F), \; n \in \mathbf{N},$$

is increasing and ultimately constant. If we denote the ultimate constant value of this sequence by $\overline{As}^*(I, F)$ then we have

$$\overline{As}^*(I, F) = \{P \in \overline{As}^*(I, R) : P \subseteq P' \; for \; some \; P' \in Coass_R(F)\}.$$

Proof. We have

$$F/(I^n)^{*(M)}F \simeq R/(I^n)^{*(M)} \otimes_R F.$$

Now the result follows from 2.13, and 4.1.

Corollary 4.3. Let R be a reduced ring and let M be an $R-$module. Suppose that F is a flat $R-$module. Then for every ideal I of R, the sequence of sets

$$Ass_R(F/(I^n)^{*(M)}F), \; n \in \mathbf{N},$$

is increasing and ultimately constant in each of the following cases.

(i) R is an Artinian ring.

(ii) M is a $R-$module with $Ass_R(M) \subseteq Ass(R)$.

(iii) M is a faithful multiplication $R-$module with $Ass_R(M) \subseteq Max(R)$.

Proof. This follows from 2.1, and 4.2.

Theorem 4.4. Let M be an $R-$module with $Ass_R(M) \subseteq Ass(R)$ and let $Ass_R(M)$ have reduced property. Suppose that I is an ideal of R which contains a non-zero divisor on R. Then the sequence of sets

$$Ass_R((I^n)^{*(M)}/I^n)\ ,\ n \in \mathbf{N},$$

is increasing and ultimately constant.

Proof. We can apply the technique used in [5, 11.16]. By [5, Lemma 8.1], there exists a positive integer k such that for $n \geq k$, $(I^{n+1} :_R I) = I^n$. Let $n \geq k$ and let $P \in Ass_R((I^n)^{*(M)}/I^n)$. Then there exists an element $c \in (I^n)^{*(M)}$ such that $P = (I^n :_R c)$. This implies that $P = (I^{n+1} :_R cI)$. Now we show that

$$cI \subseteq (I^n)^{*(M)}I \subseteq (I^{n+1})^{*(M)}.$$

To see this let $xy \in (I^n)^{(*M)}I$, where $x \in (I^n)^{*(M)}$ and $y \in I$. Then by 3.10, there exists a positive integer s such that

$$(0 :_M \sum_{i=1}^{s} x^{s-i}(I^n)^i) \subseteq (0 :_M x^s).$$

So we get

$$(0 :_M \sum_{i=1}^{s}(xy)^{s-i}y^i(I^n)^i) = ((0 :_M \sum_{i=1}^{s} x^{s-i}(I^n)^i) :_M y^s)$$
$$\subseteq (0 :_M (xy)^s).$$

Since $y^i(I^n)^i \subseteq (I^{n+1})^i$,

$$(0 :_M \sum_{i=1}^{s}(xy)^{s-i}(I^{n+1})^i) \subseteq (0 :_M \sum_{i=1}^{s}(xy)^{s-i}y^i(I^n)^i)$$
$$\subseteq (0 :_M (xy)^s).$$

Hence we have $xy \in (I^{n+1})^{*(M)}$. So by the above arguments, we have $P \in Ass_R((I^{n+1})^{*(M)}/I^{n+1})$. Hence for $n \geq k$,

$$Ass_R((I^n)^{*(M)}/I^n)\ ,\ n \in \mathbf{N},$$

becomes an increasing sequence. Now the result follows from the fact that

$$Ass_R((I^n)^{*(M)}/I^n) \subseteq Ass_R(R/I^n) \subseteq As^*(I,R).$$

So the proof is complete.

Corollary 4.5. Let the situation be as in 4.4 and let F be a flat $R-$module. Then the sequence of sets

$$Ass_R((I^n)^{*(M)}F/I^nF)\ ,\ n \in \mathbf{N},$$

is increasing and ultimately constant.

Proof. It is easy to see that

$$(I^n)^{*(M)}F/I^nF \simeq (I^n)^{*(M)}/I^n \otimes F.$$

Now the result follows from 2.13 and 4.4.

Corollary 4.6. Let R be a reduced ring and let M be an $R-$module. Suppose that F is a flat $R-$module. Then for every ideal I of R which is contains a non-zero divisor on R , the sequence of sets

$$Ass_R((I^n)^{*(M)}F/I^nF)\ ,\ n \in \mathbf{N},$$

is increasing and ultimately constant in each of the following cases.

(i) R is an Artinian ring.

(ii) M is a $R-$module with $Ass_R(M) \subseteq Ass(R)$.

(iii) M is a faithful multiplication $R-$module with $Ass_R(M) \subseteq Max(R)$.

Proof. This follows from 2.1, and 4.5.

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H. Ansari-Toroghy
Department of Mathematics,
Faculty of Science, University of Guilan,
P. O. Box 1914, Rasht, Iran
E-mail: ansari@guilan.ac.ir

F. Dorostkar
Department of Mathematics,
Faculty of Science, University of Guilan,
P. O. Box 1914, Rasht, Iran
E-mail: dorostkar@guilan.ac.ir