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ON δgs-CLOSED SETS AND ALMOST WEAKLY HAUSDORFF SPACES

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Abstract. The aim of this paper is to introduce the class of δgs-closed sets and obtain characterizations of almost weakly Hausdorff spaces due to Dontchev and Ganster. We also introduce the notion of δgs-continuity and investigate the relationships between it and other types of continuity.

1. Introduction and preliminaries

Many topological properties (including separation axioms, connectedness and covering properties) have been characterized and generalized by means of several "generalized closed sets". The initiation of the study of g-closed sets, i.e. sets whose closure belongs to every open superset, was done by Levine [21] in 1970. The spaces in which the concepts of g-closed and closed sets coincide called $T_{1/2}$ spaces [21]. Dunham [11] showed that $T_{1/2}$ spaces are precisely the spaces in which singletons are open or closed. The concept of g-closed sets has been modified and studied in last ten years by weaker forms of open sets such as α-open sets [24], semi-open sets [21], preopen sets [22] and semi-preopen sets [1]. The majority of the modifications are in fact weaker than g-closedness. Recently, Dontchev and Ganster [9] introduced and studied the concept

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of δg-closed sets which is a slightly stronger form of g-closedness properly placed between δ-closedness and g-closedness, by using the semi-regularization of a given topology and the associated δ-closure operator, and introduced the notion of almost weakly Hausdorff spaces to solve the problem of finding the spaces (X, τ) in which the g-closed sets of (X, τ_s) are δg-closed in (X, τ). An example of such space is the digital line or the so called the Khalimsky line [16, 18] which is widely used in the applications of point-set topology in computer graphics. Dontchev et al. [10] introduced and studied two classes of $g\delta$-closed and δg^*-closed sets and used those concepts to give characterizations of almost weakly Hausdorff spaces. In a recent work Elnaschie [13] derived quantum gravity from set theory. Thus the study of g-closed sets will give the possible applications in computer graphics [15, 16, 17, 18] and quantum physics [12, 13]. In this paper, we introduce and study the class of δgs-closed sets, which contains the class of $g\delta s$-closed sets and the class of δg^*-closed sets, respectively. The relations with other notions directly or indirectly connected with g-closed sets are investigated. We use it to obtain new characterizations of almost weakly Hausdorff spaces and introduce the notion of δgs-continuity and investigate the relationships between it and other types of continuity.

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$, respectively. A subset A is said to be *regular open* (resp. *regular closed*) if $A = \mathrm{int}(\mathrm{cl}(A))$ (resp. $A = \mathrm{cl}(\mathrm{int}(A))$). Since the intersection of two regular open sets is regular open, the collection of all regular open sets forms a base for a coarser topology τ_s than the original one τ. The family τ_s is called the semi-regularization of τ. A space (X, τ) is called *semi-regular* if $\tau = \tau_s$. The δ-interior [33] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by

δ-int(A). The subset A is called *δ-open* [33] if $A = \delta$-int(A), i.e. a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called *δ-closed.* Alternatively, a set $A \subset (X, \tau)$ is called δ-closed [33] if $A = \delta$-cl(A), where $\delta\text{-cl}(A) = \{x \in X : \text{int}(\text{cl}(U)) \cap A \neq \emptyset,\ U \in \tau \text{ and } x \in U\}$. The family of all δ-open sets forms a topology on X and is denoted by τ_δ. It is well known that $\tau_s = \tau_\delta$.

A subset A of X is called *semiopen* [21] (resp. *α-open* [24], *δ-semiopen* [19], *preopen* [22], *δ-preopen* [30], *semi-preopen* [1]) if $A \subset \text{cl}(\text{int}(A))$ (resp. $A \subset \text{int}(\text{cl}(\text{int}(A)))$, $A \subset \text{cl}(\delta\text{-int}(A))$, $A \subset \text{int}(\text{cl}(A))$, $A \subset \text{int}(\delta\text{-cl}(A))$, $A \subset \text{cl}(\text{int}(\text{cl}(A)))$) and the complement of a semiopen (resp. α-open, δ-semiopen, preopen, δ-preopen, semi-preopen) set is called *semiclosed* (resp. *α-closed*, *δ-semiclosed*, *preclosed*, *δ-preclosed*, *semi-preclosed*). A subset A is called *δ-semiregular* [31] if it is δ-semiopen and δ-semiclosed. The intersection of all semiclosed (resp. δ-semiclosed) sets containing A is called the *semi-closure* [4] (resp. *δ-semi-closure* [28]) of A and is denoted by scl(A) (resp. δ-scl(A)). Dually, the *semi-interior* (resp. *δ-semi-interior*) of A is defined to be the union of all semiopen (resp. δ-semiopen) sets contained in A and is denoted by sint(A) (resp. δ-sint(A)). Note that $\delta\text{-scl}(A) = A \cup \text{int}(\delta\text{-cl}(A))$ and $\delta\text{-sint}(A) = A \cap \text{cl}(\delta\text{-int}(A))$ [28].

We recall the following definitions used in sequel.

Definition 1.1. A subset A of a space (X, τ) is said to be:

(a) *generalized closed* [21] (briefly, *g-closed*) if $\text{cl}(A) \subset U$ whenever $A \subset U$ and U is open in X,

(b) *generalized semi-closed* [2] (briefly, *gs-closed*) if $\text{scl}(A) \subset U$ whenever $A \subset U$ and U is open in X,

(c) *generalized δ-semi-closed* [29] (briefly *$g\delta s$-closed*) if $\delta\text{-scl}(A) \subset U$ whenever $A \subset U$ and U is open in X,

(d) *δ-generalized closed* [9] (briefly *δg-closed*) if $\delta\text{-cl}(A) \subset U$ whenever $A \subset U$ and U is open in X,

(e) *generalized δ-closed* [10] (briefly *$g\delta$-closed*) if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is δ-open in X,

(f) *δg^*-closed* [10] if δ-$\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is δ-open in X,

(g) *g-open* (resp. *gs-open*, *$g\delta s$-open*) if the complement of A is g-closed (resp. gs-closed, $g\delta s$-closed).

Definition 1.2. A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(a) *semi-continuous* [20] (resp. *irresolute* [5]) if $f^{-1}(F)$ is semiclosed in (X, τ) for every closed (resp. semiclosed) set F of (Y, σ),

(b) *δ-continuous* [25] (resp. *δ-semi-continuous* [31], *δ-semi-irresolute*) if $f^{-1}(F)$ is δ-closed (resp. δ-semiclosed, δ-semiclosed) in (X, τ) for every δ-closed (resp. closed, δ-semiclosed) set F of (Y, σ),

(c) *g-continuous* [3] (resp. *gs-continuous* [7], *gs-irresolute* [7]) if $f^{-1}(F)$ is g-closed (resp. gs-closed, gs-closed) in (X, τ) for every closed (resp. closed, gs-closed) set F of (Y, σ),

(d) *$g\delta s$-continuous* [29] (resp. *$g\delta s$-irresolute* [29]) if $f^{-1}(F)$ is $g\delta s$-closed in (X, τ) for every closed (resp. $g\delta s$-closed) set F of (Y, σ),

(e) *δ-open* [23] if $f(U)$ is δ-open in (Y, σ) for every δ-open set U of (X, τ),

(f) *δ-closed* [26] (resp. *δ-semiclosed*) if $f(F)$ is δ-closed (resp. δ-semiclosed) in (Y, σ) for every δ-closed (δ-semiclosed) set F of (X, τ).

2. Basic properties of δgs-closed sets

Definition 2.1. A subset A of a space (X, τ) is called *δgs-closed* if δ-$\mathrm{scl}(A) \subset U$ whenever $A \subset U$ and U is δ-open in (X, τ).

Remark 2.1. For a subset of a space, from definitions stated above, we have the following diagram of implications:

$$\begin{array}{ccccc} & & \text{semiclosed} & & \\ & \nearrow & & \searrow & \\ \delta\text{-semiclosed} & & & & gs\text{-closed} \\ & \searrow & & \nearrow & \\ & & g\delta s\text{-closed} & \rightarrow & \delta gs\text{-closed} \end{array}$$

where none of these implications is reversible as shown by examples of [28], [2] and [29] and the following example.

Example 2.1. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then

(a) Set $A = \{a, c, d\}$. Then A is δgs-closed but neither gs-closed nor $g\delta s$-closed in (X, τ).

(b) Set $B = \{b, c\}$. Then B is gs-closed but not δgs-closed in (X, τ).

(c) Set $C = \{c\}$. Then C is δ-semiclosed (hence δgs-closed) but not δg^*-closed in (X, τ).

Now, we observe that in semi-regular spaces the notions of δgs-closed and gs-closed sets coincide. Recall that a space (X, τ) is called T_d [8] (resp. T_b [8]) if every gs-closed set is g-closed (resp. closed).

Theorem 2.1. *Let A be a subset of a semi-regular space (X, τ). Then:*

(a) *A is δgs-closed if and only if A is gs-closed.*

(b) *If, in addition, (X, τ) is T_b (resp. T_d), then A is δgs-closed if and only if A is closed (resp. g-closed).*

The previous observations lead to the problem of finding the spaces (X, τ) in which the gs-closed sets of (X, τ_s) are δgs-closed in (X, τ). Recall that a space with semi-$T_{1/2}$ semi-regularization is called *semi weakly Hausdorff* [29].

Theorem 2.2. *For a subset A of a semi weakly Hausdorff space (X, τ) the following are equivalent:*

(a) *A is gs-closed in* (X, τ_s).

(b) *A is* δ*-semiclosed in* (X, τ).

(c) *A is* δgs*-closed in* (X, τ).

Proof. (a)$\Rightarrow$(b) Let $A \subset X$ be a gs-closed subset of (X, τ_s). Let $x \in \delta\text{-scl}(A)$. If $\{x\}$ is δ-semiopen, then $x \in A$. If not then $X \setminus \{x\}$ is δ-semiopen, since X is semi weakly Hausdorff. Assume that $x \notin A$. Since A is gs-closed in (X, τ_s), then $\delta\text{-scl}(A) \subset X \setminus \{x\}$, i.e. $x \notin \delta\text{-scl}(A)$. By contradiction $x \in A$. Thus $\delta\text{-scl}(A) = A$ or equivalently A is δ-semiclosed in (X, τ).

(b)$\Rightarrow$(c) Clear.

(c)$\Rightarrow$(a) Let $A \subset U$, where U is open in (X, τ_s). Then U is δ-open in (X, τ) and since A is δgs-closed in (X, τ), by Lemma 7.3 of [27], $\text{scl}(A) \subset U$ in (X, τ_s). Thus A is gs-closed in (X, τ_s). □

Theorem 2.3. *For a space* (X, τ) *the following are equivalent:*

(a) *Every* δ*-open set of X is* δ*-semiclosed.*

(b) *Every subset of X is* δgs*-closed.*

Proof. (a)$\Rightarrow$(b) Let $A \subset U$, where U is δ-open and A is an arbitrary subset of X. By (a), then U is δ-semiclosed and thus $\delta\text{-scl}(A) \subset \delta\text{-scl}(U) = U$.

(b)$\Rightarrow$(a) If $U \subset X$ is δ-open, then by (b) $\delta\text{-scl}(U) = U$ or equivalently U is δ-semiclosed. □

Remark 2.2. (a) Finite union of δgs-closed sets need not to be δgs-closed.

(b) Finite intersection of δgs-closed sets need not be δgs-closed.

Example 2.2. (a) Let (X, τ) be a space given in Example 2.1. Put $A = \{a, b\}$ and $B = \{c\}$. Then A and B are δgs-closed but $A \cup B = \{a, b, c\}$ is not δgs-closed in (X, τ).

(b) Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Put $A = \{a, b, c\}$ and $B = \{a, b, d\}$. Then A and B is δgs-closed but $A \cap B = \{a, b\}$ is not δgs-closed in (X, τ).

However, we have

Theorem 2.4. *The intersection of a δgs-closed set and a δ-semiclosed set is δgs-closed.*

Proof. Let A be δgs-closed and F be δ-closed. If U is δ-open set with $A \cap F \subset U$, then $A \subset U \cup (X \setminus F)$ and so δ-scl$(A) \subset U \cup (X \setminus F)$. Now, by Theorem 2.8 of [19], we have δ-scl$(A \cap F) \subset \delta$-scl$(A) \cap F \subset U$ and thus $A \cap F$ is δgs-closed. □

Recall that for a subset A of (X, τ), a point $x \in X$ is called a *δ-limit point* (resp. *δ-semi-limit point* [19]) of A if every δ-open (resp. δ-semiopen) set containing x contains a point of A different from x. The set of all δ-limit points (resp. δ-semi-limit points) of A is called the *δ-derived set* (resp. *δ-semi-derived set* [19]) of A and is denoted by δ-$D(A)$ (resp. δ-$D_s(A)$).

Theorem 2.5. *If A and B are δgs-closed sets of X such that δ-$D(A) \subset \delta$-$D_s(A)$ and δ-$D(B) \subset \delta$-$D_s(B)$, then $A \cup B$ is δgs-closed.*

Proof. Since δ-$D_s(C) \subset \delta$-$D(C)$ for any subset C of X, by hypothesis δ-$D_s(A) = \delta$-$D(A)$ and δ-$D_s(B) = \delta$-$D(B)$, i.e. δ-cl$(A) = \delta$-scl(A) and δ-cl$(B) = \delta$-scl(B). Let $A \cup B \subset U$, where U is δ-open in X. Then $A \subset U$ and $B \subset U$. Since A and B are δgs-closed, δ-scl$(A) \subset U$ and δ-scl$(B) \subset U$. Now, δ-cl$(A \cup B) = \delta$-cl$(A) \cup \delta$-cl$(B) = \delta$-scl$(A) \cup \delta$-scl$(B) \subset U$. But, since δ-scl$(A \cup B) \subset \delta$-cl$(A \cup B)$, δ-scl$(A \cup B) \subset U$ and hence $A \cup B$ is δgs-closed. □

Theorem 2.6. (a) *If A is δgs-closed subset of (X, τ), then δ-scl$(A) \setminus A$ contains no non-empty δ-closed set.*

(b) *If A is δgs-closed subset of (X, τ) and $A \subset B \subset \delta\text{-scl}(A)$, then $\delta\text{-scl}(B) \setminus B$ contains no non-empty δ-closed set.*

Proof. (a) Let $F \subset \delta\text{-scl}(A) \setminus A$, where F is δ-closed. Then $A \subset X \setminus F$ and $X \setminus F$ is δ-open. Since A is δgs-closed, $\delta\text{-scl}(A) \subset X \setminus F$. That is $F \subset X \setminus \delta\text{-scl}(A)$. Hence $F \subset \delta\text{-scl}(A) \cap (X \setminus \delta\text{-scl}(A)) = \emptyset$, which shows $F = \emptyset$.

(b) Clear. □

Corollary 2.1. *Let A be a δgs-closed subset of (X, τ). Then A is δ-semiclosed if and only if $\delta\text{-scl}(A) \setminus A$ is δ-closed.*

Proof. It follows from Theorem 2.6 (a). □

Theorem 2.7. *If A is a δ-open and δgs-closed subset of (X, τ), then A is δ-semiclosed and hence δ-semiregular.*

Proof. Since A is δgs-closed and δ-open, $\delta\text{-scl}(A) \subset A$ and so A is δ-semiclosed. Hence A is δ-semiregular, since δ-semiopen and δ-semiclosed is δ-semiregular. □

Theorem 2.8. *Let $A \subset Y \subset X$. Then:*

(a) *If Y is open in (X, τ) and A is δgs-closed in X, then A is δgs-closed relative to Y.*

(b) *If Y is δgs-closed and δ-open in (X, τ) and A is δgs-closed relative to Y, then A is δgs-closed in X.*

Proof. (a) Let $A \subset U$ and U be δ-open relative to Y. Then by Corollary 1 of [30], $U = Y \cap V$ for some δ-open set V of X. Since A is δgs-closed in X, $\delta\text{-scl}(A) \subset V$ and by Theorem 4.2.25 of [31], $\delta\text{-scl}_Y(A) = \delta\text{-scl}(A) \cap Y \subset V \cap Y = U$. Hence A is δgs-closed relative to Y.

(b) Let $A \subset U$ and U be δ-open in X. Then by Corollary 1 of [30], $U \cap Y$ is δ-open relative to Y and since A is δgs-closed relative to Y, $\delta\text{-scl}_Y(A) \subset U \cap Y$. By Theorem 4.2.25 of [31] and Theorem 2.7,

$\delta\text{-scl}(A) = \delta\text{-scl}(A) \cap Y = \delta\text{-scl}_Y(A) \subset U$. Hence A is δgs-closed in X. □

3. On δgs-open sets

Definition 3.1. A subset A of (X, τ) is called δgs-open if its complement $X \setminus A$ is δgs-closed.

Theorem 3.1. *A subset A of (X, τ) is δgs-open if and only if $F \subset \delta\text{-sint}(A)$ whenever F is δ-closed and $F \subset A$.*

Proof. Obvious. □

Theorem 3.2. *If a subset A of (X, τ) is δgs-open, then $U = X$ whenever U is δ-open and $\delta\text{-sint}(A) \cup (X \setminus A) \subset U$.*

Proof. Let U be a δ-open set such that $\delta\text{-sint}(A) \cup (X \setminus A) \subset U$. Then $X \setminus U \subset (X \setminus \delta\text{-sint}(A)) \cap A$, i.e. $(X \setminus U) \subset \delta\text{-scl}(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus A$ is δgs-closed, by Theorem 2.6 (a), $X \setminus U = \emptyset$ and hence $U = X$. □

Theorem 3.3. *Let $A \subset Y \subset X$ and Y be δ-open and closed in (X, τ). If A is δgs-open relative to Y, then A is δgs-open in X.*

Proof. Let F be any δ-closed subset of X and $F \subset A$. Then F is δ-closed relative to Y and since A is δgs-open relative to Y, by Theorem 4.2.25 of [31], $F \subset \delta\text{-sint}_Y(A) = \delta\text{-sint}(A) \cap Y$. Hence $F \subset \delta\text{-sint}(A)$ and so A is δgs-open in X. □

Theorem 3.4. *If A is δgs-open in (X, τ) and $\delta\text{-sint}(A) \subset B \subset A$, then B is δgs-open.*

Proof. Let $F \subset B$ and F be a δ-closed subset of X. Since A is δgs-open and $F \subset A$, $F \subset \delta\text{-sint}(A)$ and then $F \subset \delta\text{-sint}(B)$. Hence B is δgs-open. □

Theorem 3.5. *If a subset A of (X, τ) is δgs-closed, then δ-scl$(A) \setminus A$ is δgs-open.*

Proof. Let $F \subset \delta$-scl$(A) \setminus A$, where F be δ-closed in X. Then by Theorem 2.6 (a), $F = \emptyset$ and so $F \subset \delta$-sint$(\delta$-scl$(A) \setminus A)$. This shows that δ-scl$(A) \setminus A$ is δgs-open. □

4. On almost weakly Hausdorff spaces

Definition 4.1. [9] A space (X, τ) is called *almost weakly Hausdorff* if its semi-regularization is $T_{1/2}$.

Theorem 4.1. *For a space (X, τ) the following conditions are equivalent:*

(a) *X is almost weakly Hausdorff.*

(b) *Every singleton of X is δ-closed or δ-open.*

(c) *Every singleton of X is δ-closed or δ-semiopen.*

(d) *Every δgs-closed subset of X is δ-semiclosed.*

Proof. (a)⇔(b) is proved [10].

(b)⇔(c) follows from the fact that every singleton is δ-semiopen if and only if it is δ-open.

(c)⇒(d) Let $A \subset X$ be δgs-closed. Let $x \in \delta$-scl(A). We consider the following two cases:

Case 1. Let $\{x\}$ be δ-semiopen. Since x belongs to the δ-semiclosure of A, then $\{x\} \cap A \neq \emptyset$. This shows that $x \in A$.

Case 2. Let $\{x\}$ be δ-closed. If we assume that $x \notin A$, then we have $x \in \delta$-scl$(A) \setminus A$, which cannot happen according to Theorem 2.6 (a). Hence $x \in A$.

So in both cases we have δ-scl$(A) \subset A$. Since the reverse inclusion is trivial, then $A = \delta$-scl(A) or equivalently A is δ-semiclosed.

(d)⇒(c) If $\{x\}$ is not δ-closed, then $X \setminus \{x\}$ is not δ-open and thus δgs-closed. By (d), $X \setminus \{x\}$ is δ-semiclosed, i.e. $\{x\}$ is δ-semiopen. □

To obtain further characterization of almost weakly Hausdorff spaces, we start with following definition.

Definition 4.2. A subset A of (X, τ) is called *δ-nowhere dense* if $\text{int}(\delta\text{-cl}(A)) = \emptyset$.

Clearly, every δ-nowhere dense set is nowhere dense but not conversely.

Example 4.1. $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{c\}\}$. Then $\{a, b\}$ is nowhere dense but not δ-nowhere dense in(X, τ).

Lemma 4.1. *For a space (X, τ), the following are valid:*
(a) *Every singleton is δ-preclosed or δ-open in X.*
(b) *Every singleton is δ-nowhere dense or δ-preopen in X.*

Theorem 4.2. *For a space (X, τ) the following conditions are equivalent:*
(a) *X is almost weakly Hausdorff.*
(b) *Every δ-preclosed singleton of X is δ-closed.*
(c) *Every non δ-open singleton of X is δ-closed.*

Proof. (a)$\Rightarrow$(b) Let $x \in X$ and $\{x\}$ be δ-preclosed in X. By above lemma, $\{x\}$ is not δ-open and hence by Theorem 4.1, $\{x\}$ is δ-closed.

(b)$\Rightarrow$(a) If $\{x\}$ is not δ-open for some $x \in X$, then by above lemma $\{x\}$ is δ-preclosed and by (b), it is δ-closed. Hence X is almost weakly Hausdorff.

(b)$\Leftrightarrow$(c) Obvious. □

A space (X, τ) is called $T_{3/4}$ [9] if every δg-closed subset of X is δ-closed and *weakly Hausdorff* [32] if the semi-regularization of X is T_1. Dontchev and Ganster [9] showed the following implications: weakly Hausdorff space $\Rightarrow$ almost weakly Hausdorff space $\Rightarrow$ $T_{3/4}$ space but not conversely and obtained some conditions that imply the reverse claim is valid. To find other conditions that imply the reverse claim is also valid, we start with the following lemma.

Lemma 4.2. *For a space (X, τ) the following are equivalent:*

(a) *Every δ-preopen singleton is δ-closed.*

(b) *Every singleton is δ-nowhere dense or δ-closed.*

Proof. (a)⇒(b) By Lemma 4.1, every singleton is either δ-nowhere dense or δ-preopen. In the first case we are done; in the second case δ-closedness follows from assumption.

(b)⇒(a) Let $\{x\}$ be δ-preopen. Assume that $\{x\}$ is not δ-closed. Then by (b) it is δ-nowhere dense. Thus $\{x\} \subset \text{int}(\delta\text{-cl}(\{x\}) = \emptyset$, which is impossible. □

Theorem 4.3. *For a space (X, τ) the following are equivalent:*

(a) *X is weakly Hausdorff.*

(b) *X is almost weakly Hausdorff and every singleton is δ-nowhere dense or δ-closed.*

(c) *X is almost weakly Hausdorff and every δ-preopen singleton is δ-closed.*

Proof. (a)⇒(b) is obvious.

(b)⇒(a) If a singleton is not δ-closed, then it must be δ-open and hence regular open, since X is almost weakly Hausdorff. Moreover, by the rest of assumption it is δ-nowhere dense. Since a non-empty regular open set cannot be δ-nowhere dense at the same time, then X is weakly Hausdorff.

(b)⇔(c) It follows from Lemma 4.2. □

Theorem 4.4. *For a space (X, τ) the following are equivalent:*

(a) *X is almost weakly Hausdorff.*

(b) *X is $T_{3/4}$ and every non δ-closed singleton is open.*

Proof. (a)⇒(b) is obvious.

(b)$\Rightarrow$(a) If a singleton is not δ-closed, it must be closed, since X is $T_{3/4}$. Moreover, by the rest of the assumption it is open and hence clopen, i.e. it is δ-open. $\square$

5. δgs-continuous and δgs-irresolute functions

Definition 5.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called *δgs-continuous* if $f^{-1}(V)$ is δgs-closed in (X, τ) for every closed set V of (Y, σ).

Definition 5.2. A function $f : (X, \tau) \to (Y, \sigma)$ is called *δgs-irresolute* if $f^{-1}(V)$ is δgs-closed in (X, τ) for every δgs-closed set V of (Y, σ).

Clearly, $f : (X, \tau) \to (Y, \sigma)$ is δgs-continuous (resp. δgs-irresolute) if and only if $f^{-1}(V)$ is δgs-open in (X, τ) for every open (resp. δgs-open) set V of (Y, σ).

Remark 5.1. For a function $f : (X, \tau) \to (Y, \sigma)$, from Definition 1.2 and Remark 2.1, we have the following diagram:

$$\begin{array}{ccccc} & & \text{semi-continuity} & & \\ & \nearrow & & \searrow & \\ \delta\text{-semi-continuity} & & & & gs\text{-continuity} \\ & \searrow & & \nearrow & \\ & & g\delta s\text{-continuity} & \rightarrow & \delta gs\text{-continuity} \end{array}$$

(a) None of these implications is reversible as shown by examples of [31], [29] and of [7] and the following example.

(b) The notions of gs-continuity and δgs-continuity are independent of each other.

(c) The notions of irresoluteness, gs-irresoluteness and δgs-irresoluteness are mutually independent.

Example 5.1. (a) Let $X = \{a, b, c, d\}$, τ be the topology given in Example 2.1 and $\sigma = \{X, \emptyset, \{a, d\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity. Then f is gs-continuous but not δgs-continuous, since $\{b, c\}$ is closed in (X, σ) and $f^{-1}(\{b, c\})$ is not δgs-closed in (X, τ).

(b) Let $X = \{a, b, c, d\}$, τ be the topology given in Example 2.1 and $\sigma = \{X, \emptyset, \{b\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity. Then f is δgs-continuous but neither $g\delta s$-continuous nor gs-continuous, since $\{a, c, d\}$ is closed in (X, σ) and $f^{-1}(\{a, c, d\})$ is neither $g\delta s$-closed nor gs-closed in (X, τ).

(c) Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{X, \emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be the identity. Then f is δgs-continuous (even gs-continuous) but not δgs-irresolute, since $\{b\}$ is δgs-closed in (X, σ) and $f^{-1}(\{b\})$ is not δgs-closed in (X, τ).

Example 5.2. (a) Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a, b\}\}$ and $\sigma = \{X, \emptyset, \{b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity. Then f is δgs-irresolute but neither $g\delta s$-irresolute nor irresolute.

(b) Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity. Then f is irresolute but neither gs-irresolute nor δgs-irresolute.

(c) Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be a function defined by $f(a) = f(c) = f(d) = b$ and $f(b) = a$. Then f is gs-irresolute but neither δgs-irresolute nor irresolute.

Theorem 5.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then:*

(a) *If f is δgs-irresolute and (X, τ) is almost weakly Hausdorff, then f is δ-semi-irresolute.*

(b) *If f is δgs-continuous and (X, τ) is almost weakly Hausdorff, then f is δ-semi-continuous.*

(c) *If (X, τ) is semi-regular, then f is δgs-continuous if and only if f is gs-continuous.*

(d) *If (X, τ) is semi-regular and T_b (resp. T_d), then f is δgs-continuous if and only if f is continuous (resp. g-continuous).*

Proof. (a) Let V be δ-semiclosed in Y. Then V is δgs-closed in Y and since f is δgs-irresolute, then $f^{-1}(V)$ is δgs-closed in X. Since X

is almost weakly Hausdorff, $f^{-1}(V)$ is δ-semiclosed in X. Hence f is δ-semi-irresolute.

(b) Similar to (a).

(c) and (d) follow from Theorem 2.1. □

Theorem 5.2. *If $f : (X, \tau) \to (Y, \sigma)$ is a δ-continuous and δ-semiclosed function, then $f(A)$ is δgs-closed in (Y, σ) for every δgs-closed set A of (X, τ).*

Proof. Let A be δgs-closed in X. Let $f(A) \subset V$, where V be any δ-open in Y. Since f is δ-continuous, $f^{-1}(V)$ is δ-open in X and $A \subset f^{-1}(V)$. Then we have $\delta\text{-scl}(A) \subset f^{-1}(V)$ and so $f(\delta\text{-scl}(A)) \subset V$. Since f is δ-semiclosed, $f(\delta\text{-scl}(A))$ is δ-semiclosed in Y and hence $\delta\text{-scl}(f(A)) \subset \delta\text{-scl}(f(\delta\text{-scl}(A))) \subset V$. This shows that $f(A)$ is δgs-closed in Y. □

The composition of two δgs-continuous functions need not be δgs-continuous. For, consider the following example:

Example 5.3. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$, $\sigma = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\rho = \{X, \phi, \{c, d\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity and $g : (X, \sigma) \to (X, \rho)$ be a function defined by $g(a) = a$, $g(b) = d$, $g(c) = b$ and $g(d) = d$. Then f and g are δgs-continuous but the composition $g \circ f$ is not δgs-continuous since $\{a, b\}$ is closed in (X, ρ) and $(g \circ f)^{-1}(\{a, b\})$ is not δgs-closed in (X, τ).

However, the following theorem holds:

Theorem 5.3. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu)$ be two functions. Then:*

(a) *If f is δgs-continuous and g is continuous, then $g \circ f$ is δgs-continuous.*

(b) *If f is δgs-irresolute and g is δgs-irresolute, then $g \circ f$ is δgs-irresolute.*

(c) *If f is δgs-irresolute and g is δgs-continuous, then $g \circ f$ is δgs-continuous.*

(d) *Let (Y, σ) be an almost weakly Hausdorff space. If f is δ-semi-irresolute and g is δgs-continuous, then $g \circ f$ is δ-semi-continuous.*

Proof. Obvious. □

Theorem 5.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be onto, δgs-irresolute and δ-semiclosed. If (X, τ) is an almost weakly Hausdorff space, then (Y, σ) is almost weakly Hausdorff.*

Proof. Let F be a δgs-closed set of Y. Since f is δgs-irresolute, then $f^{-1}(F)$ is δgs-closed in X. Since X is almost weakly Hausdorff, then $f^{-1}(F)$ is δ-semiclosed in X. By the rest of the assumption it follows that F is δ-semiclosed in Y. Hence Y is almost weakly Hausdorff. □

Theorem 5.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a δ-closed and δ-open bijection. If (X, τ) is an almost weakly Hausdorff space, then (Y, σ) is almost weakly Hausdorff.*

Proof. Let $y \in Y$. Since X is almost weakly Hausdorff and f is bijective, then by Theorem 4.1 for some $x \in X$ with $f(x) = y$, we have $\{x\}$ is δ-closed or δ-open. If $\{x\}$ is δ-closed, then $\{y\} = f(\{x\})$ is δ-closed, since f is δ-closed and injective. If $\{x\}$ is δ-open, then $\{y\}$ is δ-open, since f is δ-open. Hence Y is almost weakly Hausdorff. □

Finally, we define the concept of δgsc-homeomorphisms and prove that the set of all δgsc-homeomorphisms from (X, τ) into itself has a group structure under composition.

Definition 5.3. A function $f : (X, \tau) \to (Y, \sigma)$ is called δgsc-homeomorphism if f is a bijective δgs-irresolute and its inverse function f^{-1} is δgs-irresolute.

For a space (X,τ), we introduce the following notations: $\delta gsch(X,\tau) = \{f \mid f : (X,\tau) \to (X,\tau)$ is a δgsc-homeomorphism$\}$ and $h(X,\tau) = \{f \mid f : (X,\tau) \to (X,\tau)$ is a homeomorphism$\}$.

Theorem 5.6. (a) *The set $\delta gsch(X,\tau)$ is a group which contains $h(X,\tau)$ as a subgroup.*

(b) *If $f : (X,\tau) \to (Y,\sigma)$ is a δgsc-homeomorphism, then f induces an isomorphism from the group $\delta gsch(X,\tau)$ onto $\delta gsch(Y,\sigma)$.*

Proof. (a) A binary operation $\mu : \delta gsch(X,\tau)\times\delta gsch(X,\tau) \to \delta gsch(X,\tau)$ is well defined by $\mu(a,b) = b \circ a$ (the composition) for any $a, b \in \delta gsch(X,\tau)$. Then, it is shown that $\delta gsch(X,\tau)$ is a group with binary operation μ. Every homeomorphism is both δ-continuous and δ-semiclosed. It follows from Theorem 5.2 that every homeomorphism is δgsc-homeomorphism. Therefore, it is shown that $h(X,\tau)$ is a subgroup of $\delta gsch(X,\tau)$.

(b) The isomorphism $f_* : \delta gsch(X,\tau) \to \delta gsch(Y,\sigma)$ is induced from f by $f_*(\mu) = f\circ\mu\circ f^{-1}$ for every $\mu \in \delta gsch(X,\tau)$ by usual argument. □

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