Honam Mathematical J. **29** (2007), No. 4, pp. 495–509

RIGIDITY FOR *MPR*, THE MALVENUTO-POIRIER-REUTENAUER HOPF ALGEBRA OF PERMUTATIONS

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Abstract. In this note it is proved that *MPR* is rigid as a Hopf algebra with distinguished basis. I.e. there are no nontrivial automorphisms that preserve the multiplication and comultiplication and take the distinguished basis of all permutations into itself (as a graded set).

1. Introduction and statement of the theorem

Let S_n be the symmetric group on n letters. Permutations will be written as words, the permutation which takes i into $\sigma(i)$, $i = 1, 2, \dots, n$ being written as $(\sigma(1), \dots, \sigma(n))$. These words on $\mathbf{N} = \{1, 2, \cdots\}$ are called permutation words.

The *MPR* Hopf algebra, also known as the Hopf algebra of permutations, has as underlying Abelian group the direct sum

$$\oplus_{i=0}^{\infty} \mathbf{Z} S_n \tag{1.1}$$

Here $\mathbf{Z}S_0 = \mathbf{Z}$. The element $1 \in \mathbf{Z}S_0 = \mathbf{Z}$ together with all permutations on any number of letters serve as a distinguished basis. The element 1 is viewed as the empty (permutation) word. This Abelian group is seen as a graded one as in (1.1), i.e. the grading is by the length of permutation words; degree = length. To define the Hopf algebra structure on this graded Abelian group the Schenstedt, (see [9]),

Received June 27, 2006. Accepted August 5, 2006.
2000 Mathematics Subject Classification: 16W30, 05Exx.
Key words and phrases: *MPR* Hopf algebra, rigidity, Hopf algebra of permutations.

notion of standardization is used. Given a sequence $\alpha = (a_1, \cdots, a_n)$ of distinct positve integers its standardization is the unique permutation word $\mathrm{st}(\alpha) = (b_1, \cdots, b_n)$, $b_i \in \{1, \cdots, n\}$ with $b_i < b_j$ iff$a_i < a_j$. So, for instance, $\mathrm{st}(7,8,1,4,9) = (3,4,1,2,5)$.

On the given distinguished basis the comultiplication and multiplication are defined as follows

$$(1.2)\quad \mu(a_1, \cdots, a_n) = \sum_{i=0}^{n} (a_1, \cdots, a_n)_{\{1,\cdots,i\}} \otimes \mathrm{st}((a_1, \cdots, a_n)_{\{i+1,\cdots,n\}})$$

Here for a subset S of $\{1, \cdots, n\}$, $(a_1, \cdots, a_n)_S$ is the word obtained by removing all letters that are not in S. So, for instance,

$$\mu(4,1,2,3) = 1 \otimes (4,1,2,3) + (1) \otimes (3,1,2) + (1,2) \otimes (2,1) + (1,2,3) \otimes (1) + (4,1,2,3) \otimes 1$$

The multiplication of two permutation words α and β of lengths r and s respectively, is defined by

$$(1.3)\qquad \alpha\beta = m(\alpha \otimes \beta) = \sum u * v$$

where the sum is over all concatenations $u * v$ of two words u and v such that $u * v$ is a permutation word of length $r + s$ and $\mathrm{st}(u) = \alpha$, $\mathrm{st}(v) = \beta$. So, for instance,

$$(1,2)(2,1) = (1,2,4,3) + (1,3,4,2) + (1,4,3,2) + (2,3,4,1) + (2,4,3,1) + (3,4,2,1)$$

It is a theorem, see [7], that this multiplication and comultiplication define a bialgebra structure on (1.1), that is, moreover, obviously, graded and connected, and, hence, because the underlying Abelian group is connected graded, a Hopf algebra. Meaning that the existence of an antipode follows. A much easier proof that this multiplication and comutiplication form a bialgebra is in [2], where this is established for much more general word Hopf algebras.

These are the multiplication and comultiplication denoted $*$ and Δ in [7] and $*'$ and δ' in [8]. They are also the multiplication and comultiplication described in [6] in a different way.

Let $\mathcal{B}$ denote the distinguished basis of *MPR* consisting of 1 and all permutations on n letters, $n = 1, 2, \cdots$.

Theorem 1.4. *The only graded Hopf algebra automorphism α of MPR that preserves $\mathcal{B}$, i.e. such that $\alpha(\mathcal{B}) = \mathcal{B}$, is the identity.*

For the Hopf algebra *Symm* of symmetric functions there is a similar result. There are just two automorphisms that preserve the distinguished basis of Schur functions, the identity and the one that interchanges the elementary and complete symmetric functions. This follows from a theorem of Liulevecius [4]. I am convinced that there are similar results for such related Hopf algebras with distinguished bases as *NSymm*, the Hopf algebra of noncommutative symmetric functions, its dual, *QSymm*, the Hopf algebra of quasi-symmetric functions, and the Loday-Ronco Hopf algebra of binary planar trees [5], that plays a crucial role in renormalization [3]. These remain to be established.

2. Proof of the theorem

Let α be a graded automorphism of *MPR* that preserves the distinguished basis. This means of course that $\alpha(1) = 1$ and $\alpha((1)) = (1)$ because there is only one basis element, viz (1), of order 1. At order 2 there are, a priori, two possibilities, viz:

Case A) $\alpha((1,2)) = (1,2)$, $\alpha((2,1)) = (2,1)$

Case B) $\alpha((1,2)) = (2,1)$, $\alpha((2,1)) = (1,2)$

Case A) is taken care of by the following proposition.

Proposition 2.1. *Let α be a Hopf algebra automorphism of MPR, preserving the distinguished basis, that is the identity in degrees $\leq n$, $n \geq 2$. Then α is also the identity in degree $n+1$.*

Proof of the proposition. For the proof some lemmas are used that are perhaps of more general interest because similar things can be proved for any Hopf algebra that is like *MPR* in that it satisfies a number of axioms (that are clearly true for *MPR*). At least as regards lemma 2.2. □

Lemma 2.2. *Let σ be a permutation word of degree r and let τ be a permutation word of degree s. Then:*

(i) The product $\sigma\tau$ is the sum of precisely $\binom{r+s}{r}$ pairwise different basis words of length $r+s$.

(ii) If σ' and τ' are two (possibly) other words of lengths r and s respectively and the products $\sigma\tau$ and $\sigma'\tau'$ have a term in common, then $\sigma = \sigma'$, $\tau = \tau'$.

Proof. If the concatenation $u * v$, u of length r and v of length s occurs in both $\sigma\tau$ and $\sigma'\tau'$ then by the definition of the product, see (1.3), $\sigma = \mathrm{st}(u)$, $\sigma' = \mathrm{st}(u)$, $\tau = \mathrm{st}(v)$, $\tau' = \mathrm{st}(v)$. This proves (ii).

By definition the product $\sigma\tau$ is the sum of those permutation words of length $r+s$ that are concatenations $u * v$ of a word u of length r and a word v of length s such that $\mathrm{st}(u) = \sigma$, $\mathrm{st}(v) = \tau$. Such words are obtained as follows. From the set of natural numbers $\{1, 2, \cdots, (r+s)\}$ select a subset U of size r and let $V = \{1, 2, \cdots, (r+s)\}\backslash U$. There are precisely $\binom{r+s}{r}$ different ways of doing this. For each such pair there is precisely one way to order the members of U to a word u such that $\mathrm{st}(u) = \sigma$ and also precisely one way to order the elements of V to a word v such that $\mathrm{st}(v) = \tau$. This proves (i). □

For each permutation word σ, let $\nu(\sigma)$ be the nontrivial part of $\mu(\sigma)$, i.e.

$$\nu(\sigma) = \mu(\sigma) - 1 \otimes \sigma - \sigma \otimes 1 \tag{2.3}$$

Lemma 2.4. *Let σ and τ be two permutation words of lengths ≥ 2, and suppose that $\nu(\sigma) = \nu(\tau)$. Then they have of course equal length, say n, and* *(i) If σ (or τ) does not have the numbers 1, n as neighbors, $\sigma = \tau$*

(ii) If σ (or τ) does have 1, n as neighbors and $\sigma \neq \tau$ then τ is obtained from σ by switching 1 and n.

(And, inversely, if 1, n are neighbors in σ and τ is obtained from σ by switching 1 and n, then $\nu(\sigma) = \nu(\tau)$.)

Proof. Look at the term in $\nu(\sigma)$ of bidegree $(n-1,1)$. This one is necessarily of the form

$$(a_1, a_2, \cdots, a_{n-1}) \otimes (1) \tag{2.5}$$

where the left hand side of this tensor product is a permutation word of length $n-1$. It follows, by definition of the coproduct, see (1.2), that both σ and τ are obtained from $(a_1, a_2, \cdots, a_{n-1})$ by inserting an n somewhere. Now look at the term of bidegree $(1, n-1)$ which is necessarily of the form

$$(1) \otimes (b_1, b_2, \cdots, b_{n-1}) \tag{2.6}$$

Note that $(a_1, a_2, \cdots, a_{n-1})$ is the subpermutation word of $\sigma = (c_1, c_2, \cdots, c_n)$ obtained by removing n and $(b_1+1, b_2+1, \cdots, b_{n-1}+1)$ is the subword of $\sigma = (c_1, c_2, \cdots, c_n)$ obtained by removing 1.

Now look where n occurs in $(b_1+1, b_2+1, \cdots, b_{n-1}+1)$. If n and 1 are not neighbors this uniqely determines where n should be inserted in $(a_1, a_2, \cdots, a_{n-1})$ to obtain a permutation word which has (2.6) as the bidegree $(1, n-1)$ term if the comultiplication μ is applied to it.

If 1 and n are neighbors there is uncertainty where n should be inserted, namely just before or just after 1 in $(a_1, a_2, \cdots, a_{n-1})$. This proves the lemma. □

Note that in fact only the information on the terms of bidegrees $(1, n-1)$ and $(n-1, 1)$ is used. And in fact if these are equal to so are all the others in $\nu(\sigma)$ as is easily seen.

Some examples may make things easier to follow.

Take for example $\sigma = (4, 6, 5, 1, 2, 3)$. The bidegree $(5, 1)$ part of $\mu(\sigma)$ is $(4, 5, 1, 2, 3) \otimes (1)$ and the only permutation words of length 6 for which this is possible are the ones obtained by inserting a 6 somewhere in $(4, 5, 1, 2, 3)$. The bidegree $(1, 5)$ part of $\mu(\sigma)$ is $(1) \otimes (3, 5, 4, 1, 2)$. The element $5 = 6 - 1$ occurs between 3 and 4, which means that 6 has to occur between 4 and 5 in any permutation word of length 6 that gives rise to this term $(1) \otimes (3, 5, 4, 1, 2)$. As 4 and 5 are neighbors in $(4, 5, 1, 2, 3)$ it follows that there is only one possibility.

Now consider $\sigma = (4, 6, 1, 5, 2, 3)$. The bidegree $(5, 1)$ part of $\mu(\sigma)$ is $(4, 1, 5, 2, 3) \otimes (1)$ and the only permutation words of length 6 for which this is possible are the ones obtained by inserting a 6 somewhere in $(4, 1, 5, 2, 3)$. The bidegree $(1, 5)$ part of $\mu(\sigma)$ is $(1) \otimes (3, 5, 4, 1, 2)$. Again, it follows that 6 must be inserted between 4 and 5 in $(4, 1, 5, 2, 3)$. But now 5 and 4 are not anymore neighbors; there is a 1 in between (and this is the only thing that can happen). So there are two possibilites for a permutation word of length 6 with bidegree $(5, 1)$ and $(1, 5)$ terms $(4, 1, 5, 2, 3) \otimes (1)$ and $(1) \otimes (3, 5, 4, 1, 2)$ respectively; viz $\sigma = (4, 6, 1, 5, 2, 3)$ and $(4, 1, 6, 5, 2, 3)$.

Finally, consider $\sigma = (6, 1, 4, 5, 2, 3)$. The bidegree $(5, 1)$ part of $\mu(\sigma)$ is $(1, 4, 5, 2, 3) \otimes (1)$ and the only permutation words of length 6 for which this is possible are the ones obtained by inserting a 6 somewhere in $(1, 4, 5, 2, 3)$. The bidegree $(1, 5)$ part of $\mu(\sigma)$ is $(1) \otimes (5, 3, 4, 1, 2)$. Here 5 occurs left of 3 so 6 must beinserted to the left of 4 in $(1, 4, 5, 2, 3)$ giving again two posibilities.

Proof of proposition 2.1 (continued). So, let α be an automorphism of *MPR* that preserves the distinguished basis and that is the identity in degree $n, n \geq 2$. Let σ be a permutation word of length $n+1$. Because α preserves the comultiplication and, necessarily $\alpha((1)) = (1)$ and α is the identity in degree n it follows that the bidegree $(1, n-1)$ and $(n-1, 1)$

terms of $\nu(\sigma)$ and $\nu(\alpha(\sigma))$ are equal. So, by lemma 2.4, if 1 and $n+1$ are not neighbors in σ it must be the case that $\sigma = \alpha(\sigma)$.

But in the case that 1 and $n+1$ are neighbors there is, so far, still the posibility that $\alpha(\sigma)$ is obtained from σ by switching 1 and $n+1$. So let 1 and $n+1$ be neighbors in σ. Cut σ between 1 and $n+1$ and also cut $\alpha(\sigma)$ between 1 and $n+1$. Suppose that 1 occurs before $n+1$ in σ. Then this exhibits σ and $\alpha(\sigma)$ as concatenations

$$\begin{aligned} &\sigma = u * v,\ u = (?, \cdots, ?, 1),\ v = (n+1, ?, \cdots, ?) \\ (2.7)\quad &\alpha(\sigma) = u' * v',\ u' = (?, \cdots, ?, n+1),\ v' = (1, ?, \cdots, ?) \\ &1 \leq \text{length}(u) = \text{length}(u') \leq n,\ 1 \leq \text{length}(v) = \text{length}(v') \leq n \end{aligned}$$

The first line of (2.7) exhibits σ as a term in the product of $\sigma_1 = \text{st}(u)$ and $\sigma_2 = \text{st}(v)$. But α preserves the multiplicaton, so $\alpha(\sigma)$ must also be a term in this product.

Now suppose that the length of v is 2 or more. Then $\text{st}(v) \neq \text{st}(v')$ because $\text{st}(v')$ starts wih 1 and $\text{st}(v)$ starts with a number at least 2 because the length of v is at least 2 and $n+1$ is the highest number in v. Thus by the second line of (2.7), $\alpha(\sigma)$ also occurs in a different product, which is impossible by lemma 2.2 part (ii). The possibility that remains is that the length of v is 1. But then the length of u is 2 or more, and the same reasoning applied to u and u' concludes the proof, observing that the same reasoning works if $n+1$ occurs before 1 in σ. □

Proof of the theorem (continued). Induction and proposition 2.1 take care of case A). It remains to analyse case B). So suppose that α is a graded automorphism that preserves the distinguished basis and such that

$$(2.8)\qquad \alpha(1) = 1,\ \alpha((1)) = (1),\ \alpha((1,2)) = (2,1),\ \alpha((2,1)) = (1,2)$$

As it turns out this one can exist up to and including degree 3. And this extension is unique. This is seen as follows. Write down the nontrivial parts of the comultiplication of the six permutation words of length 3.

$$(2.9)\qquad \begin{aligned} \nu((1,2,3)) &= (1)\otimes(1,2)+(1,2)\otimes(1) \\ \nu((1,3,2)) &= (1)\otimes(2,1)+(1,2)\otimes(1) \\ \nu((2,1,3)) &= (1)\otimes(1,2)+(2,1)\otimes(1) \\ \nu((2,3,1)) &= (1)\otimes(1,2)+(2,1)\otimes(1) \\ \nu((3,1,2)) &= (1)\otimes(2,1)+(1,2)\otimes(1) \\ \nu((3,1,2)) &= (1)\otimes(2,1)+(2,1)\otimes(1) \end{aligned}$$

It follows that if α satisfies (2.8) it must be the case that

$$(2.10)\qquad \alpha((1,2,3)) = (3,2,1), \alpha((3,2,1)) = (1,2,3)$$

It also follows that $\alpha((1,3,2))$ must be equal to either $(2,1,3)$ or $(2,3,1)$. But $(1,3,2)$ occurs in the product $(1)(2,1)$. As α preserves multiplication it follows that $\alpha((1,3,2))$ must be in the product $(1)(1,2)$. It follows that (using lemma 2.2 again, or explicit computation)

$$(2.11)\qquad \alpha((1,3,2)) = (2,1,3)$$

Similarly one finds
(2.12)

$$\alpha(2,1,3) = (1,3,2),\ \alpha((2,3,1)) = (3,1,2),\ \alpha((3,1,2)) = (2,3,1)$$

Now consider $\alpha((1,4,2,3))$. The nontrivial part of the comultiplication on this element is equal to

$$\nu((1,4,2,3)) = (1)\otimes(3,1,2)+(1,2)\otimes(2,1)+(1,2,3)\otimes(1)$$

Under $\alpha\otimes\alpha$ this goes by (2.10) and (2.12) into

$$(2.13)\qquad (1)\otimes(2,3,1)+(2,1)\otimes(1,2)+(3,2,1)\otimes(1)$$

Thus, because α respects the comultiplication $\alpha((1,4,2,3))$ must be obtained from $(3,2,1)$ by inserting a 4 somewhere. But the first term of (2.13) says that $\alpha((1,4,2,3))$ is equal to $(3,4,2)$ with a 1 inserted somewhere. The only possibility is

$$\alpha((1,4,2,3)) = (3,4,2,1) \tag{2.14}$$

Now examine $\alpha((4,1,2,3))$. Because 1 and 4 are neighbors in (4, 1, 2, 3)

$$\nu((4,1,2,3)) = \nu((1,4,2,3))$$

(or do a direct calculation). It follows as just before that also

$$\alpha((4,1,2,3)) = (3,4,2,1)$$

and so α cannot be an automorphism in degree 4. This concludes the proof of the theorem. □

Remarks 2.15.

(i) If the requirement that the Hopf algebra automorphism preserve the distinguished basis is dropped there are many nontrivial automorphisms. Indeed the antipode is an anti-automorphism for both the mutiplication and the comultiplication. So the even powers of the antipode are all Hopf algebra automorphisms. As the antipode of *MPR* is of infinite order, see [1], this gives an infinite number of automorphisms.

(ii) There is precisely one automorphism *MPR* that is multiplication-preserving and anti-comultiplication-preserving and preserves the distinguished basis, viz the operation

$$(a_1, \cdots, a_n) \mapsto (n+1-a_1, \cdots, n+1-a_n)$$

There is also precisely one automorphism of *MPR* that is anti-multiplica-tion-preserving and comultiplication-preserving and preserves the distinguished basis, viz the reversal

$$(a_1, \cdots, a_n) \mapsto (a_n, \cdots, a_1)$$

(iii) And, finally, there is one distinguished basis preserving automorphism that is both anti-multiplication-preserving and anti-comultiplication-preserving, obtained by combining the two previous operations. These three together with the identity form the Klein four group.

Indeed, that these operations have the indicated properties is proved in [8]. And if there were more than one of either of the three types composing them would give a (true) distinguished basis preserving Hopf algebra automorphism unequal to the identity.

(iv) Composing the odd powers of the anipode with the morphism of (iii) above gives another infinity of automorphisms of *MPR*.

3. Relations between the (anti-) Hopf algebra automorphisms of *MPR* and those of *NSymm* and *Symm*

Let *NSymm* be the Hopf algebra of noncommutative symmetric functions. There is a natural embedding

$$NSymm \subset MPR \tag{3.1}$$

described in some detail below and the question arises whether the (anti-)auto-morphisms described above in 2.15 preserve this sub Hopf algebra and to describe the restrictions (if any).

As an algebra *NSymm* is the free associative algebra in countably infinite indeterminates over the integers and there is a natural projection onto the Hopf algebra of symmetric functions, *Symm*

$$NSymm = \mathbf{Z}\langle Z_1, Z_2, \cdots \rangle \longrightarrow Symm, \;\; Z_i \mapsto h_i \tag{3.2}$$

where the h_i are the complete symmetric functions (in a countable infinity of indeterminates).

The further question that arises is whether the induced (anti-)automorphisms of *NSymm* descend under this projection to *Symm* and which automorphisms result.

3.3 (Descent sets). Let $\sigma = (s_1, s_2, \cdots, s_n)$ be a permutation (word) on n letters. Then its descent set is defined as

$$\text{desc}(\sigma) = \{i \in \{1, 2, \cdots, n-1\}\colon s_i > s_{i+1}\} \tag{3.4}$$

viewed as a subset of $\{1, \cdots, n-1\}$. That is, which set a descent set is a subset of is part of the data: $\{1\} \subset \{1, 2, 3\}$ is a very different thing from $\{1\} \subset \{1\}$.

Now let $\alpha = [a_1, \cdots, a_m]$ be a composition of length m, i.e. a finite sequence of strictly positive integers. The weight of α is $w = \text{wt}(\alpha) = \sum_{i=1}^{m} a_i$ and α is said to be a composition of w. Associated to such a composition is a descent set in $\{1, 2, \cdots, w-1\}$ defined by

$$\text{Desc}(\alpha) = \{a_1, a_1 + a_2, \cdots, a_1 + \cdots + a_{m-1}\} \tag{3.5}$$

This sets up a bijection between descent sets in $\{1, 2, \cdots, w-1\}$ and compositions of weight w. Note that the descent set of a permutation is a rather different thing from a descent set of a composition as is refelected in the typography.

3.6 (The natural embedding of *NSymm* into *MPR*). One graded basis for the free Abelian group underlying *NSymm* consists of all monomials

$$Z_\alpha,\ \alpha \text{ any composition, including the empty composition, } Z_\varnothing = 1,$$
$$Z_\alpha = Z_{a_1} Z_{a_2} \cdots Z_{a_m} \text{ if } \alpha = [a_1, a_2, \cdots, a_m]$$

Another graded basis is formed by the socalled ribbon Schur functions R_α, α a composition, including the empty composition, $R_\varnothing = 1$. A formula is

$$R_\alpha = \sum_{\alpha \geq \beta} (-1)^{\lg(\alpha) - \lg(\beta)} S_\beta \tag{3.7}$$

where the sum is over all coarsenings β of α and where the S_n are related to the Z_n by the noncommutative Wronski relations

$$S_n - S_{n-1}Z_1 + S_{n-2}Z_2 + \cdots + (-1)^{n-1} S_1 Z_{n-1} + (-1)^n Z_n = 0 \tag{3.8}$$

so that (also) $NSymm = \mathbf{Z}\langle S_1, S_2, \cdots\rangle$. For instance $R_{[n]} = S_n$, $R_{[1,1,\cdots,1]} = Z_n$. In terms of the ribbon Schur functions the embedding of *NSymm* into *MPR* is given by

$$(3.9) \qquad NSymm \longrightarrow MPR,\ R_\alpha \mapsto \sum_{\mathrm{desc}(\sigma)=\mathrm{Desc}(\alpha)} \sigma$$

See e.g. [2] for details.

3.10 (Induced (anti-)automorphisms on *NSymm*). Let ω_n be the permutation $(n, n-1, \cdots, 2, 1)$. Denote by ψ_l the automorphism

$$(3.11)\ \ \sigma \mapsto \omega_n \circ \sigma,\ (s_1, s_2, \cdots, s_n) \mapsto (n+1-s_1, n+1-s_2, \cdots, n+1-s_n)$$

This is a multiplication preserving and anti comultiplication preserving. For a descent set $D = \{d_1, \cdots, d_m\} \subset \{1, 2, \cdots, n-1\}$ let $n - D$ and D^c be the descent sets

$$(3.12) \qquad n - D = \{n - d_1, \cdots, n - d_m\},\ D^c = \{1, 2, \cdots, n-1\}\backslash D$$

With these notations (as is easily checked)

$$(3.13) \qquad \mathrm{desc}(\psi_l(\sigma)) = \mathrm{desc}(\sigma)^c$$

The descent set of $\psi_l(\sigma)$ depends only on $\mathrm{desc}(\sigma)$ and not on σ itself. So by (3.9) it induces an automorphism of *NSymm* that is multiplication preserving and comultiplication preserving (because *NSymm* is cocommutative).

It is readily checked that this must be the automorphism that is determined (as an algebra morphism) by requiring that

$$(3.14) \qquad Z_n \mapsto S_n$$

This follows by considering what happens to the identity permutation and to ω_n and their descent sets.

Next consider the automorphism ψ_r given by

$$(3.15) \qquad \sigma \mapsto \sigma \circ \omega_n, (s_1, s_2, \cdots, s_n) \mapsto (s_n, s_{n-1}, \cdots, s_1)$$

This one is anti-multiplication preserving and comultiplication preserving. Using the notations (3.12) one readily checks

(3.16) $$\mathrm{desc}(\psi_r(\sigma)) = (n - \mathrm{desc}(\sigma))^c = n - \mathrm{desc}(\sigma)^c$$

Again the descent set of the result only depends on the descent set of σ and not on σ itself. And, again, it follows that (3.15) induces an automorphism of *NSymm* that is an anti-homomorphism of algebras and a homomorphism of coalgebras.

It follows that this is the automorphism of *NSymm* that as an anti homomorphism of algebras is determined by

(3.17) $$Z_n \mapsto S_n$$

Note that (3.14) and (3.17) are different: one is an anti-homomorphism of algebras, the other a homomorphism of algebras, so that e.g. under (3.14), $Z_1Z_2 \mapsto S_1S_2$, while under (3.17) $Z_1Z_2 \mapsto S_2S_1$.

Thirdly consider the composite $\psi_l\psi_r = \psi_r\psi_l$

(3.18) $$\sigma \mapsto \omega_n \circ \sigma \circ \omega_n$$

which is anti-multiplication preserving and anti-comultiplication preserving. For the descent sets one finds

(3.19) $$\mathrm{desc}(\psi_l(\psi_r(\sigma))) = n - \mathrm{desc}(\sigma)$$

It follows that this composite induces an automorphism of *NSymm* and that it is the anti algebra automorphism determined by

(3.20) $$Z_n \mapsto Z_n$$

3.21 (Induced automorphisms on *Symm*)**.** The natural projection *NSymm* $\longrightarrow$ *Symm* is given by

(3.22)

$Z_n \mapsto h_n$ or, equivalently, because of the Wronski relations, $S_n \mapsto e_n$

where the e_n are the elementary symmetric functions. All four of the (anti-)auto-morphisms identity, (3.14), (3.17), (3.20) descend to auto-morphisms of *Symmi*: (3.20) descends to the identity, and both (3.14) and (3.17) descend to the Hopf algebra automorphism of *Symm* that interchanges the elementary and the complete elementary symmetric functions, the only positivity preserving graded automorphism of *Symm*.

Acknowledgement

The first time I had opportunity to lecture on these things was 9, 10 Jan. 2006 at a special conference in Jeonju. I am grateful for the hospitality of IMCAG, Prof. Seul Hee Choi, and Jeonju University. I am also grateful to Prof. Yanggon Kim and the Honam Math. Society, who made things possible and Prof. Ki-Bong Nam and Prof. Moon-Ok Wang who made things work for me personally.

It is perhaps worth recording that I started investigating these things as a result of questions asked at the previous Jeonju University meeting in August 2004. For the proceedings of that meeting, see Ki-Bong Nam a.o. (eds), Advances in Algebra towards millenium problems, SAS international publicztions, 2004.

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