Honam Mathematical J. **29** (2007), No. 3, pp. 455–464

MIXED VECTOR F-IMPLICIT COMPLEMENTARITY PROBLEMS AND CORRESPONDING VECTOR VARIATIONAL INEQUALITY PROBLEMS

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Abstract. We consider existence theorems for a new class of mixed vector F-implicit complementarity problems and the corresponding mixed vector F-implicit variational inequality problems.

1. Introduction and Preliminaries

The following F-complementarity problem: Find $x \in K$ such that

$$\langle Tx, x\rangle + F(x) \ = \ 0 \text{ and } \langle Tx, y\rangle + F(y) \ \geq \ 0 \text{ for all } y \in K,$$

and the corresponding F-variational inequality problem:

Find $x \in K$ such that

$$\langle Tx, y - x\rangle + F(y) - F(x) \ \geq \ 0 \text{ for all } y \in K$$

were introduced and considered in [6], where K is a nonempty closed convex cone of a real Banach space X, $T : K \to X^{\star}$(the dual space) is a mapping and $F : K \to (-\infty, +\infty)$ is a positively homogeneous and convex function.

In 2003, Fang and Huang [2] studied a new class of vector F-complementarity problems with demi-pseudomonotone mappings in Banach spaces by considering the solvability of the problems. Huang and Li [3]

Received July 2, 2007. Accepted August 13, 2007.
2000 Mathematics Subject Classification: 49J40, 90C33.
Key words and phrases: Mixed vector F-implicit complementarity problems, mixed vector F-implicit variational inequality problems, KKM-mapping, Fan-KKM Theorem.

considered a class of scalar F-implicit complementarity problems and another class of F-implicit variational inequality problems in Banach spaces, in 2004. And then, Li and Huang [5] extended and generalized the scalar case to some vector case. The equivalence between the F-implicit complementarity problem and F-implicit variational inequality problem was presented and some new existence theorems of solutions for F-implicit variational inequality problems were also proved.

The main objective of this work is to generalize some results of [4, 5] to more generalized vector case. We introduce a new class of generalized mixed vector F-implicit complementarity problems and corresponding new class of generalized mixed vector F-implicit variational inequality problems in Banach spaces and prove the equivalence between them under certain assumptions. Furthermore, we derive some new existence theorems of solutions for the generalized mixed vector F-implicit complementarity problems and the generalized mixed vector F-implicit variational inequality problems by using Fan-KKM Theorem [1] under some suitable assumptions without any monotonicity.

An ordered Banach space (Y, P) is a real Banach space Y with an ordering defined by a closed cone $P \subseteq Y$ with an apex at the origin in the form of

$$x \geq y \Leftrightarrow x - y \in P$$

and

$$x \not\geq y \Leftrightarrow x - y \notin P$$

A mapping $F : K \to Y$ is said to be positively homogeneous if $F(\alpha x) = \alpha F(x)$ for all $x \in K$ and $\alpha \geq 0$, where X and Y are vector spaces and K a subspace of X.

2. Main Results

Definition 2.1. Let K be a nonempty subset of a topological vector space X. A mapping $G : K \to 2^X$ is called a KKM-mapping if for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of K

$$\text{conv}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i),$$

where conv denotes the convex hull.

Fan-KKM Theorem [1]. *Let K be a nonempty subset of Hausdorff topological vector space X. Let $G : K \to 2^X$ be a KKM-mapping such that for any $y \in K$ $G(y)$ is closed and $G(y^\star)$ is compact for some $y^\star \in K$, then there exists $x^\star \in K$ such that $x^\star \in G(y)$ for all $y \in K$, i.e.,*

$$\bigcap_{y \in K} G(y) \neq \emptyset.$$

Lemma 2.1 [5]. *Let (Y, P) be an ordered Banach space by a pointed closed convex cone P. If $x \geq 0$ and $y \geq 0$, then $x+y \geq 0$, for all $x, y \in Y$.*

Let X be a real Banach space, $K \subseteq X$ be a nonempty closed convex cone and (Y, P) be an ordered Banach space. Denote $L(X, Y)$ the space of all continuous linear mappings from X into Y and $\langle t, x\rangle$ the value of a linear continuous mapping $t \in L(X, Y)$ at x. Let $A, T : K \to L(X, Y)$, g, $h : K \to K$, $F : K \to Y$ and $N : L(X, Y) \times L(X, Y) \to L(X, Y)$ be mappings.

In this section, we consider the following mixed vector F-implicit complementarity problem (MVF-ICP): Find $x^\star \in K$ such that

$$\langle N(Ax^\star, Tx^\star), g(x^\star)\rangle + F(g(x^\star)) = 0$$

and

$$\langle N(Ax^\star, Tx^\star), h(y)\rangle + F(h(y)) \geq 0, \text{ for all } y \in K$$

We also introduce the following mixed vector F-implicit variational inequality problem (MVF-IVIP): Find $x^\star \in K$ such that

$$\langle N(Ax^\star, Tx^\star), h(y) - g(x^\star)\rangle + F(h(y)) - F(g(x^\star)) \geq 0, \text{ for all } y \in K.$$

Remark 2.1. (i) By putting $g = h$ in (MVF-ICP) and (MVF-IVIP), we obtain (GVF-ICP) and (GVF-IVIP) in [4].
(ii) The following vector F-implicit complementarity problem (VF-ICP) of finding $x^\star \in K$ such that

$$\langle f(x^\star), g(x^\star)\rangle + F(g(x^\star)) = 0$$

and

$$\langle f(x^\star), y\rangle + F(y) \geq 0, \text{ for all } y \in K,$$

is a particular form of (MVF-ICP) and the corresponding vector F-implicit variational inequality problem (VF-IVIP) of finding such that

$$\langle f(x^\star), y - g(x^\star)\rangle + F(y) - F(g(x^\star)) \geq 0, \text{ for all } y \in K,$$

is a particular form of (MVF-IVIP) for the identities A and T and a mapping $f : K \to L(X, Y)$ defined by $f(x) = N(x, x)$ for $x \in K$, and for the identity h, which were considered and studied by Li and Huang [5].

We first establish the equivalence between (MVF-ICP) and (MVF-IVIP).

Theorem 2.1.
(i) If $x^\star$ solves (MVF-ICP), then it solves (MVF-IVIP).

(ii) Let $F : K \to Y$ be a positively homogeneous mapping and h be a surjective mapping.
If $x^\star$ solves (MVF-IVIP), then it solves (MVF-ICP).

Proof. (i) If $x^\star \in K$ is a solution of (GVF-ICP), then

$$\langle N(Ax^\star, Tx^\star), g(x^\star)\rangle + F(g(x^\star)) = 0$$

and

$$\langle N(Ax^\star, Tx^\star), h(y)\rangle + F(h(y)) \geq 0 \text{ for all } y \in K.$$

Hence

$$\begin{aligned}
&\langle N(Ax^\star, Tx^\star), h(y) - g(x^\star)\rangle + F(h(y)) - F(g(x^\star)) \\
=&[\langle N(Ax^\star, Tx^\star), h(y)\rangle + F(h(y))] - [\langle N(Ax^\star, Tx^\star), g(x^\star)\rangle + F(g(x^\star))] \\
=&\langle N(Ax^\star, Tx^\star), h(y)\rangle + F(h(y)) \\
\geq&\ 0,
\end{aligned}$$

for all $y \in K$.

(ii) If $x^\star \in K$ is a solution of (MVF-IVIP), then

$$\langle N(Ax^\star, Tx^\star), h(y) - g(x^\star)\rangle + F(h(y)) - F(g(x^\star)) \geq 0 \text{ for all } y \in K.$$

Since $F : K \to Y$ is positively homogeneous, h is sujective and K is a convex cone, we can take $y \in K$ such that $h(y) = 2g(x^\star)$ in (MVF-IVIP), thus we have

$$\langle N(Ax^\star, Tx^\star), g(x^\star)\rangle + F(g(x^\star)) \geq 0.$$

Similarly we take $y \in K$ such that $h(y) = \frac{1}{2}g(x^\star)$, then we have

$$\langle N(Ax^\star, Tx^\star), g(x^\star)\rangle + F(g(x^\star)) \leq 0.$$

Hence,

$$\langle N(Ax^\star, Tx^\star), g(x^\star)\rangle + F(g(x^\star)) \in P \cap \{-P\}.$$

Since P is a pointed cone

$$\langle N(Ax^\star, Tx^\star), g(x^\star)\rangle + F(g(x^\star)) = 0.$$

Thus, we obtain

$$\begin{aligned}
&\langle N(Ax^\star,\, Tx^\star),\, h(y)\rangle + F(h(y)) \\
=&[\langle N(Ax^\star,\, Tx^\star),\, h(y) - g(x^\star))\rangle + F(h(y)) - F(g(x^\star))] \\
&+ [\langle N(Ax^\star,\, Tx^\star),\, g(x^\star)\rangle + F(g(x^\star))] \\
=&\langle N(Ax^\star,\, Tx^\star),\, h(y) - g(x^\star)\rangle + F(h(y)) - F(g(x^\star)) \\
\geq& 0
\end{aligned}$$

for all $y \in K$. □

If A, T and h are identity mappings on K, then we have the following result.

Corollary 2.1 [5].
(i) If $x^\star$ solves (VF-ICP), then it solves (VF-IVIP).

(ii) Let $F : K \to Y$ be positively homogeneous.
If $x^\star$ solves (VF-IVIP), then it solves (VF-ICP).

Now we consider the existence of solutions to (MVF-IVIP) and the solution sets.

Theorem 2.2. *Assume that*

(a) mappings $N : L(X,Y) \times L(X,Y) \to L(X,Y)$, $A, T : K \to L(X,Y)$ and $F : K \to Y$ are continuous, and g, $h : K \to K$ are continuous and h is surjective;

(b) there exists a mapping $i : K \times K \to Y$ such that

(i) *$i(x,x) \geq 0$ for all $x \in K$;*
(ii) *$\langle N(Ax,Tx),\, h(y) - g(x)\rangle + F(h(y)) - F(g(x)) - i(x,y) \geq 0$ for all $x, y \in K$;*
(iii) *the set $\{y \in K : i(x,y) \ngeq 0\}$ is convex for all $x \in K$;*

(c) there exists a nonempty compact convex subset C of K such that for all $x \in K \setminus C$ there exists $y \in C$ such that

$$\langle N(Ax, Tx),\, h(y) - g(x)\rangle + F(h(y)) - F(g(x)) \ \not\geq\ 0.$$

Then (MVF-IVIP) has a solution. Furthermore, the solution set is closed.

Proof. First we define a mapping $G : K \to 2^C$ by

$$G(y) \ = \{x \in C : \langle N(Ax, Tx),\, h(y) - g(x)\rangle + F(h(y)) - F(g(x)) \ \geq\ 0\}, \quad \text{for all } y \in K.$$

By the assumption (a), for any $y \in K$, $G(y)$ is closed in C. Since every element $x^\star \in \bigcap_{y \in K} G(y)$ is a solution of (MVF-IVIP), we have to show that $\bigcap_{y \in K} G(y) \neq \emptyset$. Since C is compact it is sufficient to prove that the family $\{G(y)\}_{y \in K}$ has the finite intersection property. Let $\{y_1, y_2, \ldots, y_n\}$ be a finite subset of K and set $B := \overline{\text{conv}}(C \cup \{y_1, y_2, \ldots, y_n\})$. Then B is a compact and convex subset of K.

Define mappings $F_1, F_2 : B \to 2^B$ as follows:

$$F_1(y) = \{x \in B : \langle N(Ax, Tx),\, h(y) - g(x)\rangle + F(h(y)) - F(g(x)) \ \geq\ 0\} \quad \text{for all } y \in B,$$

and

$$F_2(y) \ = \ \{x \in B : i(x, y) \ \geq\ 0\} \text{ for all } y \in B.$$

From the conditions (i) and (ii), we have

$$i(y, y) \ \geq\ 0$$

and

$$\langle N(Ay, Ty),\, h(y) - g(y)\rangle + F(h(y)) - F(g(y)) - i(y, y) \ \geq\ 0.$$

Now Lemma 2.1 implies

$$\langle N(Ay, Ty),\, h(y) - g(y)\rangle + F(h(y)) - F(g(y)) \ \geq\ 0$$

and so $F_1(y)$ is nonempty. Similarly, we can prove that for any $y \in K$, $F_1(y)$ is closed. Since $F_1(y)$ is a closed subset of a compact set B, we know that $F_1(y)$ is compact. Next, we prove that F_2 is a KKM-mapping. Suppose that there exists a finite subset $\{u_1, u_2, \ldots, u_n\}$ of B and $\lambda_i \geq 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$u = \sum_{i=1}^{n} \lambda_i u_i \notin \bigcup_{j=1}^{n} F_2(u_j).$$

Then

$$i(u, u_j) \ngeq 0, \ j = 1, 2, \ldots, n.$$

From the condition (iii), we have

$$i(u, u) \ngeq 0,$$

which contradicts the condition (i). Hence F_2 is a KKM-mapping. On the other hand, from the condition (ii), we have

$$F_2(y) \subseteq F_1(y) \text{ for all } y \in B.$$

Now $x \in F_2(y)$ implies that $i(x, y) \geq 0$ and by the condition (ii), we have

$$\langle N(Ax, Tx), h(y) - g(x) \rangle + F(h(y)) - F(g(x)) - i(x, y) \geq 0.$$

It follows from Lemma 2.1, that

$$\langle N(Ax, Tx), h(y) - g(x) \rangle + F(h(y)) - F(g(x)) \geq 0,$$

i.e., $x \in F_1(y)$. Thus F_1 is a KKM mapping. From the Fan-KKM Theorem, there exists $x^\star \in B$, such that $x^\star \in F_1(y)$ for all $y \in B$. Hence

$$\langle N(Ax^\star, Tx^\star), h(y) - g(x^\star) \rangle + F(h(y)) - F(g(x^\star)) \geq 0 \text{ for all } y \in B.$$

By assumption (c), we get $x^\star \in C$ and moreover $x^\star \in G(y_i)$, $i = 1, 2, \ldots, n$. Hence $\{G(y)\}_{y \in K}$ has the finite intersection property.

Since $A, T : K \to L(X, Y)$, g, $h : K \to K$, $F : K \to Y$ and $N : L(X, Y) \times L(X, Y) \to L(X, Y)$ are continuous, the solution set of (MVF-IVIP) is obviously closed. □

Theorem 2.3. *Assume that* $A, T : K \to L(X,Y)$, $N : L(X,Y) \times L(X,Y) \to L(X,Y)$ *and* g, $h : K \to K$ *are continuous and* h *is surjective, and* $F : K \to Y$ *is positively homogeneous and continuous. If assumptions (b) and (c) in Theorem 2.2 hold, then (MVF-ICP) has a solution. Furthermore, the solution set of (MVF-ICP) is closed.*

Proof. The conclusion follows directly from Theorems 2.1 and 2.2. □

Corollary 2.2 [5]. *Assume that*

(a) $f : K \to L(X,Y)$, $g : K \to K$ *and* $F : K \to Y$ *are continuous,*

(b) there exists a mapping $i : K \times K \to Y$ *such that*

(i) $i(x,x) \geq 0$, *for all* $x \in K$;
(ii) $\langle f(x), y - g(x)\rangle + F(y) - F(g(x)) - i(x,y) \geq 0$ *for all* x, $y \in K$;
(iii) *the set* $\{y \in K : i(x,y) \not\geq 0\}$ *is convex, for all* $x \in K$;

(c) there exists a nonempty, compact, convex subset C *of* K *such that for all* $x \in K \setminus C$, *there exists* $y \in C$ *such that*

$$\langle f(x), y - g(x)\rangle + F(y) - F(g(x)) \not\geq 0.$$

Then (VF-IVIP) has a solution. Furthermore, the solution set is closed.

Remark 2.2. Putting $g = h$ in Theorems 2.1, 2.2 and 2.3, we obtain results in [4] as corollaries.

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