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SOME MANIFOLDS WITH NONZERO EULER CHARACTERISTIC AS PL FIBRATORS

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Abstract. Approximate fibrations form a useful class of maps. By definition fibrators provide instant detection of maps in this class, and PL fibrators do the same in the PL category. We show that every closed s-hopfian t-aspherical manifold N with some algebraic conditions and $\chi(N) \neq 0$ is a codimension-$(2t+2)$ PL fibrator.

1. Introduction

Approximate fibrations form an extremely useful class of maps, due to the presence of several associated sequences displaying computable homotopical and homological relationships involving the domain, image and typical fiber. Continuing an extensive series, this paper seeks to identify homotopy types by means of which a proper map defined on an arbitrary manifold of a given dimension can be quickly recognized as an approximate fibration, simply because all point preimages have the specified homotopy type. More precisely, the goal is to present closed n-manifolds N which force proper maps $p : M \to B$ to be approximate fibrations, when M is an $(n+k)$-manifold and each $p^{-1}(b)$ has the homotopy type (or, more generally, the shape) of N. Such a manifold N is called a codimension-k fibrator.

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It is well known that every closed manifold N with hopfian $\pi_1(N)$ and $\chi(N) \neq 0$ is a codimension-2 fibrator (See [11, Proposition 2.4]). In general, these codimension-2 fibrators need not to be codimension-k ($k > 2$) fibrators. Normally cohopficity and sparsely Abelianness on $\pi_1(N)$ are indispensable for those codimension-2 fibrators to be codimension-k ($k > 2$) fibrators. For example, the real projective plane $\mathbb{RP}^2$ is not a codimension-3 fibrator although its fundamental group is hopfian and its Euler characteristic is nonzero [6].

Remarkably, at this stage of development only a few classes of manifolds are known not to be PL fibrators: those that already fail in codimension 2 and those that have as a Cartesian factor either a sphere, a real projective space, or a certain kind of 3-dimensional spherical space form that can be regarded as a coset space. The codimension-2 situation, reviewed extensively in the introduction to [4], is fairly well understood and is not treated here.

Throughout this paper, the symbols χ, $\approx$ and $\cong$ denote Euler characteristic, homeomorphism and isomorphism in that order, and homology groups will be computed with integer coefficients unless specified.

We begin by presenting the notation and fundamental terminology to be employed throughout: M is a connected $(n + k)$-manifold and $p : M \to B$ is a proper map of M to a space B such that each $p^{-1}(b)$ has the homotopy type (or, more generally, the shape) of a closed, connected n-manifold. Such a map p will be called a *codimension-k* map. When N is a fixed PL n-manifold, M is a PL manifold, B is a polyhedron, and $p : M \to B$ is a PL map, then p is said to be *N-like* if each $p^{-1}(b)$ collapses to an n-complex homotopy equivalent to N (denoted by $p^{-1}(b) \sim N$). (This PL tameness feature, which seems just as effective as requiring $p^{-1}(b)$ actually to be an n-manifold, imposes significant homotopy-theoretic relationships, revealed in [5, Lemma 2.4], between N and preimages of links in B.) We call N a *codimension-k PL fibrator* if, for every PL $(n+k)$-manifold M and N-like PL map $p : M \to B$, p is an approximate fibration. Similarly, we call N a *codimension-k orientable PL fibrator* if this holds for all orientable, PL $(n+k)$-manifolds M, which we abbreviate by writing that N is a codimension-k PL o-fibrator. Finally, if N is a codimension-k PL fibrator (respectively, codimension-k PL o-fibrator) for all $k > 0$, we simply call N a *PL fibrator* (respectively, *PL o-fibrator*).

An ANR Y is said to be *t-aspherical* if $\pi_i(Y) \cong 0$ whenever $1 < i \leq t$.

A group G is said to be: *hopfian* if each epimorphism $G \to G$ is an isomorphism; *cohopfian* if each monomorphism $G \to G$ is an isomorphism; and *normally cohopfian* if each monomorphism $G \to G$ with image a normal subgroup of G is an isomorphism. A group G is *sparsely Abelian* if it contains no nontrivial normal subgroup A such that G/A is isomorphic to a normal subgroup of G.

A closed, orientable manifold N is said to be *hopfian* if every degree 1 map $N \to N$ which induces an isomorphism at the fundamental group level is a homotopy equivalence. As a result, when $\pi_1(N)$ is a hopfian group, N is a hopfian manifold if and only if all degree 1 maps $N \to N$ are homotopy equivalences.

In this paper, without having normally cohopficity on $\pi_1(N)$, we show that a manifold N with $\chi(N) \neq 0$ can be a codimension-k $(k > 2)$ fibrator. Of course, not every manifolds N with $\chi(N) \neq 0$, such as some 4-manifolds N with $\pi_1(N) = \mathbb{Z}_2 * \mathbb{Z}_2$, have normally cohopfian $\pi_1(N)$.

By looking at the covering spaces of M instead of considering M itself, we have to take care of degree of maps between different manifolds with same dimensions (See Lemma 2.1 below and the proof of Lemma 2.4). As a result, we get the main result, Theorem 3.5, which promises that every closed s-hopfian t-aspherical manifold N with some algebraic conditions and $\chi(N) \neq 0$ is a codimension-$(2t+2)$ fibrator.

2. Degree of the map between fibers

For simplicity, throughout this section we shall assume each fiber of a codimension-k map to be an ANR(absolute neighborhood retract) having the homotopy type of a manifold.

A codimension-k map $p : M^{n+k} \to B$ is said to have Property $\mathcal{R}_i^{\cong}$ ($\mathcal{R}_i^{\geq}, \mathcal{R}_i^{\leq}$) if for each $x \in B$, a retraction $\mathcal{R} : p^{-1}(U) \to p^{-1}(x)$ defined on some open neighborhood U of x in B induces an isomorphism(epimorphism, monomorphism, respectively) $(\mathcal{R}|p^{-1}(y))_* : H_i(p^{-1}(y)) \to H_i(p^{-1}(x))$ for all $y \in U$. The (absolute) degree of a map is computed with integer coefficients and is understood to be a nonnegative number. Explicitly, a map $f : N \to N'$ between closed, orientable n-manifolds is said to have *degree* d if there are choices of generators $\gamma \in H_n(N;\mathbb{Z}), \gamma' \in H_n(N';\mathbb{Z})$ such that $f_*(\gamma) = d\gamma'$, where $d \geq 0$ is an integer.

The *continuity set* of p consists of all $x \in B$ equipped with such a neighborhood U such that the associated $\mathcal{R} : p^{-1}(U) \to p^{-1}(x)$ restricts to an isomorphism $(\mathcal{R}|p^{-1}(y))_* : H_n(p^{-1}(y)) \to H_n(p^{-1}(x))$ for all $y \in U$. Establishing that B equals the continuity set of p is a cornerstone for showing an N-like map p is an approximate fibration.

Lemma 2.1. [9] *Let $p : M^{n+k} \to \mathbb{R}^k$ be a codimension-k map from an orientable $(n+k)$-manifold M^{n+k} onto Euclidian k-space such that p is an approximate fibration over $\mathbb{R}^k \setminus \mathbf{0}$. Then p has Property $\mathcal{R}_i^{\cong}$ for all $i \leq k-3$, and $\mathcal{R}_{k-2}^{\geq}$. Furthermore, if p has Property $\mathcal{R}_{k-2}^{\leq}$ and $\mathcal{R}_{k-1}^{\geq}$, then for all $y \in \mathbb{R}^k \setminus \mathbf{0}$, the degree of map $\mathcal{R}|p^{-1}(y) : p^{-1}(y) \to p^{-1}(\mathbf{0})$ is one.*

Lemma 2.2. [5] *Suppose N and N' are closed orientable n-manifolds such that $\beta_i(N) = \beta_i(N') > 0$ for some $0 < i < n$, and suppose $f : N \to N'$ is a map that induces isomorphisms $f_*| :$ free part$\{H_i(N)\} \to$*

free part$\{H_i(N')\}$ *and* $f^*|$: *free part*$\{H^{n-i}(N')\} \to$ *free part*$\{H^{n-i}(N)\}$. *Then the degree of map* f *is one.*

Remark 2.3. *The argument actually shows that the degree of map* f *is one merely if it induces isomorphisms of free part of* H^{n-i} *and if some indivisible element* $\xi' \in H_i(N')$ *belongs to the image of* f_*.

Lemma 2.4. *Let* N *and* N' *be closed orientable* n*-manifolds. Suppose that* $p : M^{n+k} \to \mathbb{R}^k$ *is a codimension-*k *map defined on an orientable* $(n+k)$*-manifold* M^{n+k} *such that* p *is an approximate fibration over* $\mathbb{R}^k \setminus \mathbf{0}$, *and* $N' \sim p^{-1}(\mathbf{0})$. *Then for all* $y \in \mathbb{R}^k \setminus \mathbf{0}$, *the degree of map* $\mathcal{R}|p^{-1}(y) : p^{-1}(y) \to p^{-1}(\mathbf{0})$ *is one provided either*

1. $\beta_s(N') \geq \beta_s(N) > 0$ *for some* s *satisfying* $n/2 \leq s \leq k-2$, *or*
2. $\beta_s(N') \geq \beta_s(N) > 1$ *for some* s *satisfying* $n/2 < s \leq k-1$, *or*
3. n *is even,* $n < k$ *and* $\beta_s(N') \geq \beta_s(N) > 2$ *for* $s = n/2$.

Proof. (1) The case $s \leq k-3$ is similar and easier, for p has property $\mathcal{R}_i^{\cong}$ for all $i \leq k-3$ by Lemma 2.1.

Now concentrate on the case $s = k-2$. Examine the following portion $(*)$ of the long exact homology sequence for the pair $(M, M \setminus p^{-1}(\mathbf{0}))$:

$$(*) H_{k-1}(M, M\backslash p^{-1}(\mathbf{0})) \to H_{k-2}(M\backslash p^{-1}(\mathbf{0})) \to H_{k-2}(M) \to H_{k-2}(M, M\backslash p^{-1}(\mathbf{0})).$$

By the duality, $H_{k-1}(M, M\backslash p^{-1}(\mathbf{0})) \cong H^{n+1}(N') \cong 0$ and $H_{k-2}(M, M\backslash p^{-1}(\mathbf{0})) \cong 0$, while the (shape) deformation retraction $\mathcal{R} : M \to p^{-1}(\mathbf{0})$ can be regarded as producing an isomorphism $H_{k-2}(M) \to H_{k-2}(N')$.

Let $y \in \mathbb{R}^k \backslash \mathbf{0}$. Since p is an approximate fibration over the homotopy $(k-1)$-sphere $\mathbb{R}^k \setminus \mathbf{0}$, the Serre exact sequence [14, p. 519]

$$\cdots \to H_i(p^{-1}(y)) \to H_i(M \setminus p^{-1}(\mathbf{0})) \to H_i(\mathbb{R}^k \setminus \mathbf{0}) \cong 0 \ (i < k-1)$$

gives an isomorphism $H_i(p^{-1}(y)) \to H_i(M \setminus p^{-1}(\mathbf{0}))$ for $i \leq k-3$ and an epimorphism for $i = k-2$. The homology exact sequence of the pair $(M, M \setminus p^{-1}(\mathbf{0}))$ gives an isomorphism $H_i(M \setminus p^{-1}(\mathbf{0})) \to H_i(M)$ for

$i \leq k-2$. Therefore, the inclusion $p^{-1}(y) \to M$ induces a composite homomorphism

$$incl_* : H_i(p^{-1}(y)) \to H_i(M \setminus p^{-1}(\mathbf{0})) \to H_i(M),$$

which is an epimorphism for $i = k-2$ so that $\beta_{k-2}(N') \leq \beta_{k-2}(N)$. Since $\beta_{k-2}(N) \leq \beta_{k-2}(N')$, the above composite epimorphism is an isomorphism of free part of H_{k-2}. Since $n-k+2 \leq k-2$, we have an isomorphism of free part of H^{n-k+2} by the universal coefficient theorem. The conclusion follows from Lemma 2.2.

(2) Concentrate on the case $s = k-1$; the case $s \leq k-2$ is similar and easier. As in the proof of (1), the Serre exact sequence for $p|M \setminus p^{-1}(\mathbf{0})$ yields

$$H_{k-1}(M \setminus p^{-1}(\mathbf{0})) \cong \mathbb{Z} \oplus j_*(H_{k-1}(p^{-1}(y))),$$

where $y \in \mathbb{R}^k \setminus \mathbf{0}$ and $j : p^{-1}(y) \to M \setminus p^{-1}(\mathbf{0})$ is the inclusion. Furthermore, the long exact sequence for the pair $(M, M \setminus p^{-1}(\mathbf{0}))$ reveals that the inclusion-induced

$$\phi_* : H_{k-1}(M \setminus p^{-1}(\mathbf{0})) \to H_{k-1}(M)$$

is surjective, as the next term is trivial, by the duality. Since $n-k+1 < k-1$, we have $\beta_{k-1}(N) = \beta_{n-k+1}(N) = \beta_{n-k+1}(N') = \beta_{k-1}(N')$. In view of the assumption that $\beta_{k-1}(N) > 1$, there exists an indivisible element of $H_{k-1}(M) \cong H_{k-1}(p^{-1}(\mathbf{0}))$ in $\phi_* j_*(H_{k-1}(p^{-1}(y)))$. By Lemma 2.1 and the universal coefficient theorem for cohomology,

$$j^*\phi^*| : \text{free part}\{H^{n-k+1}(M)(\cong H^{n-k+1}(p^{-1}(\mathbf{0})))\} \to \text{free part}\{H^{n-k+1}(p^{-1}(y))\}$$

is an isomorphism. The conclusion now follows from Remark 2.3.

(3) Assume $n/2 = k-1$. Working with the inclusion $\phi j : p^{-1}(y) \to M, y \in \mathbb{R}^k \setminus \mathbf{0}$, described as above, we repeat the analysis there, which indicates free part $\{H_{k-1}(M) \cong H_{k-1}(p^{-1}(\mathbf{0}))\}$ is isomorphic to $\phi_* j_*(\text{free part}\{H_{k-1}(p^{-1}(z))\})$. By the universal coefficient theorem for cohomology, free part$\{H^{k-1}(p^{-1}(y))\}$ is then isomorphic to $j^*\phi^*(\text{free part}\{H^{k-1}(M)\})$. Consequently, there exists a choice of an

indivisible $\xi \in H_{k-1}(p^{-1}(\mathbf{0}))$ for which the proof of Lemma 2.2 and Remark 2.3 apply, that is, corresponding to ξ is some $\nu \in H^{k-1}(p^{-1}(\mathbf{0}))$ with $\xi = \mathcal{R}_*(\eta \frown \mathcal{R}^*(\nu))$. Just as before, the degree of map $\mathcal{R}$: $p^{-1}(y) \to p^{-1}(\mathbf{0})$ is one. □

3. Manifolds with nonzero Euler characteristic as PL fibrators

In this section, we show that every closed s-hopfian t-aspherical manifold N with some algebraic conditions and $\chi(N) \neq 0$ is a codimension-$(2t+2)$ fibrator.

Often an important step involves establishing that B is a manifold; properties of PL approximate fibrations ensure that the complement of its 0-skeleton is a manifold, but to apply the results of Section 2 it is necessary that B itself is one. The main results follow present conditions under which manifold image occurs.

Proposition 3.1. (*c.f.* [9, Lemma 2.5]) *Suppose N is a codimension-$(k-1)$ PL o-fibrator with sparsely Abelian fundamental group and $\chi(N) \neq 0$ such that N is t-aspherical, where $k \leq t+1$. Suppose that $p : M^{n+k} \to B^k$ is an N-like PL map defined on an orientable manifold M^{n+k}. Then B is a k-manifold; furthermore, B is a k-manifold for $k \leq 2t+2$, provided either*

1. $\beta_s(N) > 0$ *for some s satisfying $n/2 \leq s \leq k-2$, or*
2. $\beta_s(N) > 1$ *for some s satisfying $n/2 < s \leq k-1$, or*
3. $n/2 = k-1$ *and* $\beta_{k-1}(N) > 2$.

Proof. Focus on a star S about a typical vertex $v \in B$, with corresponding link L. It suffices to show that the link L is a homotopy $(k-1)$-sphere. Being the image of $L' = p^{-1}(L)$ under an approximate fibration, L must be a closed $(k-1)$-manifold [3, Theorem 5.4]. See [5, Lemma 2.1] about its being a sphere for $k \leq 2$. So we will assume $k > 2$.

We will show that $\pi_1(L) \cong 1$; then L is a homotopy $(k-1)$-sphere by the t-asphericity of N and the Poincaré Duality. In fact, L is a homotopy $(k-1)$-sphere when $k = 3$ or 4. When $k > 4$, $k/2 \leq min\{t+1, k-2\}$. Consider the exact sequence

$$\cdots \to \pi_i(N) \to \pi_i(L') \to \pi_i(L) \to \pi_{i-1}(N) \to \cdots .$$

For $2 \leq i < k/2$, we have $\pi_i(N) \cong 0$. In the homotopy exact sequence of the approximate fibration $p \mid L'$,

$$\cdots \to \pi_2(L) \to \pi_1(N) \to \pi_1(L') \to \pi_1(L) \to 1,$$

we have a trivial homomorphism $\pi_2(L) \to \pi_1(N)$ and a monomorphism $\pi_1(N) \to \pi_1(L')$ since $\pi_1(N)$ is sparsely Abelian. Hence,

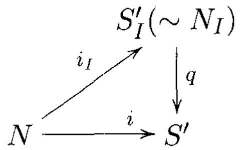
$$H_i(L) \cong \pi_i(L) \cong \pi_i(L') \cong \pi_i(S') \cong \pi_i(N) \cong 0$$

holds for the same range. Here the third isomorphism is obtained by the fact that the inclusion $L' \to S' = p^{-1}(S)$ induces isomorphisms $\pi_i(L') \to \pi_i(S')$ for $i \leq k-2$. Poincaré Duality assures that L is a homotopy $(k-1)$-sphere.

Assume that L were not simply connected. Then in the homotopy exact sequence of the approximate fibration $p|L'$, $\pi_1(p^{-1}(z)) \to \pi_1(L')$ could not be surjective for $z \in L$. If not, form the covering $q : S'_I \to S'$ of S' corresponding to the image of $\pi_1(p^{-1}(z)) \to \pi_1(L') \to \pi_1(S')$. (Recall that the final homomorphism is an isomorphism.) Let N_I denote a cover of N corresponding to the image of

$$\pi_1(p^{-1}(z)) \to \pi_1(S') \to \pi_1((p^{-1}(v)) \cong \pi_1(N).$$

The inclusion $i : p^{-1}(z) \to S'$ lifts to $i_I : p^{-1}(z) \to S'_I$, which induces a π_1-epimorphism. Consider



Case 1: $[\pi_1(S'); i_{\#}(\pi_1(N))] = \infty$.

Since $\pi_1(N)$ is sparsely Abelian, i_I induces a π_1-isomorphism. Set $L'_I = q^{-1}(L')$; L'_I is partitioned into copies of the various $p^{-1}(z)$, and the associated quotient map $\mu : L'_I \to L_I$ can be viewed as an N-like PL map, which is an approximate fibration, since locally over the base it looks just like $p|L'$. Inspection of the homotopy exact sequence for μ reveals that $\pi_1(L_I) = 1 - \pi_1(N) \to \pi_1(L'_I)$ is surjective because $\pi_1(L'_I) \to \pi_1(S'_I)$ is an isomorphism. Since N is t-aspherical, we have $\pi_i(L_I) = 0$ for $1 \leq i \leq k/2 \leq min\{t+1, k-2\}$. Being L_I an infinite covering space of L, L_I is contractible by the Poincaré Duality and the Whitehead theorem. The approximate fibration $L'_I \to L_I$ shows that L'_I has the same homotopy type as N. We will show that $H_i(N_I)$ is finitely generated for all i. For $i = 0, 1$, $H_i(N_I)$ is obviously finitely generated. Assume $H_j(N_I)$ is finitely generated for $j < i$. Consider the homology exact sequence for the pair (S'_I, L'_I);

$$H_i(L'_I) \to H_i(S'_I) \to H_i(S'_I, L'_I).$$

$H_i(L'_I) \cong H_i(N)$ and $H_i(S'_I, L'_I) \cong H_c^{n+k-i}(N_I) \cong H_{i-k}(N_I)$ are finitely generated. Hence, $H_i(S'_I) \cong H_i(N_I)$ is finitely generated. By Milnor [12], $\chi(N) = 0$. This is impossible.

Case 2: $[\pi_1(S'); i_{\#}(\pi_1(N))] < \infty$.

We divide into two cases $k \leq t+1$ and $t+2 \leq k \leq 2t+2$.

Subcase (i) : $k \leq t+1$.

From the above diagram, i_I induces a π_1-isomorphism. By the t-asphericity of N and the Poincaré Duality , L_I is a homotopy $(k-1)$-sphere and i_I induces π_i-isomorphism for $1 \leq i \leq t$. By the Whitehead theorem, $(i_I)_* : H_i(N) \to H_i(S'_I)$ is an isomorphism for $1 \leq i \leq t$. Lemma 2.1 implies that i_I is a degree one map. Since degree of i_I is 1 and degree of q is positive, $\beta_i(N) \geq \beta_i(N_I) \geq \beta_i(N)$ for all i. Then $\chi(N) = \chi(N_I)$ and q is a homeomorphism. Consequently, $i : N \to S'$

has degree 1 and induces a π_1-isomorphism. Hence, $\pi_1(L) = 1$ and L is a homotopy $(k-1)$-sphere.

Subcase (ii) : $t+1 \leq k \leq 2t+2$.

Mimic the argument of Subcase (i) by using Lemma 2.4. □

Remark 3.2. *The above result contains the stable range of* [9, Lemma 2.5], *so that we follow the proof of* [9, Lemma 2.5] *in the first part of the proof.*

Lemma 3.3. ([7, Corollary 2.6]) *Suppose N is a closed, hopfian n-manifold such that $\pi_1(N)$ is a hopfian group. Let $p : M^{n+k} \to \mathbb{R}^k$ be a proper, surjective map defined on an orientable $(n+k)$-manifold such that each fiber is homotopy equivalent to N and p is an approximate fibration over $\mathbb{R}^k \setminus \mathbf{0}$. Then p is an approximate fibration provided either*

1. $k \geq n+2$, *or*
2. $\beta_t(N) > 0$ *for some t satisfying* $n/2 \leq t \leq k-2$, *or*
3. $\beta_t(N) > 1$ *for some t satisfying* $n/2 < t \leq k-1$, *or*
4. $n/2 = k-1$ *and* $\beta_{k-1}(N) > 2$, *or*
5. $H_1(N)$ *and* $H_{k-1}(N)$ *are finite groups of relatively prime order.*

Theorem 3.4. *Suppose N is a closed hopfian n-manifold with sparsely Abelian, hopfian fundamental group and $\chi(N) \neq 0$, and N is t-aspherical, where either*

(1) $\beta_s(N) > 0$ *for some s satisfying* $n/2 \leq s \leq t$, *or*

(2) $t \geq n/2$ *and* $\beta_{t+1}(N) > 1$, *or*

(3) $t+1 = n/2$ *and* $\beta_{n/2}(N) > 2$.

Then N is a codimension-$(2t+2)$ PL o-fibrator.

Proof. Note N is a codimension-$(t+1)$ PL o-fibrator[9]. We assume N is a codimension-$(k-1)$ PL o-fibrator, $t+2 \leq k \leq 2t+2$, and consider an N-like PL map $p : M^{n+k} \to B$ defined on an orientable $(n+k)$-manifold M^{n+k}. Proposition 3.1 implies that B is a k-manifold, and so p is an approximate fibration by Lemma 3.3. □

Our concluding results address PL fibrator properties, not simply PL o-fibrator properties. It involves the following approach for investigating nonorientable manifolds introduced in [10]. Let N be a closed n-manifold which has a 2-to-1 covering. Consider the covering space N_H of N corresponding to H, where $H = \bigcap_{i \in I} H_i$ with $[\pi_1(N) : H_i] = 2$ for $i \in I = \{i : [\pi_1(N) : H_i] = 2\}$. The index set I is finite, and N_H is a closed orientable n-manifold, since every (finite) covering of an n-dimensional orientable manifold is again orientable and all non-orientable manifolds have 2-to-1 orientable coverings. A closed manifold N is *s-hopfian* if N is hopfian when N is orientable, and N_H is hopfian when N is non-orientable, where N_H is the covering space of N corresponding to $H = \bigcap_{i \in I} H_i$ with $I = \{i : [\pi_1(N) : H_i] = 2\}$. From now on, we reserve the symbols H and N_H to represent the above. Although an orientable N must be hopfian when N_H is, the converse is unknown. In a related setting, index 2 subgroups of hopfian groups need not be hopfian.

Proposition 3.5. *Suppose N is a codimension-$(k-1)$ PL fibrator with sparsely Abelian fundamental group and $\chi(N) \neq 0$ such that N is t-aspherical, where $k \leq t+1$. Suppose that $p : M^{n+k} \to B^k$ is an N-like PL map defined on a manifold M^{n+k}. Then B is a k-manifold; furthermore, B is a k-manifold for $k \leq 2t+2$, provided either*

1. *$\beta_s(N) > 0$ for some s satisfying $n/2 \leq s \leq k-2$, or*
2. *$\beta_s(N) > 1$ for some s satisfying $n/2 < s \leq k-1$, or*
3. *$n/2 = k-1$ and $\beta_{k-1}(N) > 2$.*

Proof. Mimic the proof of [9, Lemma 2.8]. □

As a result, we have the following main theorem.

Theorem 3.6. *Suppose N is a closed $s-$hopfian n-manifold with sparsely Abelian, hopfian fundamental group and $\chi(N) \neq 0$, and N is t-aspherical, where either*

(1) $\beta_s(N) > 0$ for some s satisfying $n/2 \leq s \leq t$,

(2) $t \geq n/2$ *and* $\beta_{t+1}(N) > 1$, *or*

(3) $t + 1 = n/2$ *and* $\beta_{n/2}(N) > 2$.

Then N *is a codimension-*$(2t + 2)$ *PL fibrator.*

Proof. It is known that N is a codimension-2 fibrator provided N is a closed *s-hopfian* manifold with hopfian fundamental group and $\chi(N) \neq 0$ [11, Proposition 2.4]. By induction we assume that N is a codimension-$(k-1)$ PL fibrator($k \geq 3$). Suppose $p : M^{n+k} \to B$ is an N-like PL map defined on a manifold M^{n+k}, where $k \leq t + 1$. By Proposition 3.5, B is a manifold. Upon forming the cover $\theta : M_H \to M$ corresponding to the image of $H \subset \pi_1(N) \cong \pi_1(p^{-1}(v))$ in $\pi_1(M)$, we see $p \circ \theta : M_H \to B$ is an N_H-like PL map and M_H is orientable, since it covers all possible 2-1 coverings of M. Following the proof of Lemma 3.3, $p \circ \theta$ and p are approximate fibrations. □

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