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ON STAR MOMENT SEQUENCE OF OPERATORS

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Abstract. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space. We call "an operator T acting on $\mathcal{H}$ has a star moment sequence supported on a set K" when there exist nonzero vectors u and v in $\mathcal{H}$ and a positive Borel measure μ such that $\langle T^{*j}T^k u, v\rangle = \int_K \bar{z}^j z^k d\mu$ for all $j, k \geq 0$. We obtain a characterization to find a representing star moment measure and discuss some related properties.

1. Introduction and Preliminaries

Let $\mathcal{X}$ be a (real or complex) Banach space, and denoted by $\mathcal{L}(\mathcal{X})$ be the algebra of all bounded linear operators on $\mathcal{X}$. Let $\mathcal{X}^*$ be a dual space of $\mathcal{X}$. Following [2], we say that a T in $\mathcal{L}(\mathcal{X})$ has a *moment sequence* if there exist nonzero vectors $x \in \mathcal{X}$ and $y \in \mathcal{X}^*$ and a positive Borel measure supported on the spectrum $\sigma(T)$ of T (and, of course, $\sigma(T) \subset \mathbb{R}$ if $\mathcal{X}$ is a real Banach space) such that

$$y(T^n x) = \int_{\sigma(T)} \lambda^n d\mu_{x,y}, \ n \in \mathbb{N}_0, \tag{1.1}$$

where $\mathbb{N}_0$ denotes, in usual, the set of nonnegative integers.

Atzmon and Godefroy then showed in [2] that if $\mathcal{X}$ is real and satisfies some additional conditions, that every operator in $\mathcal{L}(\mathcal{X})$ that has a moment sequence (as in (1.1)), has, in fact, a nontrivial invariant subspace. This immediately raises the following question.

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Question 1.1. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space. Which classes of operators have moment sequences?

Of course, at present, it is not known, in this context, whether having a moment sequence implies the existence of invariant subspace. This led the authors of [4] and [5] to undertake a study of this equation in the case that $\mathcal{X} = \mathcal{H}$, and they showed that various classes of operators in $\mathcal{L}(\mathcal{H})$ do have moment sequences.

In this note we consider a related question and define a new definition of moment sequence.

Definition 1.2. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space. An operator $T \in \mathcal{L}(\mathcal{H})$ has a **-moment sequence* supported on $K \subset \mathbb{C}$ if there exist nonzero vectors u and v in $\mathcal{H}$ and a positive Borel measure μ such that

$$\left\langle T^{*j} T^k u, v \right\rangle = \int_K \bar{z}^j z^k d\mu, \qquad j, k \in \mathbb{N}_0. \tag{1.2}$$

(Throughout this note K will be $\sigma(T)$ in almost cases.)

In this note, we will establish some necessary and sufficient conditions for which an operator T in $\mathcal{L}(\mathcal{H})$ has a *-moment sequence, and in condition which equivalent to *-moment sequence.

2. Some Results

Let N be a normal operator in $\mathcal{L}(\mathcal{H})$. Then obviously N has a *-moment sequence. Moreover, this fact can be improved to the case of subnormal operator as following.

Proposition 2.1. *Every subnormal operator S in $\mathcal{L}(\mathcal{H})$ have a *-moment sequence on $\sigma(S)$.*

Proof. Suppose S is a subnormal operator. Let N be the minimal normal extension of S and let E be the corresponding spectral measure to N. For $x, y \in \mathcal{H}$, define

$$\mu_{x,y}(\Delta) = \langle E(\Delta)x, y\rangle, \qquad \Delta \in \mathcal{B}(\sigma(N)),$$

where $\mathcal{B}(\sigma(N))$ is the ring of Borel subsets of $\sigma(N)$. Then $\mu_{x,y}$ is a complex-valued measure and $E(\Delta)$ is obviously a projection. In particular, if we take $x = y$, then

$$\begin{aligned}\mu_{x,x}(\Delta) &= \langle E(\Delta)x, x\rangle = \left\langle E(\Delta)^2 x, x\right\rangle \\ &= \langle E(\Delta)x, E(\Delta)x\rangle \\ &= \|E(\Delta)x\| \geq 0.\end{aligned}$$

Thus $\mu_{x,x}$ is a positive Borel measure on $\sigma(N)$ for all $x \in \mathcal{H}$. By the spectral theorem and spectral inclusion theorem, we have

$$\begin{aligned}\left\langle S^{*j}S^k x, x\right\rangle &= \left\langle N^{*j}N^k x, x\right\rangle \\ &= \int_{\sigma(N)} \bar{z}^j z^k d\mu_{x,x} = \int_{\sigma(S)} \bar{z}^j z^k d\mu_{x,x}, \; j,k \in \mathbb{N}_0.\end{aligned}$$

Hence S has a *-moment sequence. □

We now characterize operators having a *-moment sequence, which is the main theorem of this note.

Theorem 2.2. *Let $T \in \mathcal{L}(\mathcal{H})$. Then T has a *-moment sequence supported on $\sigma(T)$ if and only if there exist nonzero vectors u and v in $\mathcal{H}$ such that $\langle u, v\rangle \geq 0$ and*

$$|\langle p(T^*, T)u, v\rangle| \leq \langle u, v\rangle \, \|p\|_\infty, \qquad p \in \mathbb{C}[\bar{z}, z],$$

where $\|p\|_\infty$ (:=$\sup_{z,\bar{z}\in\sigma(T)}|p(z,\bar{z})|$ for $p \in C[\bar{z}, z]$) is the sup norm supported on $\sigma(T)$ and

$$|\langle p(T^*, T)u, v\rangle| = \sum a_{jk} T^{*j} T^k \text{ for } p(\bar{z}, z) = \sum a_{jk} \bar{z}^j z^k, \; j,k \in \mathbb{N}_0.$$

(Note that $p(T^*, T) \equiv \sum a_{jk} T^{*j} T^k$ which is not equal to $\sum a_{jk} T^k T^{*j}$ in general.)

Proof. Suppose $T \in \mathcal{L}(\mathcal{H})$ has a *-moment sequence supported on $\sigma(T)$. Then there exist nonzero vectors u and v and a positive Borel measure μ such that

$$\left\langle T^{*j}T^k u, v\right\rangle = \int_{\sigma(T)} \bar{z}^j z^k d\mu, \quad j,k \in \mathbb{N}_0.$$

For $j = k = 0$, we have

$$\langle u, v\rangle = \int_{\sigma(T)} 1 d\mu = \mu(\sigma(T)).$$

Then

$$\begin{aligned} |\langle p(T^*,T)u,v\rangle| &= \left|\int_{\sigma(T)} p(\bar{z},z)d\mu\right| \leq \int_{\sigma(T)} |p(\bar{z},z)|\, d\mu \\ &\leq \|p\|_\infty \left|\int_{\sigma(T)} 1\ d\mu\right| = \|p\|_\infty \mu(\sigma(T)) = \|p\|_\infty \langle u,v\rangle . \end{aligned}$$

Conversely, without loss of generality we may assume that there exist nonzero vectors u, v such that $\|u\| = \|v\| = 1$, $\langle u, v\rangle \geq 0$ and

$$|\langle p(T^*,T)u,v\rangle| \leq \langle u,v\rangle \|p\|_\infty, \ p \in C[\bar{z},z].$$

Define $\tau : \mathbb{C}[\bar{z},z] \to \mathbb{C}$ by

$$\tau(p(\bar{z},z)) = \langle p(T^*,T)u,v\rangle .$$

Then τ is obviously linear and $\tau(1) = \langle u,v\rangle \geq 0$. So τ is positive. By the Hahn Banach theorem, there exists a continuous linear mapping $\tau_{ext} : C(\sigma(T)) \to \mathbb{C}$ such that $\tau_{ext}|_{\mathbb{C}[\bar{z},z]} = \tau$ and $\|\tau_{ext}\| = \|\tau\| = \tau(1)$, which implies that τ_{ext} is positive. By Riesz representation theorem, there exists a positive Borel measure μ supported on $\sigma(T)$ such that

$$\tau_{ext}(p(\bar{z},z)) = \int_{\sigma(T)} p(\bar{z},z)d\mu.$$

Take $p(\bar{z},z) = \bar{z}^i z^j,\ i,j \in \mathbb{N}_0$ and since

$$\langle p(T^*,T)u,v\rangle = \tau(p(\bar{z},z)) = \tau_{ext}(p(\bar{z},z)),$$

we have that

$$\left\langle T^{*j}T^k u, v\right\rangle = \int_{\sigma(T)} \bar{z}^j z^k d\mu, \ \ j,k \in \mathbb{N}_0.$$

Hence T has a *-moment sequence supported on $\sigma(T)$. □

Corollary 2.3. *If an operator T has a *-moment sequence on $\sigma(T) \subset \mathbb{C}$, then also T^* has a *-moment sequence supported on $\overline{\sigma(T)} = \sigma(T^*)$.*

Proof. By Theorem 2.2, there exist nonzero vectors u and v in $\mathcal{H}$ such that

$$|\langle p(T^*, T)u, v\rangle| \le \langle u, v\rangle \|p\|_{\sigma(T)}, \quad p \in \mathbb{C}_{\sigma(T)}[\bar{z}, z].$$

Say $p(\bar{z}, z) = \sum_{j,k} a_{jk}\bar{z}^j z^k$, $j, k \in \mathbb{N}_0$. Then

$$\begin{aligned} |\langle p((T^*)^*, T^*)u, v\rangle| &= \left|\left\langle \sum_{j,k} a_{jk} T^j T^{*k} u, v \right\rangle\right| \\ &= \left|\left\langle \sum_{j,k} \overline{a_{jk}} T^k T^{*j} v, u \right\rangle\right| \le \langle u, v\rangle \|\overline{p}\|_{\sigma(T)} \\ &= \langle u, v\rangle \|p\|_{\overline{\sigma(T)}}, \quad \overline{\sigma(T)} = \{\bar{z} \in \mathbb{C} : z \in \sigma(T)\}. \end{aligned}$$

By Theorem 2.2, T^* has a *-moment sequence supported on $\overline{\sigma(T)} = \sigma(T^*)$. □

The following corollary follows immediately from Theorem 2.2 and Corollary 2.3.

Corollary 2.4. *If an operator T has a *-moment sequence supported on $\sigma(T) \subset \mathbb{C}$ and $p(\bar{z}, z)$ is a polynomial, then $p(T)$ has a *-moment sequence supported on $p(\sigma(T)) \cup \overline{p(\sigma(T))}$.*

Remark 2.5. If an operator T has two nonzero invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{M} \perp \mathcal{N}$, then T has a *-moment sequence supported on $\sigma(T) \subset \mathbb{C}$. (*Indeed,* take $u \in \mathcal{M}$ and $v \in \mathcal{N}$. Then $T^k u \in \mathcal{M}$ and $T^j v \in \mathcal{N}$. Then the zero measure μ satisfies

$$\left\langle T^{*j} T^k u, v \right\rangle = \left\langle T^k u, T^j v \right\rangle = 0 = \int_{\sigma(T)} \bar{z}^j z^k d\mu, \ j, k \in \mathbb{N}_0$$

for any set $\sigma(T)$. So we have this remark.)

The following Proposition 2.6 improves Proposition 1.6 in [4].

Proposition 2.6. *Let $T \in \mathcal{L}(\mathcal{H})$. Suppose that $\{u_n\}$ and $\{v_n\}$ are sequences in $\mathcal{H}$ converging in norm to nonzero vectors u_0 and v_0, respectively, and for every $n \in \mathbb{N}_0$ there exists a Borel measure μ_n supported on the compact set $\sigma(T) \subset \mathbb{C}$ such that*

$$\left\langle T^{*j}T^k u_n, v_n \right\rangle = \int_{\sigma(T)} \bar{z}^j z^k d\mu_n, \ n \in \mathbb{N}_0, \ j,k \in \mathbb{N}_0.$$

Then there exists a Borel measure μ_0 supported on compact set $\sigma(T) \subset \mathbb{C}$ such that

$$\left\langle T^{*j}T^k u_0, v_0 \right\rangle = \int_{\sigma(T)} \bar{z}^j z^k d\mu_{0,} \ \ n \in \mathbb{N}_0, \ j,k \in \mathbb{N}_0.$$

Proof. Let $C(\sigma(T))$ be a set of all continuous functions from $\sigma(T)$ into $\mathbb{R}$. Then $C(\sigma(T))$ is a Banach space with sup-norm and μ_n is contained in the dual space of $C(\sigma(T))$. So

$$\mu_n(K) = \int_{\sigma(T)} d\mu_n = \langle u_n, v_n \rangle \leq \ \|u_n\| \, \|v_n\| \,, \quad n \in \mathbb{N}_0.$$

Since $\{u_n\}$ and $\{v_n\}$ converge, $\|u_n\|$ and $\|v_n\|$ are bounded. Thus $\mu_n(K)$ is a bounded sequence in $C(\sigma(T))^*$. In weak*-topological space, there exists a subsequence μ_{n_k} of bounded sequence μ_n such that μ_{n_k} weak*-converges to μ_0. Since μ_{n_k} is positive, μ_0 is positive. Also since μ_{n_k} weak*-converges to μ, by definition of weak*-topology,

$$\int_{\sigma(T)} \bar{z}^j z^k d\mu_{n_k} \longrightarrow \int_{\sigma(T)} \bar{z}^j z^k d\mu_0, \ j,k \in \mathbb{N}_0.$$

Since $\left\langle T^{*j}T^k u_{n_k}, v_{n_k} \right\rangle$ converges to $\left\langle T^{*j}T^k u_0, v_0 \right\rangle$, we have

$$\left\langle T^{*j}T^k u_0, v_0 \right\rangle = \int_{\sigma(T)} \bar{z}^j z^k d\mu_0.$$

Hence the proof is complete. □

3. Remarks and Problem

We close this note an open problem and related some remarks. Recall from [4] that if an operator of the form $T = N + K$, where N is a normal operator and K is a compact operator, then T has a moment sequence in version of [4]. But we do not know the following.

Problem 3.1. Let $T = N + K$, where N is a normal operator and K is a compact operator. Does T have a *-moment sequence supported on $\sigma(T)$?

Let $\mathbf{K} := \mathbf{K}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$. Let $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbf{K}$ be the Calkin map. Problem 3.1 is so interesting because of the following remark.

Remark 3.2. Assume that Problem 3.1 is true. If an operator T is essentially normal (i.e., $\pi(T)$ is normal), then T has a *-moment sequence supported on $\sigma(T) \subset \mathbb{C}$ or T has a nontrivial invariant subspace. (*Indeed*, by BDF-theorem ([3]), if T is biquasitriangular, T has a *-moment sequence supported on $\sigma(T) \subset \mathbb{C}$. If T is not biquasitriangular, by AFV-theorem ([1]), T has a nontrivial invariant subspace.)

Remark 3.3. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *almost hyponomal* if $T^*T - TT^*$ can be written as $P + K$, where $P \geq 0$ and $K \in \mathcal{C}_1(\mathcal{H})$, the ideal of trace-class operators in $\mathcal{L}(\mathcal{H})$ (cf. [6]). It follows from [6] that if $T \in \mathcal{L}(\mathcal{H})$ is almost hyponormal, $X \in \mathcal{C}_2(\mathcal{H})$, where $\mathcal{C}_2(\mathcal{H})$ is the Hilbert-Schmidt class, and $T^*T - TT^* \notin \mathcal{C}_1(\mathcal{H})$, then $T + X$ has a nontrivial invariant subspace. Thus, if every operator in $\mathcal{L}(\mathcal{H})$ of the form $T + X$, where T is almost hyponormal and $X \in \mathcal{C}_2(\mathcal{H})$ admits a *-moment sequence or has a nontrivial invariant subspace. (*Indeed*, if $T^*T - TT^* \notin \mathcal{C}_1(\mathcal{H})$, $T + X$ has a nontrivial invariant subspace. If $T^*T - TT^* \in \mathcal{C}_1(\mathcal{H})$, $T + X$is essentially normal. By Remark 3.2, $T + X$ has a *-moment sequence or $T + X$ has a nontrivial invariant subspace.)

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References

[1] C. Apostol, C. Foias and D. Voiculescu, *Some results on non-quasitriangular operators*, IV, Revue Roum. de Math. Pure. Appl. **18** (1973), 487-514.

[2] A. Atzmon and G. Godefroy, *An application of the smooth variational principle to the existence of nontrivial invariant subspaces*, Comp. R. I'Acad. Sci. Paris, Serie I, Math. **332**(2001), 151-156.

[3] L. Brown, R. G. Douglas and P. Fillmore, *Extensions of C^*-algebras and K-homology*, Ann. Math. **105** (1977), 265-324.

[4] C. Foias, I. Jung, E. Ko and C. Pearcy, *Operators that admit a moment sequence*, Israel J. Math. **145** (2005), 83-91.

[5] B. Chevreau, I. Jung, E. Ko and C. Pearcy, *Operators that admit a moment sequence,II,* Proc. the Amer. Math. Soc., **135** (2007), 1763-1767.

[6] D. Voiculescu, *A note on quasitriangularity and trace-class self-commutators*, Acta Sci. Math. (Sz.) **42** (1980), 1303-1320.

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