

## A NOTE ON FLIP SYSTEMS

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**Abstract.** A dynamical system with a skew-commuting involution map is called a flip system. Every flip system on a subshift of finite type is represented by a pair of matrices, one of which is a permutation matrix. The transposition number of this permutation matrix is studied. We define an invariant, called the flip number, that measures the complexity of a flip system, and prove some results on it. More properties of flips on subshifts of finite type with symmetric adjacency matrices are investigated.

### 1. Introduction

Let  $(X, T)$  be a topological dynamical system, where  $X$  is a topological space and  $T : X \rightarrow X$  a homeomorphism. A homeomorphism  $\phi : X \rightarrow X$  is called a *flip* for  $(X, T)$  if

$$T \circ \phi = \phi \circ T^{-1} \quad \text{and} \quad \phi^2 = id.$$

The triplet  $(X, T, \phi)$  is called a *flip system*.

A flip system  $(X, T, \phi)$  is said to be *conjugate* to another system  $(Y, S, \psi)$  if there is a homeomorphism  $\gamma : X \rightarrow Y$  such that

$$\gamma \circ T = S \circ \gamma \quad \text{and} \quad \gamma \circ \phi = \psi \circ \gamma.$$

In this case,  $\gamma$  is called a *conjugacy* from  $(X, T, \phi)$  to  $(Y, S, \psi)$ .

Throughout the work, let  $\mathcal{A}$  denote an alphabet, i.e., a finite set of symbols, and be equipped with the discrete topology. For  $x \in \mathcal{A}^{\mathbb{Z}}$ , we

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define  $\sigma x$  and  $\rho x$  in  $\mathcal{A}^{\mathbb{Z}}$  by

$$(\sigma x)_i = x_{i+1} \quad \text{and} \quad (\rho x)_i = x_{-i} \quad \text{for } i \in \mathbb{Z}.$$

Then  $\sigma$  and  $\rho$  are homeomorphisms of  $\mathcal{A}^{\mathbb{Z}}$ , and satisfy

$$\sigma \circ \rho = \rho \circ \sigma^{-1} \quad \text{and} \quad \rho^2 = id.$$

Hence  $\rho$  is a flip for the full  $\mathcal{A}$ -shift  $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ . The map  $\rho$  is called the *mirror map*, and  $\sigma$  is called the *shift map*.

Let  $A$  be an  $\mathcal{A} \times \mathcal{A}$ , 0-1 matrix. The subshift of finite type  $X_A$  is the closed  $\sigma$ -invariant subset of  $\mathcal{A}^{\mathbb{Z}}$  defined by

$$X_A = \{x = (x_i) \in \mathcal{A}^{\mathbb{Z}} : A_{x_i x_{i+1}} = 1 \text{ for } i \in \mathbb{Z}\}.$$

The restriction of the shift map on  $X_A$  is denoted by  $\sigma_A$  or simply by  $\sigma$ . The mirror map restricted on  $X_A$ , denoted by  $\rho_A$ , maps  $X_A$  onto  $X_{A^T}$ . If  $A = A^T$ , then  $X_A$  is  $\rho_A$ -invariant, and hence  $\rho_A$  is a flip for  $(X_A, \sigma_A)$ .

Suppose that  $A$  and  $P$  are  $\mathcal{A} \times \mathcal{A}$ , 0-1 matrices such that

$$(1.1) \quad AP = PA^T \quad \text{and} \quad P^2 = I.$$

For each  $a \in \mathcal{A}$ , let  $\tau(a) \in \mathcal{A}$  be defined by  $P(a, \tau(a)) = 1$ . Then  $\tau : \mathcal{A} \rightarrow \mathcal{A}$  is a bijection and  $\tau^2 = id$ . For  $x \in X_A$  define  $\varphi_{A,P} x \in X_A$  by

$$(\varphi_{A,P} x)_i = \tau(x_{-i}) \quad \text{for } i \in \mathbb{Z}.$$

We see that  $\sigma_A \circ \varphi_{A,P} = \varphi_{A,P} \circ \sigma_A^{-1}$  and  $(\varphi_{A,P})^2 = id$ . Thus  $\varphi_{A,P}$  is a flip for  $X_A$ . When we refer to  $(X_A, \sigma_A, \varphi_{A,P})$  as a flip system, we always assume that  $A$  and  $P$  are 0-1 matrices satisfying (1.1).

Every flip for a subshift of finite type can be represented in this way. For proofs of the following theorem, see [3, 5].

**Theorem 1.1.** *Let  $(X, \sigma, \phi)$  be a flip system such that  $(X, \sigma)$  is a subshift of finite type. Then there are 0-1 matrices  $A$  and  $P$  satisfying (1.1) such that  $(X, \sigma, \phi)$  is conjugate to  $(X_A, \sigma_A, \varphi_{A,P})$ .*

**Definition 1.1.** Let  $P$  be an  $\mathcal{A} \times \mathcal{A}$  0-1 matrix such that  $P^2 = I$ . A state  $I \in \mathcal{A}$  is said to be *in the body* if  $P(I, I) = 1$ ; if  $P(J, J') = 1$  and

$J \neq J'$ , then the state  $J$  is said to be *in the wing* or the pair  $(J, J')$  *in the wings*. The number of pairs in the wings is called the *transposition number* of  $P$  and denoted by  $t_P$ .

For instance, if  $P$  is the identity matrix, the transposition number is equal to 0.

**Definition 1.2.** Let  $(X, \sigma, \phi)$  be a flip system such that  $(X, \sigma)$  is a subshift of finite type. The *flip number* of  $(X, \sigma, \phi)$  is defined to be

$$\min\{t_P | (X_A, \sigma_A, \varphi_{A,P}) \text{ is conjugate to } (X, \sigma, \phi)\}.$$

The flip number is obviously a conjugacy invariant that measures how complicated a flip system is. For example, simplest flip systems would be of flip number 0. Such a system can be represented by a pair of matrices  $(A, I)$  with  $A$  symmetric, and so is conjugate to  $(X_A, \sigma_A, \rho_A)$ . In this article, we attempt to answer (partially) the following natural questions.

1. Given a flip system on a subshift of finite type, what transposition numbers can its representations have?
2. Let  $A$  and  $B$  be 0-1 square matrices such that  $(X_A, \sigma_A)$  and  $(X_B, \sigma_B)$  are conjugate as subshifts. If  $(X_A, \sigma_A, \phi)$  and  $(X_B, \sigma_B, \psi)$  are flip systems with flip number zero, then are they always conjugate as flip systems?
3. Can we characterize flip systems with flip number zero?
4. Let  $A$  and  $P$  be 0-1 matrices satisfying (1.1), and  $(X_A, \sigma_A, \varphi_{A,P})$  be the corresponding flip system. For each  $k \in \mathbb{Z}$ , what is the flip number of the system  $(X_A, \sigma_A, \sigma_A^k \circ \varphi_{A,P})$ ?

## 2. Transposition numbers

Let  $(X_A, \sigma_A)$  be a subshift of finite type with  $\mathcal{A} \times \mathcal{A}$  adjacency matrix  $A$ . Throughout the rest of the paper we always assume that  $X_A$  has

infinitely many points. For a state  $I \in \mathcal{A}$ , the set of outgoing edges from  $I$  is denoted by  $\mathcal{E}_I$  and the set of incoming edges into  $I$  by  $\mathcal{E}^I$ . We note that there exists  $I \in \mathcal{A}$  for which either  $\mathcal{E}_I$  or  $\mathcal{E}^I$  contains more than one edge since  $X_A$  has infinitely many points.

We prove the following theorem using the standard state splitting method for subshifts of finite type. See [4, 6] for state splitting techniques.

**Theorem 2.1.** *Let  $A, P$  be 0-1 square matrices satisfying (1.1) and  $(X_A, \sigma_A, \varphi_{A,P})$  be the corresponding flip system. Then there is a flip system  $(X_B, \sigma_B, \varphi_{B,Q})$  conjugate to  $(X_A, \sigma_A, \varphi_{A,P})$  of which the transposition number  $t_Q$  is equal to  $t_P + 1$ .*

*Proof.* We may assume that there exists a state  $I$  such that  $\mathcal{E}_I$  has more than one edge. We divide the proof into several cases.

**Case 1.** Suppose that  $(I, I')$  is in the wings and there exist distinct pairs  $(J, J'), (K, K')$  in the wings such that  $A_{IJ} = A_{IK} = A_{J'I'} = A_{K'I'} = 1$ . (It may be allowed that  $I = J$  or  $I = K$ , that is,  $I$  has a loop.)

We denote by  $e$  the edge from  $I$  to  $J$ , and by  $e'$  the edge from  $J'$  to  $I'$ .

$$J \xleftarrow{e} I \xrightarrow{f} K \qquad K' \xrightarrow{f'} I' \xleftarrow{e'} J'$$

We out-split the state  $I$  into  $I_1$  and  $I_2$  using the partition  $\mathcal{P} = \{\{e\}, \mathcal{E}_I \setminus \{e\}\}$ , and at the same time in-split the state  $I'$  into  $I'_1$  and  $I'_2$  using the partition  $\mathcal{P}' = \{\{e'\}, \mathcal{E}^{I'} \setminus \{e'\}\}$ .

$$J \xleftarrow{e} I_1 \quad I_2 \xrightarrow{f} K \qquad K \xrightarrow{f'} I'_2 \quad I'_1 \xleftarrow{e'} J'$$

Let  $B$  be the adjacency matrix of this newly obtained graph. The matrix  $Q$  is defined as  $Q(M, N) = P(M, N)$  if  $M$  and  $N$  are states of the old graph,  $Q(I_i, I'_i) = Q(I'_i, I_i) = 1$  for  $i = 1, 2$  and  $Q(M, N) = 0$  otherwise. It is routine to check that  $BQ = QB^\top$ ,  $Q^2 = I$  and that the

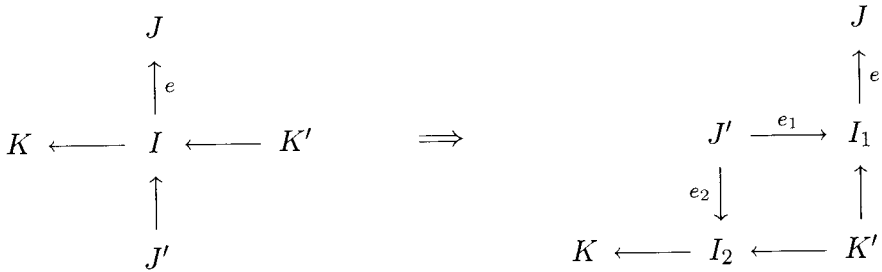
flip system  $(X_B, \sigma_B, \varphi_{B,Q})$  is conjugate to  $(X_A, \sigma_A, \varphi_{A,P})$ . Note that the pairs  $(I_1, I'_1)$  and  $(I_2, I'_2)$  are in the wings of  $Q$ , and hence  $t_Q = t_P + 1$ .

**Case 2.** Suppose that  $(I, I')$  is in the wings and there exist  $J$  and  $K, J \neq K$ , such that  $A_{IJ} = A_{IK} = 1$ , where  $J$  or  $K$  is in the body.

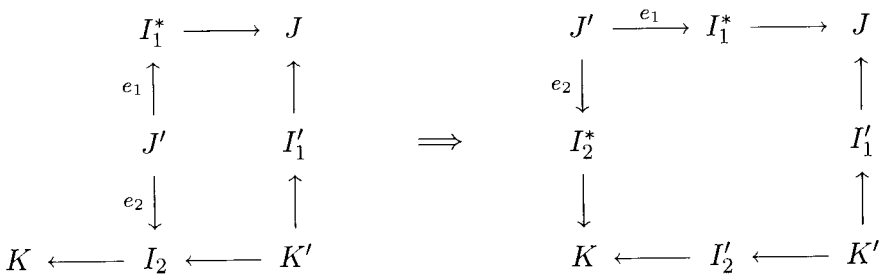
The proof is similar to Case 1.

**Case 3.** Suppose that  $I$  is in the body and that there exist distinct pairs  $(J, J'), (K, K')$  in the wings for which  $A_{IJ} = A_{J'I} = A_{IK} = A_{K'I} = 1$ .

Let  $e$  denote the edge from  $I$  to  $J$ . First, we out-split the state  $I$  into  $I_1$  and  $I_2$ , using the partition  $\mathcal{P} = \{\{e\}, \mathcal{E}_I \setminus \{e\}\}$ .



We denote the edge from  $J'$  to  $I_1$  by  $e_1$  and the edge from  $J'$  to  $I_2$  by  $e_2$ . Now in-split the state  $I_1$  into  $I_1^*$  and  $I'_1$ , using the partition  $\mathcal{P}_1 = \{\{e_1\}, \mathcal{E}^{I_1} \setminus \{e_1\}\}$ . Finally we in-split the state  $I_2$  into  $I_2^*$  and  $I'_2$ , using the partition  $\mathcal{P}_2 = \{\{e_2\}, \mathcal{E}^{I_2} \setminus \{e_2\}\}$ .



Let  $B$  the adjacency matrix of the new graph. The permutation matrix  $Q$  is defined as  $Q(M, N) = P(M, N)$  if  $M$  and  $N$  are states of the old graph,  $Q(I_1^*, I_1^*) = Q(I'_2, I'_2) = 1$ ,  $Q(I'_1, I_2^*) = Q(I_2^*, I'_1) = 1$  and  $Q(M, N) = 0$  otherwise. It can be seen easily that the flip system  $(X_B, \sigma_B, \varphi_{B,Q})$  is conjugate to  $(X_A, \sigma_A, \varphi_{A,P})$ . Since  $I_1^*$  and  $I'_2$  are states

in the body of  $Q$  and  $(I_2^*, I_1')$  is a pair in the wings of  $Q$ , we conclude that  $t_Q = t_P + 1$ .

**Case 4.** Suppose that  $I$  is in the body, that there exist pair  $(J, J')$  in the wings such that  $A_{IJ} = A_{J'I} = 1$ , and that there is a loop at  $I$ .

The loop at the state  $I$  is denoted by  $e$ . First, we out-split the state  $I$  into  $I_1$  and  $I_2$  using the partition  $\mathcal{P} = \{\{e\}, \mathcal{E}_I \setminus \{e\}\}$ . Then the new state  $I_1$  also has a loop, say  $e_1$ . Now we in-split the state  $I_1$  into  $I_1^*$  and  $I_1'$ , using the partition  $\mathcal{P}_1 = \{\{e_1\}, \mathcal{E}^{I_1} \setminus \{e_1\}\}$ . The edge from  $I_1^*$  to  $I_2$  is denoted by  $e_2^*$  and the edge from  $I_1'$  to  $I_2$  by  $e_2'$ . Finally, we in-split the state  $I_2$  into  $I_2^*$  and  $I_2'$ , using the partition  $\mathcal{P}_2 = \{\{e_2^*, e_2'\}, \mathcal{E}^{I_2} \setminus \{e_2^*, e_2'\}\}$ .

Let  $B$  the adjacency matrix of the new graph. The permutation matrix  $Q$  is defined as in Case 3. Once again it can be routinely checked that the flip system  $(X_B, \sigma_B, \varphi_{B,Q})$  is conjugate to  $(X_A, \sigma_A, \varphi_{A,P})$ . Since  $I_1^*$  and  $I_2'$  are states in the body of  $Q$  and  $(I_2^*, I_1')$  is a pair in the wings of  $Q$ , we see that  $t_Q = t_P + 1$ .

**Case 5.** Suppose that  $I$  is in the body and there exist  $J$  and  $K$ ,  $J \neq K$ , such that  $A_{IJ} = A_{IK} = 1$ , where  $J$  or  $K$  is in the body.

If  $I$  does not have a loop, the proof is similar to Case 3; if  $I$  has one, the proof is similar to Case 4. □

By an inductive argument using the above theorem, we can show that the flip system on a subshift of finite type has a matrix representation with arbitrarily large transposition number.

**Corollary 2.2.** *Let  $A, P$  be 0-1 square matrices satisfying (1.1) and  $(X_A, \sigma_A, \varphi_{A,P})$  be the corresponding flip system. Then for each integer  $k \geq t_P$  there is a flip system  $(X_{B_k}, \sigma_{B_k}, \varphi_{B_k, Q_k})$  conjugate to  $(X_A, \sigma_A, \varphi_{A,P})$  such that  $t_{Q_k} = k$ .*

### 3. Flip maps on symmetric matrices

We begin the section with a simple observation. Let  $(X, \sigma, \phi)$  be a flip system. Then it can be seen that  $(X, \sigma, \sigma^k \circ \phi)$  is also a flip system for each integer  $k$ . Let  $k$  and  $l$  be integers of the same parity. We find that

$$\sigma^{(-k+l)/2} \circ (\sigma^k \circ \phi) = \sigma^l \circ \sigma^{(k-l)/2} \circ \phi = (\sigma^l \circ \phi) \circ \sigma^{(-k+l)/2},$$

and so that  $\sigma^{(-k+l)/2}$  is a conjugacy from  $(X, \sigma, \sigma^k \circ \phi)$  to  $(X, \sigma, \sigma^l \circ \phi)$ . However, if two integers have different parity, the corresponding flip systems may not be conjugate.

**Lemma 3.1.** *Let  $A$  and  $B$  be any 0-1 symmetric matrices. Then the flip systems  $(X_A, \sigma_A, \sigma_A \circ \rho_A)$  and  $(X_B, \sigma_B, \rho_B)$  are not conjugate.*

*Proof.* Let  $\gamma : (X_B, \sigma_B, \rho_B) \rightarrow (X_A, \sigma_A, \sigma_A \circ \rho_A)$  be a conjugacy. We notice that the mirror map fixes each point of period 2. Let  $y \in X_A$  be a point of the least period 2 (such a point does exist since  $X_A$  is infinite) and put  $x = \gamma^{-1}(y) \in X_B$ . Since  $\gamma$  preserves the least period,

$$(\sigma_A \circ \rho_A \circ \gamma)(x) = \sigma_A(y) \neq y = (\gamma \circ \rho_B)(x),$$

which is a contradiction. Therefore the two flip systems are not conjugate. □

The following lemma asserts that the mirror map has a kind of uniqueness property. The proof is similar to that of the previous lemma.

**Lemma 3.2.** *Let  $A$  be a 0-1 symmetric matrix and  $(X_A, \sigma_A, \varphi_{A,P})$  a flip system. If there is a symmetric matrix  $B$  such that  $(X_B, \sigma_B, \rho_B)$  is conjugate to  $(X_A, \sigma_A, \varphi_{A,P})$ , then  $P = I$ .*

*Proof.* Let  $\phi_P : X_A \rightarrow X_A$  denote the symbolic conjugacy defined by  $\varphi_{A,P} = \phi_P \circ \rho_A$ . Let  $\gamma$  be a conjugacy from the flip system  $(X_B, \sigma_B, \rho_B)$  to  $(X_A, \sigma_A, \varphi_{A,P})$ . Then we see that  $\phi_P \circ \rho_A \circ \gamma = \gamma \circ \rho_B$ . Assume that  $P \neq I$ . Then there exists a point  $y$  in  $X_A$  of the least period 2 such that

$\phi_P(y) \neq y$ . We put  $x = \gamma^{-1}(y) \in X_B$ . Since  $y$  and  $x$  are respectively fixed by  $\rho_A$  and  $\rho_B$ , we find that

$$(\phi_P \circ \rho_A \circ \gamma)(x) = \phi_P(y) \neq y = (\gamma \circ \rho_B)(x).$$

This contradiction leads to the conclusion that  $P = I$ . □

**Lemma 3.3.** *Let  $A$  be a 0-1 symmetric matrix and let  $(X_A, \sigma_A, \varphi_{A,P})$  be a flip system. Then the flip system  $(X_A, \sigma_A, \sigma_A \circ \varphi_{A,P})$  is not conjugate to  $(X_B, \sigma_B, \rho_B)$  for any symmetric matrix  $B$ .*

*Proof.* If  $P = I$ , then the result follows from Lemma 3.1. We may assume that  $P \neq I$ . Suppose that there exist a symmetric matrix  $B$  and a conjugacy  $\gamma$  from  $(X_B, \sigma_B, \rho_B)$  to  $(X_A, \sigma_A, \sigma_A \circ \varphi_{A,P})$ . Let  $\phi_P$  be as in the proof of Lemma 3.2. Note that  $\sigma_A \circ \phi_P \circ \rho_A \circ \gamma = \gamma \circ \rho_B$ . Since  $P \neq I$ , there exist states  $a, a^*, a \neq a^*$ , of  $X_A$  for which  $P(a, a^*) = P(a^*, a) = 1$ .

We claim that there is  $y \in X_A$  such that  $\sigma_A^2(y) = y$ , and  $\phi_P(y) \neq \sigma_A(y)$  or equivalently  $(\sigma_A \circ \phi_P)(y) \neq y$ . To prove the claim, first assume that  $X_A$  has only two states  $a$  and  $a^*$ , and hence is contained in the full 2-shift  $\{a, a^*\}^{\mathbb{Z}}$ . Then both  $a^\infty$  and  $a^{*\infty}$  are points in  $X_A$  since  $X_A$  is infinite and  $P(a, a^*) = P(a^*, a) = 1$ . (In fact,  $X_A = \{a, a^*\}^{\mathbb{Z}}$ .) Setting  $y = a^\infty \in X_A$  we have  $(\sigma_A \circ \phi_P)(y) = (a^*)^\infty \neq y$ . In case where  $X_A \not\subseteq \{a, a^*\}^{\mathbb{Z}}$ , there is  $b \notin \{a, a^*\}$  such that  $ab, ba$  are allowed 2-blocks in  $X_A$ . We denote by  $b^*$  the unique state such that  $P(b, b^*) = 1$ . Put  $y = \dots bab.aba \dots \in X_A$ . It follows that

$$\phi_P(y) = \dots b^* a^* b^* . a^* b^* a^* \dots \neq \dots aba.bab \dots = \sigma_B(y)$$

since  $a^* \neq b$ .

Now we put  $x = \gamma^{-1}(y) \in X_B$ , and observe that

$$(\sigma_A \circ \phi_P \circ \rho_A \circ \gamma)(x) = (\sigma_A \circ \phi_P)(y) \neq y = (\gamma \circ \rho_B)(x),$$

which is a contradiction. This completes the proof. □



For any symmetric matrix  $A$ , the natural flip system  $(X_A, \sigma_A, \rho_A)$  has flip number zero. The following theorem says that there exist flip systems obtained from symmetric matrices with nonzero flip numbers. The theorem is an immediate consequence of Lemma 3.3 and the comment at the beginning of the section.

**Theorem 3.4.** *Let  $A, P$  be 0-1 matrices satisfying (1.1) and assume that  $A$  is symmetric. Then the flip systems  $(X_A, \sigma_A, \sigma_A^k \circ \varphi_{A,P})$ ,  $k$  odd, have a nonzero flip number.*

**Remark 3.1.** Let  $A$  be a 0-1 non-symmetric square matrix with a flip system  $(X_A, \sigma_A, \varphi_{A,P})$ . Suppose that there exists  $y \in X_A$  such that  $\sigma_A^2(y) = y$  (then also  $y \in X_{A^T}$ ) and  $\phi_P(y) \neq y$ . Using the same argument as in the proof of Lemma 3.2, we can see that  $(X_A, \sigma_A, \varphi_{A,P})$  is not conjugate to  $(X_B, \sigma_B, \rho_B)$  for any symmetric matrix  $B$ . Therefore the system  $(X_A, \sigma_A, \varphi_{A,P})$  has a nonzero flip number.

On the other hand, we assume that there exists  $y \in X_A$  such that  $\sigma_A^2(y) = y$  (then also  $y \in X_{A^T}$ ) and  $\phi_P(y) \neq \sigma_A(y)$ . Then by an argument analogous to the one in the proof of Lemma 3.3, we see that the flip system  $(X_A, \sigma_A, \sigma_A \circ \varphi_{A,P})$  is not conjugate to  $(X_B, \sigma_B, \rho_B)$  for any symmetric matrix  $B$ . Thus we conclude that the flip number of  $(X_A, \sigma_A, \sigma_A \circ \varphi_{A,P})$  is nonzero.

**Example 3.5.** We consider the flip system  $(X_A, \sigma_A, \sigma_A \circ \rho_A)$  for the symmetric matrix  $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ . (The subshift of finite type  $(X_A, \sigma_A)$  is called the golden mean shift.) It follows from Theorem 3.4 that the flip number of the system is positive. It can be shown that  $(X_A, \sigma_A, \sigma_A \circ \rho_A)$

is conjugate to  $(X_B, \sigma_B, \varphi_{B,Q})$  where

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since  $t_Q = 3$ , the flip number of  $(X_A, \sigma_A, \sigma_A \circ \rho_A)$  is not greater than 3. However, the exact flip number is not known yet.

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