Honam Mathematical J. **29** (2007), No. 3, pp. 465–474

A NOTE ON COMPLICATED SUBTRACTION ALGEBRAS

Eun Hwan Roh

Abstract. We provided characterizations of complicated in a subtraction algebra, and we showed that the complicated subtraction algebra is a ring.

1. Introduction

B. M. Schein [11] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [12] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [5] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [4], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Y. B. Jun and K. H. Kim [6] introduced the notion of prime and irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also provided a condition for an ideal to be a prime/irreducible ideal. Y. B. Jun, Y. H. Kim and K. A. Oh [7] discussed subtraction algebras

Received July 5, 2007. Accepted August 9, 2007.
2000 Mathematics Subject Classification: 03G25, 06B10, 06D99.
Key words and phrases: complicated subtraction algebra, ring.

with additional conditions, so called complicated subtraction algebra (c-subtraction algebra, for short), and investigated several properties. they provided characterizations of ideals in c-subtraction algebras, and showed that the set of all ideals in a complicated subtraction algebra is a complete lattice. In this paper, we provided characterizations of complicated in a subtraction algebra, and we showed that the c-subtraction algebra is a ring.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;

(S2) $x - (x - y) = y - (y - x)$;

(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [5, 6]):

(p1) $(x - y) - y = x - y$.

(p2) $x - 0 = x$ and $0 - x = 0$.

(p3) $(x - y) - x = 0$.

(p4) $x - (x - y) \leq y$.
(p5) $(x - y) - (y - x) = x - y$.
(p6) $x - (x - (x - y)) = x - y$.
(p7) $(x - y) - (z - y) \leq x - z$.
(p8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(p9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(p10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(p11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.
(p12) $(x - y) - z = (x - z) - (y - z)$.

Definition 2.1. [5] A nonempty subset A of a subtraction algebra X is called an *ideal* of X if it satisfies

- $0 \in A$
- $(\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$.

Lemma 2.2. [6] *An ideal A of a subtraction algebra X has the following property:*

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

Let X be a subtraction algebra. For any $a, b \in X$, we define

$$\mathfrak{G}(a, b) = \{x \in X \mid x - a \leq b\}.$$

Note that $\mathfrak{G}(a, b)$ is nonempty since $0, a, b \in \mathfrak{G}(a, b)$.

Definition 2.3. [7] A subtraction algebra X is said to be *complicated* if for any $a, b \in X$ the set $\mathfrak{G}(a, b)$ has the greatest element.

The greatest element of $\mathfrak{G}(a, b)$ is denoted by $a + b$.

Proposition 2.4. [7] *If X is a complicated subtraction algebra (c-subtraction algerba, for short), then*

(i) $(\forall a, b \in X)\ (a \leq a + b,\ b \leq a + b)$,
(ii) $(\forall a \in X)\ (a + 0 = a = 0 + a)$,
(iii) $(\forall a, b \in X)\ (a + b = b + a)$.

(iv) $(\forall x, y, z \in X)\ (x \le y \Rightarrow x + z \le y + z)$.

Theorem 2.5. [7] *If X is a c-subtraction algebra, then $(X, +, 0)$ is a commutative monoid.*

Proposition 2.6. [7] *Any c-subtraction algebra X satisfies the following axioms:*

(i) $(\forall x, y, z \in X)\ ((x - y) - z = x - (y + z))$,
(ii) $(\forall x, y, z \in X)\ (x - y \le (x - z) + (z - y))$,
(iii) $(\forall x, y, z \in X)\ ((x + z) - (y + z) \le x - y \le x + y)$,
(iv) $(\forall x, y \in X)\ (x \le y \Rightarrow x + y = y)$,
(v) $(\forall x, y, z \in X)\ ((x + y) - z = (x - z) + (y - z))$,
(vi) $(\forall x, y \in X)$ ($x + y$ is the least upper bound of x and y),
(vii) $(\forall x, y \in X)\ (x + y = x + (y - x))$.

3. Complicated subtraction algebras

We provide characterization of a c-subtraction algebra.

Theorem 3.1. *A subtraction algebra X is complicated if and only if there exists a binary operation $+$ on X such that $(x-y)-z = x-(y+z)$ for any $x, y, z \in X$.*

Proof. By Proposition 2.6(i), we need to show the sufficiency. Assume that there exists a binary operation $+$ on X such that $(x - y) - z = x - (y + z)$ for any $x, y, z \in X$, then

$$((x + y) - x) - y = (x + y) - (x + y) = 0,$$

and so $x + y$ is a solution of the equation

$$(u - x) - y = 0 \quad (1)$$

with u as the unknown. Also, for any solution t of (1), we have $t \le x+y$ since $t - (x + y) = (t - x) - y = 0$. Hence $x + y$ is the greatest solution of (1). Therefore X is complicated. □

The above theorem leads us to give an axiom system as follows.

Theorem 3.2. *An algebra* $(X; -, +, 0)$ *of type* $(2, 2, 0)$ *is a c-subtraction algebra if and only if the following hold:* $\forall x, y, z \in X$,

(S1) $x - (y - x) = x$;

(S2) $x - (x - y) = y - (y - x)$;

(S3) $(x - y) - z = (x - z) - y$;

(C) $(x - y) - z = x - (y + z)$.

Now we give some characterizations in a subtraction algebra.

Theorem 3.3. *Let* X *be a c-subtraction algebra. Then the following are equivalence:*

(i) $(\forall x, y \in X)\ (x \le y \Rightarrow x + y = y)$;

(ii) $(\forall x \in X)\ (x + x = x)$,

(iii) $(\forall x, y, z \in X)\ ((x + y) - z = (x - z) + (y - z))$;

(iv) $(\forall x, y \in X)\ (x + y = x + (y - x))$.

Proof. (i)⇒(ii). Let $x \in X$. Since $x \le x$, it follows (i) that $x + x = x$.

(ii)⇒(iii). Let $x, y, z \in X$. Then by Proposition 2.4(i) and (p9), we obtain

$$x - z \le (x + y) - z \text{ and } y - z \le (x + y) - z.$$

Then by Proposition 2.4(iv) and (ii) we have

$$(x - z) + (y - z) \le ((x + y) - z) + ((x + y) - z) = (x + y) - z.$$

Also, using (ii) and Proposition 2.6(i) we get

$$(x + y) - z = (x + y) - (z + z) = ((x + y) - z) - z.$$

Then

$$\begin{aligned} ((x + y) - z) - (x - z) &= (((x + y) - z) - z) - (x - z) \\ &= (((x + y) - z) - (x - z)) - z \\ &= (((x + y) - x) - z) - z \\ &\le ((x + y) - x) - z \qquad \text{by (p7) and (p12)} \\ &\le y - z. \end{aligned}$$

Thus we get $(x+y)-z \leq (x-z)+(y-z)$. Therefore $(x+y)-z = (x-z)+(y-z)$.

(iii)⇒(iv). Let $x, y \in X$. Since $y-x \leq y$, we have $x+(y-x) \leq x+y$. Also, by (iii) we get

$$(x+y)-x \leq (x-x)+(y-x) = y-x \leq y-x.$$

i.e., $x+y \leq x+(y-x)$. Hence $x+y = x+(y-x)$.

(iv)⇒(i). By (iv) we have $y+y = y+(y-y) = y$ for all $y \in X$. Let $x, y \in X$ be such that $x \leq y$. Then be (p1) and Proposition 2.4(iv) we have

$$(x+y)-y = (y+x)-y = ((y+x)-y)-y \leq x-y = 0.$$

i.e., $x+y \leq y$. Also we have $y \leq x+y$ by Proposition 2.4(i). Hence we obtain $x+y = y$. □

Next, we show that the c-subtraction algebra is actually a ring. We first give some propositions as follows.

Proposition 3.4. *Let X be a c-subtraction algebra. Then*

$$(\forall x, y \in X)(x = (x-y)+(x-(x-y))).$$

Proof. Let $x, y \in X$. Since $x-y \leq x$, we have $x = (x-y)+x$ by Proposition 2.6(iv). Thus by Proposition 2.6(vii), we have

$$(x-y)+x = (x-y)+(x-(x-y)).$$

□

Proposition 3.5. *Let X be a c-subtraction algebra. Then we have*

$$(\forall x, y, z \in X)(x-(y-z) = ((x-y)-z)+(z-(z-x))).$$

Proof. Let $x, y, z \in X$. Then we have $x = (x-z)+(x-(x-z))$ by Proposition 3.4, and so by (S2) we get $x = (x-z)+(z-(z-x))$. Thus we obtain

$$x-(y-z) = ((x-z)+(z-(z-x)))-(y-z).$$

Hence by Proposition 2.6 implies

$$\begin{aligned} & x-(y-z) \\ = & ((x-z)-(y-z))+((z-(z-x))-(y-z)) \\ = & ((x-y)-z)+((z-(y-z))-(z-x)) \quad \text{by (p12) and (S3)} \\ = & ((x-y)-z)+(z-(z-x)) \quad \text{by (S1)}. \end{aligned}$$

□

Proposition 3.6. *Let X be a subtraction algebra. Then we have*

(i) $(\forall x, y, z \in X)\ (x \wedge (y-z) = (x \wedge y) - z)$,

(ii) $(\forall x, y, z \in X)\ (x \wedge (y-z) = (x \wedge y) - (x \wedge z))$,

where $x \wedge y := y-(y-x)$ for all $x, y \in X$.

Proof. (i) Let $x, y, z \in X$. Then we get

$$\begin{aligned} x \wedge (y-z) &= (y-z)-((y-z)-x) \\ &= (y-z)-((y-x)-z) \quad \text{by (S3)} \\ &= (y-(y-x))-z \quad \text{by (p12)} \\ &= (x \wedge y)-z. \end{aligned}$$

(ii) Let $x, y, z \in X$. Then we have

$$\begin{aligned} (x \wedge y)-(x \wedge z) &= (x-(x-y))-(x-(x-z)) \quad \text{by (S2)} \\ &= (x-(x-(x-z)))-(x-y) \quad \text{by (S3)} \\ &= (x-z)-(x-y) \quad \text{by (p6)} \\ &= (x-(x-y))-z \quad \text{by (S3)} \\ &= (x \wedge y)-z \\ &= x \wedge (y-z). \end{aligned}$$

□

Definition 3.7. A ring $(X; \oplus, \odot, 0)$ is called *idempotent* if it satisfies the idempotent law: $x \odot x = x$ for all $x \in X$.

Theorem 3.8. *Let $(X; -, +, 0)$ be a c-subtraction algebra. Define a binary operations $\oplus$ and $\odot$ on X as follows: $\forall x, y \in X$,*

$$x \oplus y := (x-y)+(y-x) \text{ and } x \odot y := x \wedge y,$$

where $x \wedge y := y-(y-x)$. Then $(X; \oplus, \odot, 0)$ is an idempotent ring.

Proof. By Proposition 2.4(iii), we have

$$x \oplus y = (x - y) + (y - x) = (y - x) + (x - y) = y \oplus x,$$

and so the operation $\oplus$ satisfies the commutative law. Also, 0 is the zero element of $\oplus$ since $x \oplus 0 = (x - 0) + (0 - x) = x + 0 = x$ by (p2)and Proposition 2.4(ii). Moreover, the negative element of any element $x$in X is just x itself since $x \oplus x = (x - x) + (x - x) = 0 - 0 = 0$ by Proposition 2.4(ii). To prove the associative law of $\oplus$, we first note from Proposition 3.5 that

$$z - (x - y) = ((z - x) - y) + (y - (y - z)).$$

Hence by Proposition 2.6(i), we have

$$(z - (x - y)) - (y - x) = ((z - (x + y)) + (z \wedge y)) - (y - x).$$

Using Proposition 2.6(v), it follows

$$(2)\quad (z - (x - y)) - (y - x) = ((z - (x + y)) - (y - x)) + ((z \wedge y) - (y - x)).$$

Also, since $y - x \leq x + y$, we have $(x + y) + (y - x) = x + y$ by Proposition 2.6(iv). Thus Proposition 2.6(i) gives

$$(3)\quad (z - (x - y)) - (y - x) = z - ((x + y) + (y - x)) = z - (x + y).$$

By Proposition 3.6(i) and (S2), we get

$$(4)\qquad (z \wedge y) - (y - x) = z \wedge (y - (y - x)) = x \wedge y \wedge z.$$

Then by (2), (3) and (4), we obtain

$$(5)\qquad (z - (x - y)) - (y - x) = (z - (x + y)) + (x \wedge y \wedge z).$$

Hence Proposition 2.6(i),(v) together with (5), give

$$\begin{aligned} & (x \oplus y) \oplus z \\ = \; & (((x - y) + (y - x)) - z) + (z - ((x - y) + (y - x))) \\ = \; & ((x - y) - z) + ((y - x) - z) + ((z - (x - y)) - (y - x)) \\ = \; & (x - (y + z)) + (y - (z + x)) + (z - (x + y)) + (x \wedge y \wedge z). \end{aligned}$$

Note that the last expression is symmetric and the operation $\oplus$ is commutativity, we have the associative law of $\oplus$ holds. Therefore $(X; \oplus, 0)$ is an Abelian group.

Next, it is clear that the operation $\odot$ satisfies the commutative and associative laws. Also, by Proposition 3.6(ii), the following hold:

$$x \wedge (y - z) = (x \wedge y) - (x \wedge z) \text{ and } x \wedge (z - y) = (x \wedge z) - (x \wedge y).$$

Now, the distributive law of the operation $\odot$ to the operation $\oplus$ is got by

$$\begin{aligned} & x \odot (y \oplus z) \\ = \; & x \wedge ((y - z) + (z - y)) \\ = \; & (x \wedge (y - z)) + (x \wedge (z - y)) \\ = \; & ((x \wedge y) - (x \wedge z)) + ((x \wedge z) - (x \wedge y)) \\ = \; & (x \odot y) \oplus (x \odot z), \end{aligned}$$

since $(X; \wedge, \vee)$ is a distributive lattice. Therefore $(X; \oplus, \odot, 0)$ is a ring.

Finally, noticing the fact that $x \odot x = x \wedge x = x$. The proof is complete. □

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Eun Hwan Roh,
Department of Mathematics Education,
Chinju National University of Education,
Jinju 660-756, Korea
E-mail: ehroh@cue.ac.kr, ehroh0923@gmail.com