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A NOTE ON PRECONVEXITY SPACES

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Abstract. In this paper, we introduce the concepts of the co-convexity hull and co-convex sets on preconvexity spaces. We study some properties for the co-convexity hull and characterize c-convex functions and c-concave functions by using the co-convexity hull and the convexity hull.

1. Introduction

In [1], Guay introduced the concept of preconvexity spaces defined by a binary relation on the power set $P(X)$ of a set X and investigated some properties. He showed that a preconvexity on a set yields a convexity space in the same manner as a proximity[3] yields a topological space.

In this paper, we define the concept of the co-convexity hull on preconvexity spaces and study some basic properties. And we characterize c-convex functions and c-concave functions by using the co-convexity hull and the convexity hull.

Definition 1.1 ([1]). Let X be a nonempty set. A binary relation σ on $P(X)$ is called a preconvexity on X if the relation satisfies the following properties; we write $x\sigma A$ for $\{x\}\sigma A$:

1. If $A \subseteq B$, then $A\sigma B$.
2. If $A\sigma B$ and $B = \emptyset$, then $A = \emptyset$.

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3. If $A\sigma B$ and $b\sigma C$ for all $b \in B$, then $A\sigma C$.
4. If $A\sigma B$ and $x \in A$, then $x\sigma B$.

The pair (X, σ) is called a preconvexity space. In a preconvexity space (X, σ), $G(A) = \{x \in X : x\sigma A\}$ is called the convexity hull of a subset A. A is called convex[1] if $G(A) = A$.

Theorem 1.2 ([1]). *For a preconvexity space (X, σ),*

1. $G(\emptyset) = \emptyset$.
2. $A \subseteq G(A)$ *for all* $A \subseteq X$.
3. *If* $A \subseteq B$, *then* $G(A) \subseteq G(B)$.
4. $G(G(A)) = G(A)$ *for* $A \subseteq X$.

Theorem 1.3 ([1]). *Let σ be a preconvexity on X and $A, B \subseteq X$. Then*

1. $A\sigma B$ *iff* $A \subseteq G(B)$.
2. $A\sigma B$ *iff* $G(A)\sigma G(B)$.

2. Main Results

Definition 2.1. Let (X, σ) be a preconvexity space and $A \subseteq X$. $I_\sigma(A) = \{x \in A : x \not\sigma (X - A)\}$ (simply, $I(A)$) is called the co-convexity hull of a subset A.

Theorem 2.2. *Let (X, σ) be a preconvexity space and $A \subseteq X$. Then*

1. $I(A) = X - G(X - A)$.
2. $G(A) = X - I(X - A)$.

Proof. (1) Let $x \in I(A)$; then by Definition 2.1,
$x \in I(A)$ iff $x \not\sigma (X - A)$ iff $x \notin G(X - A)$ iff $x \in X - G(X - A)$. Thus we get the result.

(2) By (1), it is obvious. □

From Definition 2.1 and Theorem 2.2. we get the following theorem.

Theorem 2.3. *Let (X, σ) be a preconvexity space and $A \subseteq X$.*

1. *$A\sigma B$ iff $I(X - B) \subset X - A$.*
2. *$A\sigma B$ iff $I(X - B)\sigma I(X - A)$.*

Theorem 2.4. *Let (X, σ) be a preconvexity space and $A, B \subseteq X$.*

1. *$I(X) = X$.*
2. *$I(A) \subseteq A$.*
3. *If $A \subseteq B$, then $I(A) \subseteq I(B)$.*
4. *$I(I(A)) = I(A)$.*

Proof. (1)It follows that $I(X) = X - G(X - X) = X - G(\emptyset) = X$ by Theorem 2.2 (1) and Theorem 1.2 (1).

(2) It is obvious.

(3) Let $x \in I(A)$; then $x \not\sigma(X - A)$. Since $X - B \subset X - A$, it is $x \not\sigma(X - B)$. Thus $x \in I(B)$.

(4) By Theorem 2.2 (1) and Theorem 1.2 (4), $I(I(A)) = I(X - G(X - A)) = X - G(G(X - A)) = X - G(X - A) = I(A)$. □

Definition 2.5. Let (X, σ) be a preconvexity space and $A \subseteq X$. A is called a co-convex set if $I(A) = A$.

Theorem 2.6. *Let (X, σ) be a preconvexity space and $A \subseteq X$. Then A is a co-convex set iff A^c is a convex set.*

Proof. By Theorem 2.2, it is $X - I(A) = G(X - A)$. Thus we get the result because $G(X - A)$ is a convex set iff $I(A) = A$. □

Theorem 2.7. *Let (X, σ) be a preconvexity space.*

1. *The union of co-convex sets is a co-convex set.*
2. *X, $\emptyset$ are co-convex sets.*

Proof. (1) Let $\mathbf{A}$ be an arbitrary family of co-convex sets and let $x \in \cup\mathbf{A}$; then there exists a subset $A \in \mathbf{A}$ such that $x \in A = I(A)$. Since $A \subset \cup\mathbf{A}$, it is that $I(A) \subset I(\cup\mathbf{A})$. Thus we get $\cup\mathbf{A} \subset I(\cup\mathbf{A})$.

(2) By Theorem 2.4, it is obvious. □

Let (X, σ) be a preconvexity space. Let $\mathcal{I}(X) = \{A \subset X : I(A) = A\}$ and $\mathcal{G}(X) = \{A \subset X : G(A) = A\}$.

Theorem 2.8. *Let (X, σ) be a preconvexity space and $A \subseteq X$. Then $I(A) = \cup\{U : U \subset A, U \in \mathcal{I}(X)\}$.*

Proof. Since $I(A) \in \mathcal{I}(X)$ and $I(A) \subset A$, $I(A) \subset \cup\{U : U \subset A, U \in \mathcal{I}(X)\}$. Conversely, since $\cup\{U : U \subset A, U \in \mathcal{I}(X)\} \subset A$, from Theorem 2.7(1) and Theorem 2.4(3) it follows $\cup\{U : U \subset A, U \in \mathcal{I}(X)\} \subset I(A)$. □

Let (X, σ_1) and (Y, σ_2) be two preconvexity spaces. A function $f : X \to Y$ is said to be c-convex [1] if $A\sigma_1 B$ implies $f(A)\sigma_2 f(B)$.

Theorem 2.9. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then the following things are equivalent:*

1. *f is c-convex*
2. *$f(G_\sigma(A)) \subset G_\mu(f(A))$ for all $A \subset X$.*
3. *$G_\sigma(f^{-1}(B)) \subset f^{-1}(G_\mu(B))$ for all $B \subset Y$.*
4. *$f^{-1}(I_\mu(B)) \subset I_\sigma(f^{-1}(B))$ for all $B \subset Y$.*
5. *For each $U \in \mathcal{I}(Y)$, $f^{-1}(U) \in \mathcal{I}(X)$.*
6. *For each $C \in \mathcal{G}(Y)$, $f^{-1}(U) \in \mathcal{G}(X)$.*

Proof. (1) ⇒ (2) For each $A \subset X$, since $G(A)\sigma A$ and f is c-convex, $f(G(A))\mu f(A)$. Thus by Theorem 1.3(1), we get $f(G_\sigma(A)) \subset G_\mu(f(A))$.

(2) ⇒ (1) Let $A\sigma B$ for $A, B \subset X$; then $A \subset G_\sigma(A) \subset G_\sigma(B)$. From (2), it follows $f(A) \subset f(G_\sigma(A)) \subset f(G_\sigma(B)) \subset G_\mu(f(B))$. Thus $f(A) \subset G_\mu(f(B))$, so $f(A)\mu f(B)$ by Theorem 1.3(1).

(2) ⇒ (3) For $B \subset Y$, from (2) it is $f(G_\sigma(f^{-1}(B))) \subset G_\mu(f(f^{-1}(B))) \subset G_\mu(B)$. Thus we get $G_\sigma(f^{-1}(B)) \subset f^{-1}(G_\mu(B))$.

(3) $\Rightarrow$ (4) For $B \subset Y$, it is $G_\sigma(f^{-1}(Y - B)) \subset f^{-1}(G_\mu(Y - B))$ by (3). Then by Theorem 2.2, $G_\sigma(X - f^{-1}(B)) \subset f^{-1}(Y - I_\mu(B)) = X - f^{-1}(I_\mu(B))$. Thus $f^{-1}(I_\mu(B)) \subset X - G_\sigma(X - f^{-1}(B)) = I_\sigma(f^{-1}(B))$.

(4) $\Rightarrow$ (5) Let $U \in \mathcal{I}(X)$; by (4) $f^{-1}(U) = f^{-1}(I_\mu(U)) \subset I_\sigma(f^{-1}(U))$. Thus it is $f^{-1}(U) = I_\sigma(f^{-1}(U))$ by Theorem 2.4(2).

(5) $\Rightarrow$ (6) It is obvious.

(6) $\Rightarrow$ (2) For $A \subset X$, it is $G_\mu(f(A)) \in \mathcal{G}(Y)$ by Theorem 1.2(4). Then by (6) and Theorem 1.2(3), $G_\sigma(A) \subset f^{-1}(G_\mu(f(A)))$. Thus we get $f(G_\sigma(A)) \subset G_\mu(f(A))$. □

We recall that the notion of c-concave function: Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be c-concave[2] if for $C, D \subseteq Y$ whenever $C\mu D$, $f^{-1}(C)\sigma f^{-1}(D)$.

Theorem 2.10. *Let $f : X \to Y$ be a function on two preconvexities (X, σ) and (Y, μ). Then the following things are equivalent:*

1. *f is c-concave*
2. *$f^{-1}(G_\mu(A)) \subset G_\sigma(f^{-1}(A))$ for all $A \subset Y$.*
3. *$I_\sigma(f^{-1}(A)) \subset f^{-1}(I_\mu(A))$ for all $A \subset Y$.*

Proof. (1) $\Rightarrow$ (2) Let f be c-concave and $A \subset Y$.
Since $G_\mu(A)\mu A$ and f is c-concave, $f^{-1}(G_\mu(A))\sigma f^{-1}(A)l$. Thus $f^{-1}(G_\mu(A)) \subset G_\sigma(f^{-1}(A))$ by Theorem 1.3(1).

(2) $\Rightarrow$ (1) If $C\mu D$ for $C, D \subset Y$, then $C \subset G_\mu(D)$. By (2) $f^{-1}(C) \subset f^{-1}(G_\mu(D)) \subset G_\sigma(f^{-1}(D))$. From Theorem 1.3(1), it follows that $f^{-1}(C)\sigma f^{-1}(D)$.

(2) $\Rightarrow$ (3) For $A \subset Y$, we get $f^{-1}(G_\mu(Y - A)) \subset G_\sigma(f^{-1}(Y - A))$ by (2). From Theorem 2.2, it follows $I_\sigma(f^{-1}(A)) \subset f^{-1}(I_\mu(A))$.

Similarly, we get (2) from Theorem 2.2 and (3). □

Definition 2.11. Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then

1. f is I-convex iff for each $U \in \mathcal{I}(X)$, $f(U) \in \mathcal{I}(y)$.
2. f is G-convex iff for each $C \in \mathcal{G}(X)$, $f(U) \in \mathcal{G}(y)$.

Theorem 2.12. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then f is G-convex iff $G_\mu(f(A)) \subset f(G_\sigma(A))$ for all $A \subset X$.*

Proof. ($\Rightarrow$) Let f be G-convex; then for $A \subset X$ $f(A) \subset f(G_\sigma(A))$. Since f is G-convex and $G_\sigma(A) \in \mathcal{G}(X)$, $G_\mu(f(A)) \subset f(G_\sigma(A))$.

($\Leftarrow$)It is obvious. □

Lemma 2.13. *Let (X, σ) be a preconvexity space and $A, B \subset X$. Then $A\sigma B$ and $B\sigma A$ iff $G(A) = G(B)$.*

Proof. From Theorem 1.3, it follows that $A\sigma B$ iff $G(A) \subset G(B)$. Thus $A\sigma B$ and $B\sigma A$ iff $G(A) = G(B)$. □

Definition 2.14. Let (X, σ) be a preconvexity space. X is called a strong preconvexity space if for $A, B \subset X$ whenever $G(A) = G(B)$, $A = B$.

Theorem 2.15. *Let $f : X \to Y$ be c-convex on two preconvexity spaces (X, σ) and (Y, μ). Let Y be a strong preconvexity space. Then f is G-convex map iff $G_\mu(f(A))\mu f(G_\sigma(A))$ for all $A \subset X$.*

Proof. Assume that $G_\mu(f(A))\mu f(G_\sigma(A))$ for all $A \subset X$. Since f is a c-convex function, $f(G_\sigma(A))\mu G_\mu(f(A))$ for all $A \subset X$ by Theorem 2.9(2). And by Lemma 2.13, we get $G_\mu(f(A)) = f(G_\sigma(A))$. Thus f is G-convex.

The converse is obvious by Theorem 2.12. □

Theorem 2.16. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then*

f is I-convex iff $f(I_\sigma(A)) \subset I_\mu(f(A))$ for all $A \subset X$.

Proof. See Theorem 2.12 □

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