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INTUITIONISTIC FUZZINESS OF IMPLICATIVE IDEALS IN BCK-ALGEBRAS

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Abstract. After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Aranassov is one among them. In this paper, we apply the concept of an intuitionistic fuzzy set to implicative ideals in BCK-algebras. The notion of an intuitionistic fuzzy implicative ideal of a BCK-algebra is introduced, and some related properties are investigated. An extension property for intuitionistic fuzzy implicative ideals is established. Characterizations of an intuitionistic fuzzy implicative ideal are given. Conditions for an intuitionistic fuzzy ideal to be an intuitionistic fuzzy implicative ideal are given. Using a collection of implicative ideals, intuitionistic fuzzy implicative ideals are established.

1. Introduction

Logic appears in a 'sacred' form (resp. a 'profane' form) which is dominant in proof theory (resp. model theory). The role of logic in mathematics and computer science is two-fold – as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic including many-valued logic, fuzzy logic, etc, takes the advantage of the classical logic to handle information with various facets of uncertainty (see [9] for a generalized theory of uncertainty), such as fuzziness, randomness, and so on. Non-classical logic has become a formal and

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useful tool for computer science to deal with fuzzy information and uncertain information. Many-valued logic which is a great extension and development of classical logic [4] has emerged as a useful direction in non-classic logic. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life. Fuzzy sets, which were introduced by Zadeh [8], deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets (IFSs) introduced by Atanassov [1] is one among them. While fuzzy sets give the degree of membership of an element in a given set, intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. As for fuzzy sets, the degree of membership is a real number between 0 and 1. This is also the case for the degree of non-membership, and furthermore the sum of these two degrees is not greater than 1. For more details on intuitionistic fuzzy sets, we refer the reader to [1, 2]. Since then, a great number of theoretical and practical results appeared in the area of IFSs. There are numerical applications of IFSs in various areas of computer science, for example, in artificial intelligence, as well as in medicine, chemistry, economics, astronomy, etc. As applications of intuitionistic fuzzy sets, Davvaz et al. [5] applied the concept of an intuitionistic fuzzy set to H_v-modules. They introduced the notion of an intuitionistic fuzzy H_v-submodule of an H_v-module, and investigated related properties. They provided characterizations of intuitionistic fuzzy H_v-submodules. Jun et al. [7] discussed the intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras, and investigated some of their properties. They introduced the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BCK-algebra and investigated some related results. In this paper, we apply the concept of an intuitionistic fuzzy set to implicative ideals in BCK-algebras. We introduce the notion of an intuitionistic fuzzy implicative ideal of a BCK-algebra, and investigate some related properties. We establish an extension property for intuitionistic fuzzy implicative ideals. We give characterizations of an intuitionistic fuzzy implicative ideal. We also give conditions for an intuitionistic fuzzy ideal to be an intuitionistic fuzzy implicative ideal. Using a collection of implicative ideals, we establish intuitionistic fuzzy implicative ideals.

2. Preliminaries

By a *BCK-algebra* we mean an algebra $(X; *, 0)$ of type (2, 0) satisfying the axioms:

(a1) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(a2) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(a3) $(\forall x \in X)\ (x * x = 0, 0 * x = 0)$,

(a4) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$.

In any BCK-algebra X, the following hold:

(b1) $(\forall x \in X)\ (x * 0 = x)$,

(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,

(b3) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$,

(b4) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$.

A nonempty subset I of a BCK-algebra X is called an *ideal* of X if it satisfies

(c1) $0 \in I$,

(c2) $(\forall x \in X)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$.

A nonempty subset I of a BCK-algebra X is called an *implicative ideal* of X if it satisfies (c1) and

(c3) $(\forall x, y, z \in X)\ ((x * y) * z \in I, y * z \in I \Rightarrow x * z \in I)$.

A mapping $\mu : X \to [0, 1]$, where X is an arbitrary nonempty set, is called a *fuzzy set* in X. For any fuzzy set μ in X and any $t \in [0, 1]$ we define two sets

$$U(\mu; t) = \{x \in X \mid \mu(x) \geq t\} \text{ and } L(\mu; t) = \{x \in X \mid \mu(x) \leq t\},$$

which are called an *upper* and *lower t-level cut* of μ and can be used to the characterization of μ.

A fuzzy set μ in a BCK-algebra X is called a *fuzzy implicative ideal* of X if it satisfies

- $(\forall x \in X)\ (\mu(0) \geq \mu(x))$.
- $(\forall x, y, z \in X)\ (\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\})$.

As an important generalization of the notion of fuzzy sets in X, Atanassov [1, 2] introduced the concept of an *intuitionistic fuzzy set* (IFS for short) defined on a nonempty set X as objects having the form

$$A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle \mid x \in X\},$$

where the functions $\alpha_A : X \to [0, 1]$ and $\beta_A : X \to [0, 1]$ denote the *degree of membership* (namely $\alpha_A(x)$) and the *degree of nonmembership* (namely $\beta_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \alpha_A(x) + \beta_A(x) \leq 1$ for all $x \in X$.

Such defined objects are studied by many authors (see for example two journals: 1. *Fuzzy Sets and Systems* and 2. *Notes on Intuitionistic Fuzzy Sets*) and have many interesting applications not only in mathematics (see Chapter 5 in the book [3]).

For the sake of simplicity, we shall use the symbol $A = \langle X, \alpha_A, \beta_A\rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle \mid x \in X\}$.

3. Intuitionistic fuzzy ideals

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 3.1. [7] An IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in a BCK-algebra X is called an *intuitionistic fuzzy ideal* of X if it satisfies the following assertions:

(d1) $(\forall x \in X)$ $(\alpha_A(0) \geq \alpha_A(x),\ \beta_A(0) \leq \beta_A(x))$,

(d2) $(\forall x, y \in X)$ $(\alpha_A(x) \geq \min\{\alpha_A(x * y), \alpha_A(y)\})$,

(d3) $(\forall x, y \in X)$ $(\beta_A(x) \leq \max\{\beta_A(x * y), \beta_A(y)\})$.

For any $w \in X$ and any IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in X, we let

$$A(w) = \{x \in X \mid \alpha_A(x) \geq \alpha_A(w), \beta_A(x) \leq \beta_A(w)\}.$$

Obviously $w \in A(w)$. If $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy ideal of X, then $0 \in A(w)$. The following is our question: *For an IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in X satisfying* (d1), *is $A(w)$ an ideal of X*? But the following example provides a negative answer, that is, there exists an element $w \in X$ such that $A(w)$ is not an ideal of X.

Example 3.2. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X defined by

$$A = \left\langle X, \left(\tfrac{0}{0.9}, \tfrac{a}{0.7}, \tfrac{b}{0.3}, \tfrac{c}{0.2}, \tfrac{d}{0.5}\right), \left(\tfrac{0}{0.02}, \tfrac{a}{0.03}, \tfrac{b}{0.5}, \tfrac{c}{0.6}, \tfrac{d}{0.3}\right) \right\rangle.$$

Then $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies (d1), and it is not an intuitionistic fuzzy ideal of X because

$$\alpha_A(b) < \min\{\alpha_A(b * d),\ \alpha_A(d)\}$$

and/or

$$\beta_A(b) > \max\{\beta_A(b * d),\ \beta_A(d)\}.$$

Then $A(d) = \{0, a, d\}$ is not an ideal of X since $b * d = 0 \in A(d)$ and $d \in A(d)$, but $b \notin A(d)$. Note that $A(b) = \{0, a, b, d\}$ is an ideal of X.

We give conditions for the set $A(w)$ to be an ideal.

Theorem 3.3. *Let $w \in X$. If $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy ideal of X, then $A(w)$ is an ideal of X.*

Proof. Recall that $0 \in A(w)$. Let $x, y \in X$ be such that $x * y \in A(w)$ and $y \in A(w)$. Then $\alpha_A(w) \leq \alpha_A(x * y)$, $\beta_A(w) \geq \beta_A(x * y)$, $\alpha_A(w) \leq \alpha_A(y)$ and $\beta_A(w) \geq \beta_A(y)$. Since $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy ideal of X, it follows from (d2) and (d3) that

$$\alpha_A(x) \geq \min\{\alpha_A(x * y), \alpha_A(y)\} \geq \alpha_A(w),$$

$$\beta_A(x) \leq \max\{\beta_A(x * y), \beta_A(y)\} \leq \beta_A(w)$$

so that $x \in A(w)$. Therefore $A(w)$ is an ideal of X. □

Theorem 3.4. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X and $w \in X$.*

(i) *If $A(w)$ is an ideal of X, then $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies the following implications for all $x, y, z \in X$,*

$$\text{(3.1)} \quad \begin{array}{l} \alpha_A(x) \leq \min\{\alpha_A(y * z), \alpha_A(z)\} \Rightarrow \alpha_A(x) \leq \alpha_A(y) \\ \beta_A(x) \geq \max\{\beta_A(y * z), \beta_A(z)\} \Rightarrow \beta_A(x) \geq \beta_A(y) \end{array}$$

(ii) *If $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies* (d1) *and* (3.1), *then $A(w)$ is an ideal of X.*

Proof. (i) Assume that $A(w)$ is an ideal of X for each $w \in X$. Suppose that $\alpha_A(x) \leq \min\{\alpha_A(y*z), \alpha_A(z)\}$ and $\beta_A(x) \geq \max\{\beta_A(y*z), \beta_A(z)\}$ for all $x, y, z \in X$. Then $y * z \in A(x)$ and $z \in A(x)$. Since $A(x)$ is an ideal of X, it follows that $y \in A(x)$, that is, $\alpha_A(x) \leq \alpha_A(y)$ and $\beta_A(x) \geq \beta_A(y)$.

(ii) Suppose that $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies (d1) and (3.1). For each $w \in X$, let $x, y \in X$ be such that $x * y \in A(w)$ and $y \in A(w)$. Then $\alpha_A(x * y) \geq \alpha_A(w)$, $\beta_A(x * y) \leq \beta_A(w)$, $\alpha_A(y) \geq \alpha_A(w)$, and $\beta_A(y) \leq \beta_A(w)$, which imply that $\alpha_A(w) \leq \min\{\alpha_A(x * y), \alpha_A(y)\}$ and

$\beta_A(w) \geq \max\{\beta_A(x*y), \beta_A(y)\}$. Using (3.1), we have $\alpha_A(w) \leq \alpha_A(x)$ and $\beta_A(w) \geq \beta_A(x)$, and so $x \in A(w)$. Since $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies (d1), it follows that $0 \in A(w)$. Therefore $A(w)$ is an ideal of X. □

Lemma 3.5. [7] *Every intuitionistic fuzzy ideal $A = \langle X, \alpha_A, \beta_A \rangle$ of X satisfies the following implication.*

$$(\forall x, y \in X)\,(x \leq y \Rightarrow \alpha_A(x) \geq \alpha_A(y),\ \beta_A(x) \leq \beta_A(y)).$$

Proposition 3.6. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an intuitionistic fuzzy ideal of X. Then the following assertions are equivalent.*

(d4) $(\forall x, y \in X)\ (\alpha_A(x*y) \geq \alpha_A((x*y)*y), \beta_A(x*y) \leq \beta_A((x*y)*y))$.

(d5) $(\forall x, y, z \in X)\ (\alpha_A((x*z)*(y*z)) \geq \alpha_A((x*y)*z), \beta_A((x*z)*(y*z)) \leq \beta_A((x*y)*z))$.

Proof. Assume that the condition (d4) is valid. Note that

$$((x*(y*z))*z)*z = ((x*z)*(y*z))*z \leq (x*y)*z$$

for all $x, y, z \in X$ by using (b2), (b3), and (b4). It follows from Lemma 3.5 that

$$\alpha_A((x*y)*z) \leq \alpha_A(((x*(y*z))*z)*z),\ \beta_A((x*y)*z) \geq \beta_A(((x*(y*z))*z)*z)$$

so from (b2) and (d4) that

$$\alpha_A((x*z)*(y*z)) = \alpha_A((x*(y*z))*z) \geq \alpha_A(((x*(y*z))*z)*z) \geq \alpha_A((x*y)*z),$$

$$\beta_A((x*z)*(y*z)) = \beta_A((x*(y*z))*z) \leq \beta_A(((x*(y*z))*z)*z) \leq \beta_A((x*y)*z).$$

Thus (d5) holds. Now suppose that (d5) is valid. If we replace z by y in (d5), then

$$\alpha_A(x*y) = \alpha_A((x*y)*0) = \alpha_A((x*y)*(y*y)) \geq \alpha_A((x*y)*y),$$

$$\beta_A(x*y) = \beta_A((x*y)*0) = \beta_A((x*y)*(y*y)) \leq \beta_A((x*y)*y)$$

which proves (d4). □

Lemma 3.7. *An IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in X is an intuitionistic fuzzy ideal of X if and only if for all $x, y, z \in X$, $(x * y) * z = 0$ implies $\alpha_A(x) \geq \min\{\alpha_A(y), \alpha_A(z)\}$ and $\beta_A(x) \leq \max\{\beta_A(y), \beta_A(z)\}$.*

Proof. The necessity follows from [7, Lemma 3.6]. Conversely let $x, y, z \in X$ be such that if $(x*y)*z = 0$ then $\alpha_A(x) \geq \min\{\alpha_A(y), \alpha_A(z)\}$ and $\beta_A(x) \leq \max\{\beta_A(y), \beta_A(z)\}$. Since $(0 * y) * y = 0$, we have $\alpha_A(0) \geq \min\{\alpha_A(y), \alpha_A(y)\} = \alpha_A(y)$ and $\beta_A(0) \leq \max\{\beta_A(y), \beta_A(y)\} = \beta_A(y)$. Thus $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies the condition (d1). Note that $(x * (x * y)) * y = 0$ for all $x, y \in X$. It follows from the hypothesis that $\alpha_A(x) \geq \min\{\alpha_A(x * y), \alpha_A(y)\}$ and $\beta_A(x) \leq \max\{\beta_A(x * y), \beta_A(y)\}$. Hence $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy ideal of X. □

As a generalization of Lemma 3.7, we have the following results.

Theorem 3.8. *If an IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in X is an intuitionistic fuzzy ideal of X, then*

(d6) *for all $x, w_1, w_2, \cdots, w_n \in X$, $\prod_{i=1}^{n} x * w_i = 0$ implies*

$$\alpha_A(x) \geq \min\{\alpha_A(w_i) \mid i = 1, 2, \cdots, n\},$$

$$\beta_A(x) \leq \max\{\beta_A(w_i) \mid i = 1, 2, \cdots, n\}$$

where $\prod_{i=1}^{n} x * w_i = (\cdots((x * w_1) * w_2) * \cdots) * w_n$.

Proof. The proof is by induction on n. Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an intuitionistic fuzzy ideal of X. Lemmas 3.5 and 3.7 show that the condition (d6) is valid for $n = 1, 2$. Assume that $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies the condition (d6) for $n = k$, that is, for all $x, w_1, w_2, \cdots, w_k \in X$, $\prod_{i=1}^{k} x * w_i = 0$ implies $\alpha_A(x) \geq \min\{\alpha_A(w_i) \mid i = 1, 2, \cdots, k\}$ and $\beta_A(x) \leq \max\{\beta_A(w_i) \mid i = 1, 2, \cdots, k\}$. Let $x, w_1, w_2, \cdots, w_k, w_{k+1} \in X$ be such that $\prod_{i=1}^{k+1} x * w_i = 0$. Then

$$\alpha_A(x * w_1) \geq \min\{\alpha_A(w_j) \mid j = 2, 3, \cdots, k+1\},$$

$$\beta_A(x * w_1) \leq \max\{\beta_A(w_j) \mid j = 2, 3, \cdots, k+1\}.$$

Since $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy ideal of X, it follows from (d2) and (d3) that

$$\begin{aligned}\alpha_A(x) &\geq \min\{\alpha_A(x * w_1), \alpha_A(w_1)\} \\ &\geq \min\{\alpha_A(w_1), \min\{\alpha_A(w_j) \mid j = 2, 3, \cdots, k+1\}\} \\ &= \min\{\alpha_A(w_i) \mid i = 1, 2, \cdots, k+1\}.\end{aligned}$$

$$\begin{aligned}\beta_A(x) &\leq \max\{\beta_A(x * w_1), \beta_A(w_1)\} \\ &\leq \max\{\beta_A(w_1), \max\{\beta_A(w_j) \mid j = 2, 3, \cdots, k+1\}\} \\ &= \max\{\beta_A(w_i) \mid i = 1, 2, \cdots, k+1\}.\end{aligned}$$

This completes the proof. □

Now we consider the converse of Theorem 3.8.

Theorem 3.9. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X satisfying the condition* (d6). *Then $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy ideal of X.*

Proof. Note that $\big(\cdots((0 * \underbrace{x) * x) * \cdots\big) * x}_{n \text{ times}} = 0$ for all $x \in X$. It follows from (d6) that $\alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(0) \leq \beta_A(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $x * y \leq z$. Then

$$0 = (x * y) * z = \big(\cdots(((x * y) * z) * \underbrace{0) * \cdots\big) * 0}_{n-2 \text{ times}},$$

and so

$$\alpha_A(x) \geq \min\{\alpha_A(y), \alpha_A(z), \alpha_A(0)\} = \min\{\alpha_A(y), \alpha_A(z)\},$$

$$\beta_A(x) \leq \max\{\beta_A(y), \beta_A(z), \beta_A(0)\} = \max\{\beta_A(y), \beta_A(z)\}.$$

Hence, by Lemma 3.7, we conclude that $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy ideal of X. □

4. Intuitionistic fuzzy implicative ideals

Definition 4.1. An IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in X is called an *intuitionistic fuzzy implicative ideal* of X if it satisfies the condition (d1) and the following assertions:

(d7) $(\forall x, y, z \in X)$ $(\alpha_A(x * z) \geq \min\{\alpha_A((x * y) * z), \alpha_A(y * z)\})$,

(d8) $(\forall x, y, z \in X)$ $(\beta_A(x * z) \leq \max\{\beta_A((x * y) * z), \beta_A(y * z)\})$.

Example 4.2. Let $X = \{0, a, b\}$ be a set in which the operation $*$ is defined by the following table:

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	b	0

Then $(X; *, 0)$ is a BCK-algebra. Let $(t_0, s_0), (t_1, s_1), (t_2, s_2) \in [0, 1] \times [0, 1]$ satisfy $(t_0, s_0) > (t_1, s_1) > (t_2, s_2)$, that is, $t_0 > t_1 > t_2$, $s_0 < s_1 < s_2$, and $t_i + s_i \leq 1$ for $i = 1, 2, 3$. Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X given by

$$A = \left\langle X, \left(\tfrac{0}{t_0}, \tfrac{a}{t_1}, \tfrac{b}{t_2}\right), \left(\tfrac{0}{s_0}, \tfrac{a}{s_1}, \tfrac{b}{s_2},\right)\right\rangle,$$

that is, $\alpha_A(0) = t_0$, $\alpha_A(a) = t_1$, $\alpha_A(b) = t_2$, $\beta_A(0) = s_0$, $\beta_A(a) = s_1$, and $\beta_A(b) = s_2$. By routine calculations, we know that $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X.

Taking $z = 0$ in (d7) and (d8) and using (b1), we have the following theorem.

Theorem 4.3. *Every intuitionistic fuzzy implicative ideal is an intuitionistic fuzzy ideal.*

The following example shows that the converse of Theorem 4.3 is not true in general.

Example 4.4. Let $X = \{0, a, b, c\}$ be a set in which the operation $*$ is defined by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then $(X;*,0)$ is a BCK-algebra. Let $(t_0,s_0),(t_1,s_1),(t_2,s_2) \in [0,1] \times [0,1]$ satisfy $(t_0,s_0) > (t_1,s_1) > (t_2,s_2)$, that is, $t_0 > t_1 > t_2$, $s_0 < s_1 < s_2$, and $t_i + s_i \leq 1$ for $i = 1,2,3$. Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X given by

$$A = \left\langle X, \left(\tfrac{0}{t_0}, \tfrac{a}{t_1}, \tfrac{b}{t_1}, \tfrac{c}{t_2}\right), \left(\tfrac{0}{s_0}, \tfrac{a}{s_1}, \tfrac{b}{s_1}, \tfrac{c}{s_2}\right)\right\rangle.$$

By routine calculations, we know that $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy ideal of X, but not an intuitionistic fuzzy implicative ideal of X.

We now provide conditions for an intuitionistic fuzzy ideal to be an intuitionistic fuzzy implicative ideal.

Theorem 4.5. *An intuitionistic fuzzy ideal $A = \langle X, \alpha_A, \beta_A \rangle$ of X is an intuitionistic fuzzy implicative ideal of X if and only if it satisfies the condition* (d4).

Proof. Assume that $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X. If z is replaced by y in (d7) and (d8), then

$$\begin{aligned} \alpha_A(x*y) &\geq \min\{\alpha_A((x*y)*y), \alpha_A(y*y)\} \\ &= \min\{\alpha_A((x*y)*y), \alpha_A(0)\} \\ &= \alpha_A((x*y)*y), \end{aligned}$$

$$\begin{aligned} \beta_A(x*y) &\leq \max\{\beta_A((x*y)*y), \beta_A(y*y)\} \\ &= \max\{\beta_A((x*y)*y), \beta_A(0)\} \\ &= \beta_A((x*y)*y) \end{aligned}$$

which proves (d4). Conversely let $A = \langle X, \alpha_A, \beta_A \rangle$ be an intuitionistic fuzzy ideal of X satisfying the condition (d4). Note that

$$((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$$

for all $x,y,z \in X$. Using Lemma 3.5, we have

$$\alpha_A((x*y)*z) \leq \alpha_A(((x*z)*z)*(y*z)),$$

$$\beta_A((x*y)*z) \geq \beta_A(((x*z)*z)*(y*z)).$$

It follows from (d2), (d3) and (d4) that

$$\begin{aligned} \alpha_A(x*z) &\geq \alpha_A((x*z)*z) \\ &\geq \min\{\alpha_A(((x*z)*z)*(y*z)), \alpha_A(y*z)\} \\ &\geq \min\{\alpha_A((x*y)*z), \alpha_A(y*z)\}, \end{aligned}$$

$$\begin{aligned} \beta_A(x*z) &\leq \beta_A((x*z)*z) \\ &\leq \max\{\beta_A(((x*z)*z)*(y*z)), \beta_A(y*z)\} \\ &\leq \max\{\beta_A((x*y)*z), \beta_A(y*z)\}. \end{aligned}$$

Thus $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X. □

Combining Proposition 3.6 and Theorem 4.5, we have the following characterization of an intuitionistic fuzzy implicative ideal.

Theorem 4.6. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an intuitionistic fuzzy ideal of X. Then it is an intuitionistic fuzzy implicative ideal of X if and only if it satisfies the condition* (d5).

Theorem 4.7. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an intuitionistic fuzzy set in X. Then $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X if and only if it satisfies the condition* (d1) *and*

(d9) $(\forall x, y, z \in X)$ $(\alpha_A(x*y) \geq \min\{\alpha_A(((x*y)*y)*z), \alpha_A(z)\}, \beta_A(x*y) \leq \max\{\beta_A(((x*y)*y)*z), \beta_A(z)\})$.

Proof. Suppose that $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X. Then $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy ideal of X by Theorem 4.3, and so (d1) is true. From Theorem 4.6 it follows that $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies the condition (d5). Thus

$$\begin{aligned} \alpha_A(x*y) &\geq \min\{\alpha_A((x*y)*z), \alpha_A(z)\} \\ &= \min\{\alpha_A(((x*z)*y)*(y*y), \alpha_A(z)\} \\ &\geq \min\{\alpha_A(((x*z)*y)*y), \alpha_A(z)\} \\ &= \min\{\alpha_A(((x*y)*y)*z), \alpha_A(z)\}, \end{aligned}$$

$$\begin{aligned} \beta_A(x*y) &\leq \max\{\beta_A((x*y)*z), \beta_A(z)\} \\ &= \max\{\beta_A(((x*z)*y)*(y*y), \beta_A(z)\} \\ &\leq \max\{\beta_A(((x*z)*y)*y), \beta_A(z)\} \\ &= \max\{\beta_A(((x*y)*y)*z), \beta_A(z)\} \end{aligned}$$

which proves (d9). Conversely let $A = \langle X, \alpha_A, \beta_A \rangle$ be an intuitionistic fuzzy set in X satisfying conditions (d1) and (d9). Then

$$\alpha_A(x) = \alpha_A(x*0) \geq \min\{\alpha_A(((x*0)*0)*z), \alpha_A(z)\} = \min\{\alpha_A(x*z), \alpha_A(z)\},$$

$$\beta_A(x) = \beta_A(x*0) \leq \max\{\beta_A(((x*0)*0)*z), \beta_A(z)\} = \max\{\beta_A(x*z), \beta_A(z)\}.$$

Thus $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy ideal of X. Now taking $z = 0$ in (d9) and using (b1) and (d1), we have

$$\begin{aligned} \alpha_A(x * y) &\geq \min\{\alpha_A(((x * y) * y) * 0), \alpha_A(0)\} \\ &= \min\{\alpha_A((x * y) * y), \alpha_A(0)\} \\ &= \alpha_A((x * y) * y), \end{aligned}$$

$$\begin{aligned} \beta_A(x * y) &\leq \max\{\beta_A(((x * y) * y) * 0), \beta_A(0)\} \\ &= \max\{\beta_A((x * y) * y), \beta_A(0)\} \\ &= \beta_A((x * y) * y). \end{aligned}$$

It follows from Theorem 4.5 that $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X. □

Summarizing the above-mentioned results, we have a characterization of an intuitionistic fuzzy implicative ideal of X.

Theorem 4.8. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an intuitionistic fuzzy set in X. Then the following assertions are equivalent.*

(i) $A = \langle X, \alpha_A, \beta_A \rangle$ *is an intuitionistic fuzzy implicative ideal of* X.

(ii) $A = \langle X, \alpha_A, \beta_A \rangle$ *is an intuitionistic fuzzy ideal of* X *satisfying the condition* (d4).

(iii) $A = \langle X, \alpha_A, \beta_A \rangle$ *is an intuitionistic fuzzy ideal of* X *satisfying the condition* (d5).

(iv) $A = \langle X, \alpha_A, \beta_A \rangle$ *satisfies conditions* (d1) *and* (d9).

Theorem 4.9. *Let $w \in X$. If $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X, then $A(w)$ is an implicative ideal of X.*

Proof. Recall that $0 \in A(w)$. Let $x, y, z \in X$ be such that $(x * y) * z \in A(w)$ and $y * z \in A(w)$. Then $\alpha_A(w) \leq \alpha_A((x * y) * z)$, $\beta_A(w) \geq \beta_A((x * y) * z)$, $\alpha_A(w) \leq \alpha_A(y * z)$, and $\beta_A(w) \geq \beta_A(y * z)$.

Since $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X, it follows from (d7) and (d8) that

$$\alpha_A(x * z) \geq \min\{\alpha_A((x * y) * z),\, \alpha_A(y * z)\} \geq \alpha_A(w)$$

$$\beta_A(x * z) \leq \max\{\beta_A((x * y) * z),\, \beta_A(y * z)\} \leq \beta_A(w)$$

so that $x * z \in A(w)$. Therefore $A(w)$ is an implicative ideal of X. □

Theorem 4.10. *If $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X, then*

(d10) *for all $x, y, a, b \in X$, $((x * y) * y) * a \leq b$ implies $\alpha_A(x * y) \geq \min\{\alpha_A(a),\, \alpha_A(b)\}$ and $\beta_A(x * y) \leq \max\{\beta_A(a),\, \beta_A(b)\}$.*

(d11) *for all $x, y, z, a, b \in X$, $((x*y)*z)*a \leq b$ implies $\alpha_A((x*z)*(y*z)) \geq \min\{\alpha_A(a),\, \alpha_A(b)\}$ and $\beta_A((x * z) * (y * z)) \leq \max\{\beta_A(a),\, \beta_A(b)\}$.*

Proof. Let $x, y, a, b \in X$ be such that $((x*y)*y)*a \leq b$. Using Lemma 3.7, we have $\alpha_A((x * y) * y) \geq \min\{\alpha_A(a),\, \alpha_A(b)\}$ and $\beta_A((x * y) * y) \leq \max\{\beta_A(a),\, \beta_A(b)\}$. It follows that

$$\begin{aligned} \alpha_A(x * y) &\geq \min\big\{\alpha_A((x * y) * y),\, \alpha_A(y * y)\big\} \\ &= \min\big\{\alpha_A((x * y) * y),\, \alpha_A(0)\big\} \\ &= \alpha_A\big((x * y) * y\big) \\ &\geq \min\{\alpha_A(a),\, \alpha_A(b)\} \end{aligned}$$

and

$$\begin{aligned} \beta_A(x * y) &\leq \max\big\{\beta_A((x * y) * y),\, \beta_A(y * y)\big\} \\ &= \max\big\{\beta_A((x * y) * y),\, \beta_A(0)\big\} \\ &= \beta_A\big((x * y) * y\big) \\ &\leq \max\{\beta_A(a),\, \beta_A(b)\}. \end{aligned}$$

Now let $x, y, z, a, b \in X$ be such that $\big((x * y) * z\big) * a \leq b$, that is,

$$\big(((x * y) * z) * a\big) * b = 0.$$

Since $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X, it follows from Theorem 4.6 and Lemma 3.7 that

$$\alpha_A\big((x * z) * (y * z)\big) \geq \alpha_A\big((x * y) * z\big) \geq \min\{\alpha_A(a),\, \alpha_A(b)\}$$

and

$$\beta_A\big((x * z) * (y * z)\big) \leq \beta_A\big((x * y) * z\big) \leq \max\{\beta_A(a),\, \beta_A(b)\}.$$

This completes the proof. □

Theorem 4.11. *Let* $A = \langle X, \alpha_A, \beta_A \rangle$ *be an IFS in* X *satisfying the condition* (d10). *Then* $A = \langle X, \alpha_A, \beta_A \rangle$ *is an intuitionistic fuzzy implicative ideal of* X*.*

Proof. We first prove that $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy ideal of X. Let $x, y, z \in X$ be such that $x * y \leq z$. Then

$$\big(((x * 0) * 0) * y\big) * z = (x * y) * z = 0, \text{ that is, } ((x * 0) * 0) * y \leq z,$$

which implies from (b1) and (d10) that $\alpha_A(x) = \alpha_A(x*0) \geq \min\{\alpha_A(y),\, \alpha_A(z)\}$ and $\beta_A(x) = \beta_A(x*0) \leq \max\{\beta_A(y),\, \beta_A(z)\}$. Therefore, by Lemma 3.7, we know that $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy ideal of X. Note that $\big(((x * y) * y) * ((x * y) * y)\big) * 0 = 0$ for all $x, y \in X$. Using (d10) and (d1), we have

$$\alpha_A(x * y) \geq \min\big\{\alpha_A((x * y) * y),\, \alpha_A(0)\big\} = \alpha_A\big((x * y) * y\big),$$

$$\beta_A(x * y) \leq \max\big\{\beta_A((x * y) * y),\, \beta_A(0)\big\} = \beta_A\big((x * y) * y\big),$$

and so $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X by Theorem 4.5. □

Theorem 4.12. *Let* $A = \langle X, \alpha_A, \beta_A \rangle$ *be an IFS in* X *satisfying* (d11). *Then* $A = \langle X, \alpha_A, \beta_A \rangle$ *is an intuitionistic fuzzy implicative ideal of* X*.*

Proof. Let $x, y, a, b \in X$ be such that $\big((x * y) * y\big) * a \leq b$, that is,

$$\big(((x * y) * y) * a\big) * b = 0.$$

Then

$$\begin{aligned} \alpha_A(x * y) &= \alpha_A\big((x * y) * 0\big) = \alpha_A\big((x * y) * (y * y)\big) \\ &\geq \min\{\alpha_A(a),\, \alpha_A(b)\}, \\ \beta_A(x * y) &= \beta_A\big((x * y) * 0\big) = \beta_A\big((x * y) * (y * y)\big) \\ &\leq \max\{\beta_A(a),\, \beta_A(b)\}, \end{aligned}$$

and so $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X by Theorem 4.11. □

Corollary 4.13. *If $A = \langle X, \alpha_A, \beta_A\rangle$ is an intuitionistic fuzzy implicative ideal of X, then $\alpha_A\big((x*z)*(y*z)\big) \geq \min\{\alpha_A(w_i) \mid i = 1, 2, \cdots, n\}$ and $\beta_A\big((x*z)*(y*z)\big) \leq \max\{\beta_A(w_i) \mid i = 1, 2, \cdots, n\}$ whenever $\prod_{i=1}^{n}\big((x*y)*z\big)*w_i = 0$ for all $x, y, z, w_1, \cdots, w_n \in X$.*

Proof. Let $x, y, z, w_1, \cdots, w_n \in X$ be such that $\prod_{i=1}^{n}\big((x*y)*z\big)*w_i = 0$. Then

$$\alpha_A\big((x*z)*(y*z)\big) \geq \alpha_A\big((x*y)*z\big) \geq \min\{\alpha_A(w_i) \mid i = 1, 2, \cdots, n\}$$

and

$$\beta_A\big((x*z)*(y*z)\big) \leq \beta_A\big((x*y)*z\big) \leq \max\{\beta_A(w_i) \mid i = 1, 2, \cdots, n\}.$$

This completes the proof. □

Theorem 4.14. (Extension property) *Let $A = \langle X, \alpha_A, \beta_A\rangle$ and $B = \langle X, \alpha_B, \beta_B\rangle$ be intuitionistic fuzzy ideals of X such that $A(0) = B(0)$, i.e., $\alpha_A(0) = \alpha_B(0)$ and $\beta_A(0) = \beta_B(0)$, and $A \subset B$, i.e., $\alpha_A(x) \leq \alpha_B(x)$ and $\beta_A(x) \geq \beta_B(x)$ for all $x \in X$. If $A = \langle X, \alpha_A, \beta_A\rangle$ is an intuitionistic fuzzy implicative ideal of X, then so is $B = \langle X, \alpha_B, \beta_B\rangle$.*

Proof. Assume that $A = \langle X, \alpha_A, \beta_A\rangle$ is an intuitionistic fuzzy implicative ideal of X. For any $x, y, z \in X$, we have

$$\begin{aligned}
&\alpha_B(((x*z)*(y*z))*((x*y)*z)) \\
&= \alpha_B(((x*z)*((x*y)*z))*(y*z)) \\
&= \alpha_B(((x*((x*y)*z))*z)*(y*z)) \\
&\geq \alpha_A(((x*((x*y)*z))*z)*(y*z)) \\
&\geq \alpha_A(((x*((x*y)*z))*y)*z) \\
&= \alpha_A(((x*y)*((x*y)*z))*z) \\
&= \alpha_A(((x*y)*z)*((x*y)*z)) \\
&= \alpha_A(0) = \alpha_B(0)
\end{aligned}$$

and

$$\begin{aligned}
&\beta_B(((x*z)*(y*z))*((x*y)*z)) \\
&\quad = \beta_B(((x*z)*((x*y)*z))*(y*z)) \\
&\quad = \beta_B(((x*((x*y)*z))*z)*(y*z)) \\
&\quad \leq \beta_A(((x*((x*y)*z))*z)*(y*z)) \\
&\quad \leq \beta_A(((x*((x*y)*z))*y)*z) \\
&\quad = \beta_A(((x*y)*((x*y)*z))*z) \\
&\quad = \beta_A(((x*y)*z)*((x*y)*z)) \\
&\quad = \beta_A(0)=\beta_B(0).
\end{aligned}$$

It follows from (d1), (d2) and (d3) that

$$\begin{aligned}
&\alpha_B((x*z)*(y*z)) \\
&\quad \geq \min\{\alpha_B(((x*z)*(y*z))*((x*y)*z)), \alpha_B((x*y)*z)\} \\
&\quad \geq \min\{\alpha_B(0), \alpha_B((x*y)*z)\} \\
&\quad = \alpha_B((x*y)*z)
\end{aligned}$$

and

$$\begin{aligned}
&\beta_B((x*z)*(y*z)) \\
&\quad \leq \max\{\beta_B(((x*z)*(y*z))*((x*y)*z)), \beta_B((x*y)*z)\} \\
&\quad \leq \max\{\beta_B(0), \beta_B((x*y)*z)\} \\
&\quad = \beta_B((x*y)*z)
\end{aligned}$$

for all $x, y, z \in X$. Hence, by Theorem 4.6, $B = \langle X, \alpha_B, \beta_B\rangle$ is an intuitionistic fuzzy implicative ideal of X. □

Lemma 4.15. *An IFS $A = \langle X, \alpha_A, \beta_A\rangle$ is an intuitionistic fuzzy implicative ideal of X if and only if the fuzzy sets α_A and $\overline{\beta_A}$ are fuzzy implicative ideals of X, where $\overline{\beta_A}$ is a fuzzy set in X defined by $\overline{\beta_A}(x) = 1 - \beta_A(x)$ for all $x \in X$.*

Proof. Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an intuitionistic fuzzy implicative ideal of X. Clearly α_A is a fuzzy implicative ideal of X. Let $x, y, z \in X$. Then

$$\overline{\beta_A}(0) = 1 - \beta_A(0) \geq 1 - \beta_A(x) = \overline{\beta_A}(x),$$

$$\begin{aligned} \overline{\beta_A}(x * z) &= 1 - \beta_A(x * z) \ \geq \ 1 - \max\{\beta_A((x * y) * z), \beta_A(y * z)\} \\ &= \min\{1 - \beta_A((x * y) * z), 1 - \beta_A(y * z)\} \\ &= \min\{\overline{\beta_A}((x * y) * z), \overline{\beta_A}(y * z)\}. \end{aligned}$$

Hence $\overline{\beta_A}$ is a fuzzy implicative ideal of X. Conversely, assume that α_A and $\overline{\beta_A}$ are fuzzy implicative ideals of X. For every $x \in X$, we have $\alpha_A(0) \geq \alpha_A(x)$ and

$$1 - \beta_A(0) = \overline{\beta_A}(0) \geq \overline{\beta_A}(x) = 1 - \beta_A(x),$$

which shows that $\beta_A(0) \leq \beta_A(x)$. This shows that $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies the condition (d1). Now let $x, y, z \in X$. Then

$$\alpha_A(x * z) \geq \min\{\alpha_A((x * y) * z), \alpha_A(y * z)\}$$

and

$$\begin{aligned} 1 - \beta_A(x * z) &= \overline{\beta_A}(x * z) \ \geq \ \min\{\overline{\beta_A}((x * y) * z), \overline{\beta_A}(y * z)\} \\ &= \min\{1 - \beta_A((x * y) * z), 1 - \beta_A(y * z)\} \\ &= 1 - \max\{\beta_A((x * y) * z), \beta_A(y * z)\}, \end{aligned}$$

and so $\beta_A(x*z) \leq \max\{\beta_A((x*y)*z), \beta_A(y*z)\}$. Hence $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X. □

Theorem 4.16. *An IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in X is an intuitionistic fuzzy implicative ideal of X if and only if $\Box A := \langle X, \alpha_A, \overline{\alpha_A} \rangle$ and $\Diamond A := \langle X, \overline{\beta_A}, \beta_A \rangle$ are intuitionistic fuzzy implicative ideals of X.*

Proof. It is straightforward by Lemma 4.15. □

Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X and let $m, n \in [0, 1]$ be such that $m + n \leq 1$. Then the set

$$X_A^{(m,n)} := \{x \in X \mid \alpha_A(x) \geq m, \ \beta_A(x) \leq n\}$$

is called an (m, n)*-level subset* of $A = \langle X, \alpha_A, \beta_A \rangle$.

Theorem 4.17. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an intuitionistic fuzzy implicative ideal of X. Then for each $m, n \in [0,1]$ with $m \leq \alpha_A(0)$, $n \geq \beta_A(0)$ and $m + n \leq 1$, the (m,n)-level subset $X_A^{(m,n)}$ is an implicative ideal of X.*

Proof. Obviously $0 \in X_A^{(m,n)}$. Let $x, y, z \in X$ be such that $(x*y)*z \in X_A^{(m,n)}$ and $y*z \in X_A^{(m,n)}$. Then $\alpha_A((x*y)*z) \geq m$, $\beta_A((x*y)*z) \leq n$, $\alpha_A(y*z) \geq m$, and $\beta_A(y*z) \leq n$. It follows from (d7) and (d8) that

$$\alpha_A(x*z) \geq \min\{\alpha_A((x*y)*z), \alpha_A(y*z)\} \geq m,$$

$$\beta_A(x*z) \leq \max\{\beta_A((x*y)*z), \beta_A(y*z)\} \leq n$$

so that $x*z \in X_A^{(m,n)}$. Hence $X_A^{(m,n)}$ is an implicative ideal of X. □

Theorem 4.18. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X. Assume that $X_A^{(m,n)}$ is an implicative ideal of X for every $m, n \in [0,1]$ with $m+n \leq 1$. Then $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X.*

Proof. For any $x \in X$, let $\alpha_A(x) = m$ and $\beta_A(x) = n$, where $m + n \leq 1$. Since $0 \in X_A^{(m,n)}$, it follows that $\alpha_A(0) \geq m = \alpha_A(x)$ and $\beta_A(0) \leq n = \beta_A(x)$. For any $x, y, z \in X$, let $A((x*y)*z) = (m_1, n_1)$ and $A(y*z) = (m_2, n_2)$, i.e., $\alpha_A((x*y)*z) = m_1$, $\beta_A((x*y)*z) = n_1$, $\alpha_A(y*z) = m_2$, and $\beta_A(y*z) = n_2$, where $m_i + n_i \leq 1$ for $i = 1, 2$. Then $(x*y)*z \in X_A^{(m_1,n_1)}$ and $y*z \in X_A^{(m_2,n_2)}$. It follows that $(x*y)*z, y*z \in X_A^{(\min(m_1,m_2),\max(n_1,n_2))}$. Then $y*z \in X_A^{(\min(m_1,m_2),\max(n_1,n_2))}$. Using (c3), we have $x*z \in X_A^{(\min(m_1,m_2),\max(n_1,n_2))}$, and so

$$\alpha_A(x*z) \geq \min(m_1, m_2) = \min\{\alpha_A((x*y)*z), \alpha_A(y*z)\},$$

$$\beta_A(x*z) \leq \max(n_1, n_2) = \max\{\beta_A((x*y)*z), \beta_A(y*z)\}.$$

Therefore $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X. □

Note that

$$\begin{aligned} X_A^{(m,n)} &= \{x \in X \mid \alpha_A(x) \geq m, \beta_A(x) \leq n\} \\ &= \{x \in X \mid \alpha_A(x) \geq m\} \cap \{x \in X \mid \beta_A(x) \leq n\} \\ &= U(\alpha_A; m) \cap L(\beta_A; n). \end{aligned}$$

Hence we have the following corollary.

Corollary 4.19. *An IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in X is an intuitionistic fuzzy implicative ideal of X if and only if $U(\alpha_A; m)$ and $L(\beta_A; n)$ are implicative ideals of X for all $m \in [0, \alpha_A(0)]$ and $n \in [\beta_A(0), 1]$ with $m + n \leq 1$.*

Corollary 4.20. *Let I be an implicative ideal of X and let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X defined by*

$$\alpha_A(x) := \begin{cases} m_0 & \text{if } x \in I, \\ m_1 & \text{otherwise,} \end{cases} \qquad \beta_A(x) := \begin{cases} n_0 & \text{if } x \in I, \\ n_1 & \text{otherwise,} \end{cases}$$

for all $x \in X$ where $0 \leq m_1 < m_0$, $0 \leq n_0 < n_1$ and $m_i + n_i \leq 1$ for $i = 1, 2$. Then $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X.

Theorem 4.21. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an IFS in X and*

$$\mathrm{Im}(A) = \{(m_0, n_0), (m_1, n_1), \cdots, (m_k, n_k)\}$$

where $(m_i, n_i) < (m_j, n_j)$, that is, $m_i < m_j$ and $n_i > n_j$ whenever $i > j$. Let $\{G_r \mid r = 0, 1, \cdots, k\}$ be a family of implicative ideals of X such that

- *$G_0 \subset G_1 \subset \cdots \subset G_k = X$,*
- *$A(G_r^*) = (m_r, n_r)$, i.e., $\alpha_A(G_r^*) = m_r$ and $\beta_A(G_r^*) = n_r$, where $G_r^* = G_r \setminus G_{r-1}$, $G_{-1} = \emptyset$ for $r = 0, 1, \cdots, k$.*

Then $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X.

Proof. Since $0 \in G_0$, we have $\alpha_A(0) = m_0 \geq \alpha_A(x)$ and $\beta_A(0) = n_0 \leq \beta_A(x)$ for all $x \in X$. Let $x, y, z \in X$. To prove that $A = \langle X, \alpha_A, \beta_A \rangle$ satisfies conditions (d7) and (d8), we discuss the following cases: If $(x * y) * z \in G_r^*$ and $y * z \in G_r^*$, then $x * z \in G_r$ because G_r is an implicative ideal of X. Thus

$$\alpha_A(x * z) \geq m_r = \min\{\alpha_A((x * y) * z), \alpha_A(y * z)\},$$

$$\beta_A(x * z) \leq n_r = \max\{\beta_A((x * y) * z), \beta_A(y * z)\}.$$

If $(x * y) * z \notin G_r^*$ and $y \notin G_r^*$, then the following four cases arise:

1. $(x * y) * z \in X \setminus G_r$ and $y * z \in X \setminus G_r$,
2. $(x * y) * z \in G_{r-1}$ and $y * z \in G_{r-1}$,
3. $(x * y) * z \in X \setminus G_r$ and $y * z \in G_{r-1}$,
4. $(x * y) * z \in G_{r-1}$ and $y * z \in X \setminus G_r$.

But, in either case, we know that

$$\alpha_A(x * z) \geq \min\{\alpha_A((x * y) * z), \alpha_A(y * z)\},$$

$$\beta_A(x * z) \leq \max\{\beta_A((x * y) * z), \beta_A(y * z)\}.$$

If $(x*y)*z \in G_r^*$ and $y \notin G_r^*$, then either $y*z \in G_{r-1}$ or $y*z \in X \setminus G_r$. It follows that either $x * z \in G_r$ or $x * z \in X \setminus G_r$. Thus

$$\alpha_A(x * z) \geq \min\{\alpha_A((x * y) * z), \alpha_A(y * z)\},$$

$$\beta_A(x * z) \leq \max\{\beta_A((x * y) * z), \beta_A(y * z)\}.$$

If $(x * y) * z \notin G_r^*$ and $y * z \in G_r^*$, then by similar process we have

$$\alpha_A(x * z) \geq \min\{\alpha_A((x * y) * z), \alpha_A(y * z)\},$$

$$\beta_A(x * z) \leq \max\{\beta_A((x * y) * z), \beta_A(y * z)\}.$$

Therefore $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X. □

Theorem 4.22. *Let $\{G_m \mid m \in \Lambda \subseteq [0, \frac{1}{2}]\}$ be a finite collection of implicative ideals of X such that $X = \bigcup_{m\in\Lambda} G_m$, and for every $m, n \in \Lambda$, $m < n$ if and only if $G_n \subset G_m$. Then an IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in X defined by*

$$\alpha_A(x) = \sup\{m \in \Lambda \mid x \in G_m\} \text{ and } \beta_A(x) = \inf\{m \in \Lambda \mid x \in G_m\}$$

for all $x \in X$ is an intuitionistic fuzzy implicative ideal of X.

Proof. According to Corollary 4.19, it is sufficient to show that the nonempty sets $U(\alpha_A; m)$ and $L(\beta_A; n)$ are implicative ideals of X for every $m, n \in [0, 1]$ with $m + n \leq 1$. In order to show that $U(\alpha_A; m)$ is an implicative ideal, we divide into the following two cases:

$$\text{(i) } m = \sup\{k \in \Lambda \mid k < m\} \text{ and (ii) } m \neq \sup\{k \in \Lambda \mid k < m\}.$$

Case (i) implies that

$$\begin{aligned} x \in U(\alpha_A; m) &\Leftrightarrow x \in G_k \text{ for all } k < m \\ &\Leftrightarrow x \in \bigcap_{k<m} G_k, \end{aligned}$$

so that $U(\alpha_A; m) = \bigcap_{k<m} G_k$, which is an implicative ideal of X. For the case (ii), we claim that $U(\alpha_A; m) = \bigcup_{k \geq m} G_k$. If $x \in \bigcup_{k \geq m} G_k$, then $x \in G_k$ for some $k \geq m$. It follows that $\alpha_A(x) \geq k \geq m$ so that $x \in U(\alpha_A; m)$. This proves that $\bigcup_{k \geq m} G_k \subset U(\alpha_A; m)$. Now assume that $x \notin \bigcup_{k \geq m} G_k$. Then $x \notin G_k$ for all $k \geq m$. Since $m \neq \sup\{k \in \Lambda \mid k < m\}$, there exists $\varepsilon > 0$ such that $(m - \varepsilon, m) \cap \Lambda = \emptyset$. Hence $x \notin G_k$ for all $k > m - \varepsilon$, which means that if $x \in G_k$ then $k \leq m - \varepsilon$. Thus $\alpha_A(x) \leq m - \varepsilon < m$, and so $x \notin U(\alpha_A; m)$. Therefore $U(\alpha_A; m) = \bigcup_{k \geq m} G_k$. Next we show that $L(\beta_A; n)$ is an implicative ideal of X for all $n \in [\beta_A(0), 1]$. We consider the following two cases:

$$\text{(iii) } n = \inf\{k \in \Lambda \mid n < k\} \text{ and (iv) } n \neq \inf\{k \in \Lambda \mid n < k\}.$$

For the case (iii) we have

$$\begin{aligned} x \in L(\beta_A; n) &\Leftrightarrow x \in G_k \text{ for all } n < k \\ &\Leftrightarrow x \in \bigcap_{n<k} G_k, \end{aligned}$$

and hence $L(\beta_A; n) = \bigcap_{n<k} G_k$, which is an implicative ideal of X. For the case (iv), we will show that $L(\beta_A; n) = \bigcup_{n \geq k} G_k$. If $x \in \bigcup_{n \geq k} G_k$, then $x \in G_k$ for some $n \geq k$. It follows that $\beta_A(x) \leq k \leq n$ so that $x \in L(\beta_A; n)$. Hence $\bigcup_{n \geq k} G_k \subset L(\beta_A; n)$. Conversely, if $x \notin \bigcup_{n \geq k} G_k$ then $x \notin G_k$ for all $k \leq n$. Since $n \neq \inf\{k \in \Lambda \mid n < k\}$, there exists $\varepsilon > 0$ such that $(n, n + \varepsilon) \cap \Lambda = \emptyset$, which implies that $x \notin G_k$ for all $k < n + \varepsilon$, that is, if $x \in G_k$ then $k \geq n + \varepsilon$. Thus $\beta_A(x) \geq n + \varepsilon > n$, that is, $x \notin L(\beta_A; n)$. Therefore $L(\beta_A; n) \subset \bigcup_{n \geq k} G_k$ and consequently $L(\beta_A; n) = \bigcup_{n \geq k} G_k$. This completes the proof. □

Theorem 4.23. *Let $A = \langle X, \alpha_A, \beta_A \rangle$ be an intuitionistic fuzzy implicative ideal of X with the finite image. Then every descending chain of implicative ideals of X terminates at finite step.*

Proof. Suppose that there exists a strictly descending chain $G_0 \supset G_1 \supset G_2 \supset \cdots$ of implicative ideals of X which does not terminate at finite step. Define an IFS $A = \langle X, \alpha_A, \beta_A \rangle$ in X by

$$\alpha_A(x) := \begin{cases} \frac{n}{n+1} & \text{if } x \in G_n \setminus G_{n+1}, n = 0, 1, 2, \cdots, \\ 1 & \text{if } x \in \bigcap_{n=0}^{\infty} G_n, \end{cases}$$

$$\beta_A(x) := \begin{cases} \frac{1}{n+1} & \text{if } x \in G_n \setminus G_{n+1}, n = 0, 1, 2, \cdots, \\ 0 & \text{if } x \in \bigcap_{n=0}^{\infty} G_n, \end{cases}$$

where G_0 stands for X. We prove that $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X. Clearly $\alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(x) \leq \beta_A(0)$ for all $x \in X$. Let $x, y, z \in X$. Assume that $(x * y) * z \in G_n \setminus G_{n+1}$ and $y * z \in G_k \setminus G_{k+1}$ for $n = 0, 1, 2, \cdots; k = 0, 1, 2, \cdots$. Without loss of generality, we may assume that $n \leq k$. Then obviously $y * z \in G_n$, and so $x * z \in G_n$ because G_n is an implicative ideal of X. Hence

$$\alpha_A(x * z) \geq \tfrac{n}{n+1} = \min\{\alpha_A((x * y) * z), \alpha_A(y * z)\},$$

$$\beta_A(x * z) \leq \tfrac{1}{n+1} = \max\{\beta_A((x * y) * z), \beta_A(y * z)\}.$$

If $(x * y) * z, y * z \in \bigcap_{n=0}^{\infty} G_n$, then $x * z \in \bigcap_{n=0}^{\infty} G_n$. Thus

$$\alpha_A(x * z) = 1 = \min\{\alpha_A((x * y) * z), \alpha_A(y * z)\},$$

$$\beta_A(x * z) = 0 = \max\{\beta_A((x * y) * z), \beta_A(y * z)\}.$$

If $(x * y) * z \notin \bigcap_{n=0}^{\infty} G_n$ and $y * z \in \bigcap_{n=0}^{\infty} G_n$, then there exists $k \in \mathbb{N}$ such that $(x * y) * z \in G_k \setminus G_{k+1}$. It follows that $x * z \in G_k$ so that

$$\alpha_A(x * z) \geq \tfrac{k}{k+1} = \min\{\alpha_A((x * y) * z), \alpha_A(y * z)\},$$

$$\beta_A(x * z) \leq \tfrac{1}{k+1} = \max\{\beta_A((x * y) * z), \beta_A(y * z)\}.$$

Finally suppose that $(x * y) * z \in \bigcap_{n=0}^{\infty} G_n$ and $y \notin \bigcap_{n=0}^{\infty} G_n$. Then $y * z \in G_r \setminus G_{r+1}$ for some $r \in \mathbb{N}$. Hence $x * z \in G_r$, and so

$$\alpha_A(x * z) \geq \tfrac{r}{r+1} = \min\{\alpha_A((x * y) * z), \alpha_A(y * z)\},$$

$$\beta_A(x * z) \leq \tfrac{1}{r+1} = \max\{\beta_A((x * y) * z), \beta_A(y * z)\}.$$

Consequently, we conclude that $A = \langle X, \alpha_A, \beta_A \rangle$ is an intuitionistic fuzzy implicative ideal of X and $A = \langle X, \alpha_A, \beta_A \rangle$ has infinite number of different values. This is a contradiction, and the proof is complete. □

Finally, we consider the converse of Theorem 4.23.

Theorem 4.24. *Let X be a BCK-algebra in which every descending chain of implicative ideals terminates at finite step. For an intuitionistic fuzzy implicative ideal $A = \langle X, \alpha_A, \beta_A \rangle$ of X, if a sequence of elements of $\mathrm{Im}(A)$ is strictly intuitionistic increasing, that is, a sequence of elements of $\mathrm{Im}(\alpha_A)$ is strictly increasing and a sequence of elements of $\mathrm{Im}(\beta_A)$ is strictly decreasing, then $A = \langle X, \alpha_A, \beta_A \rangle$ has finite number of intuitionistic values, that is, α_A and β_A have finite number of values.*

Proof. Suppose that $\mathrm{Im}(\alpha_A)$ is not finite. Let $\{m_r\}$ be a strictly increasing sequence of elements of $\mathrm{Im}(\alpha_A)$. Then $0 \leq m_1 < m_2 < \cdots \leq 1$. Define

$$U(\alpha_A; t) := \{x \in X \mid \alpha_A(x) \geq m_t\}$$

for $t = 2, 3, \cdots$. Then $U(\alpha_A; t)$ is an implicative ideal of X. Let $x \in U(\alpha_A; t)$. Then $\alpha_A(x) \geq m_t > m_{t-1}$, which implies that $x \in U(\alpha_A; t-1)$. Hence $U(\alpha_A; t) \subseteq U(\alpha_A; t-1)$. Since $m_{t-1} \in \mathrm{Im}(\alpha_A)$, there exists $x_{t-1} \in X$ such that $\alpha_A(x_{t-1}) = m_{t-1}$. It follows that $x_{t-1} \in U(\alpha_A; t-1)$, but $x_{t-1} \notin U(\alpha_A; t)$. Thus $U(\alpha_A; t)$ is a proper subset of $U(\alpha_A; t-1)$, and so we obtain a strictly descending chain

$$U(\alpha_A; 1) \supset U(\alpha_A; 2) \supset U(\alpha_A; 3) \supset \cdots$$

of implicative ideals of X which is not terminating. This is a contradiction. Now assume that $\mathrm{Im}(\beta_A)$ is not finite. Let $\{n_r\}$ be a strictly decreasing sequence of elements of $\mathrm{Im}(\beta_A)$. Then $1 \geq n_1 > n_2 > n_3 > \cdots \geq 0$. Note that

$$L(\beta_A; k) := \{x \in X \mid \beta_A(x) \leq n_k\}$$

is an implicative ideal of X for $k = 2, 3, \cdots$. If $y \in L(\beta_A; k)$, then $\beta_A(y) \leq n_k < n_{k-1}$ and so $y \in L(\beta_A; k-1)$. This shows that $L(\beta_A; k) \subseteq L(\beta_A; k-1)$. Since $n_{k-1} \in \mathrm{Im}(\beta_A)$, we have $\beta_A(y_{k-1}) = n_{k-1}$ for some $y_{k-1} \in X$. Hence $y_{k-1} \in L(\beta_A; k-1)$, but $y_{k-1} \notin L(\beta_A; k)$. Therefore $L(\beta_A; k)$ is a proper subset of $L(\beta_A; k-1)$, and thus we get a strictly descending chain

$$L(\beta_A; 1) \supset L(\beta_A; 2) \supset L(\beta_A; 3) \supset \cdots$$

of implicative ideals of X which is not terminating. This is impossible, and the proof is complete. □

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