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APPLICATIONS OF SEMI-OPEN SETS

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Abstract. The aim of this paper is to introduce and study topological properties of semi-limit points, semi-derived, semi-interior, semi-closure, semi-border, and semi-frontier of a set using the concept of semi-open set.

1. Introduction

The notion of α-open set was introduced by Njåstad [13]. Since then it has been widely investigated in several literatures (see [1, 3, 4, 5, 6, 7, 8, 9, 11, 14]). In [2], Caldas introduced and studied topological properties of α-derived, α-border, α-frontier, and α-exterior of a set using the concept of α-open sets. The notion of semi-open set was introduced by Levin [12]. In this paper, we introduce the notions of semi-limit point, semi-derived set, semi-interior, semi-closure, semi-border, and semi-frontier of a set using the concept of semi-open set, and study their topological properties.

2. Preliminaries

Through this paper, $(X, \mathscr{T})$ and $(Y, \mathscr{K})$ (simply X and Y) always mean topological spaces. A subset A of X is said to be *semi-open* [12]

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(respectively, *α-open* [13] and *pre-open* [10]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (respectively, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ and $A \subset \mathrm{Int}(\mathrm{Cl}(A))$). The complement of a pre-open set (respectively, an α-open set and a semi-open set) is called a *pre-closed set* (respectively, an *α-closed set* and a *semi-closed set*). The intersection of all pre-closed sets (respectively, α-closed sets and semi-closed sets) containing A is called the *pre-closure* (respectively, *α-closure* and *semi-closure*) of A, denoted by $\mathrm{Cl}_p(A)$ (respectively, $\mathrm{Cl}_\alpha(A)$ and $\mathrm{Cl}_s(A)$). A subset A is also pre-closed (respectively, α-closed and semi-closed) if and only if $A = \mathrm{Cl}_p(A)$ (respectively, $A = \mathrm{Cl}_\alpha(A)$ and $A = \mathrm{Cl}_s(A)$). We denote the family of pre-open sets (respectively, α-open sets and semi-open sets) of $(X, \mathscr{T})$ by $\mathscr{T}^p$ (respectively, $\mathscr{T}^\alpha$ and $\mathscr{T}^s$). Obviously, we have the following relations.

open set (closed set)

↓

α-open set (α-closed set)

↙ ↘

pre-open set (pre-closed set) semi-open set (semi-closed set)

None of these implications is reversible in general.

3. Semi-open sets and α-open sets

Definition 3.1. [13, 12] A subset A of X is said to be *semi-open* (respectively, *α-open*) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (respectively, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$). The complement of a semi-open set (respectively, an α-open set) is called a *semi-closed set* (respectively, an *α-closed set*). The intersection of all semi-closed sets (respectively, α-closed sets) containing A is called the *semi-closure* (respectively, *α-closure*) of A, denoted by $\mathrm{Cl}_s(A)$ (respectively, $\mathrm{Cl}_\alpha(A)$). A subset A is also semi-closed (respectively, α-closed) if and only if $A = \mathrm{Cl}_s(A)$ (respectively, $A = \mathrm{Cl}_\alpha(A)$). We denote the family of semi-open sets (respectively, α-open sets) of $(X, \mathscr{T})$ by $\mathscr{T}^s$ (respectively, $\mathscr{T}^\alpha$).

Note that every open set is an α-open set and every α-open set is a semi-open set. Hence, obviously, $\mathscr{T} \subset \mathscr{T}^\alpha \subset \mathscr{T}^s$ for every topology $\mathscr{T}$ on a set X. But the reverse inclusions are not true in general as seen in the following example.

Example 3.2. Let $\mathscr{T} = \{\emptyset, X, \{a\}, \{c,d\}, \{a,c,d\}\}$ be a topology on $X = \{a,b,c,d,e\}$. Then we have

$$\begin{aligned} \mathscr{T}^s &= \mathscr{T} \cup \{\{a,b\}, \{a,e\}, \{a,b,e\}, \{b,c,d\}, \{c,d,e\}, \{a,b,c,d\}, \\ &\qquad \{a,c,d,e\}, \{b,c,d,e\}\}, \\ \mathscr{T}^\alpha &= \mathscr{T} \cup \{\{a,b,c,d\}, \{a,c,d,e\}\}. \end{aligned}$$

Remark 3.3. In general, $\mathscr{T}^s$ is not a topology on X. In fact, we know that $\{b,c,d\} \in \mathscr{T}^s$ and $\{a,b,e\} \in \mathscr{T}^s$, but $\{b,c,d\} \cap \{a,b,e\} = \{b\} \notin \mathscr{T}^s$ in Example 3.2.

Proposition 3.4. *If* $\mathscr{I}$ (resp. $\mathscr{D}$) *is the indiscrete* (resp. *discrete*) *topology on a set* X, *then* $\mathscr{I}^s$ (resp. $\mathscr{D}^s$) *and* $\mathscr{I}^\alpha$ (resp. $\mathscr{D}^\alpha$) *are topologies on* X.

Proof. We not that $\mathscr{I}^s = \mathscr{I}^\alpha = \mathscr{I}$ and $\mathscr{D}^s = \mathscr{D}^\alpha = \mathscr{D}$. □

Theorem 3.5. *If a topology* $\mathscr{T}$ *on a set* X *contains only* $\emptyset$, X, *and* $\{a\}$ *for* $a \in X$, *then* $\mathscr{T}^s = \mathscr{T}^\alpha$.

Proof. Let $a \in X$ and let A be an element of $\mathscr{T}^s$. If $A \in \mathscr{T}$, then obviously $A \in \mathscr{T}^\alpha$. Assume that $A \notin \mathscr{T}$. If $a \notin A$, then $A \not\subset \mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Cl}(\emptyset) = \emptyset$ and so $A \notin \mathscr{T}^s$. Therefore $a \in A$, which implies that

$$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = \mathrm{Int}(\mathrm{Cl}(\{a\})) = \mathrm{Int}(X) = X \supset A.$$

Hence $A \in \mathscr{T}^\alpha$. Since $\mathscr{T}^\alpha \subset \mathscr{T}^s$ in general, we have $\mathscr{T}^\alpha = \mathscr{T}^s$. □

Example 3.6. Let $\mathscr{T} = \{X, \emptyset, \{a,b\}, \{a,b,e\}, \{a,b,c,d\}\}$ be a topology on $X = \{a,b,c,d,e\}$. Then the intersection of all nonempty proper open sets is nonempty, and we know that $\mathscr{T}^s = \mathscr{T}^\alpha$.

In the following theorem, we generalize Example 3.6.

Theorem 3.7. *Let $\mathscr{T}$ be a topology on a set X such that*

$$\cap\{G \mid G \in \mathscr{T} \setminus \{\emptyset\}\} \neq \emptyset.$$

Then $\mathscr{T}^s = \mathscr{T}^\alpha$.

Proof. Let $A \in \mathscr{T}^s$. If $A \in \mathscr{T}$, then clearly $A \in \mathscr{T}^\alpha$. Assume that $A \notin \mathscr{T}$. Then

$$\cap\{G \mid G \in \mathscr{T} \setminus \{\emptyset\}\} \subset \mathrm{Int}(A),$$

and so $X = \mathrm{Cl}(\cap\{G \mid G \in \mathscr{T} \setminus \{\emptyset\}\}) \subset \mathrm{Cl}(\mathrm{Int}(A))$. It follows that $\mathrm{Cl}(\mathrm{Int}(A)) = X$ so that $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = \mathrm{Int}(X) = X \supset A$. Thus $A \in \mathscr{T}^\alpha$. This completes the proof. □

Theorem 3.8. *Let $\mathscr{T}$ be a topology on a set X. If every element of $\mathscr{T}$ is closed, then $\mathscr{T}^s \subset \mathscr{T}$, and so $\mathscr{T} = \mathscr{T}^s = \mathscr{T}^\alpha$.*

Proof. Let $B \notin \mathscr{T}$. Then $\mathrm{Int}(B) \neq B$. Since $\mathrm{Int}(B) \in \mathscr{T}$, $\mathrm{Int}(B)$ is closed by assumption. Thus $\mathrm{Cl}(\mathrm{Int}(B)) = \mathrm{Int}(B)$, and so $B \notin \mathscr{T}^s$. Therefore $\mathscr{T}^s \subset \mathscr{T}$. □

Example 3.9. Let $\mathscr{T} = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}\}$ be a topology on $X = \{a,b,c,d\}$. Note that every open set in $\mathscr{T}$ is also a closed set. We know that $\mathscr{T}^s = \mathscr{T} = \mathscr{T}^\alpha$.

Corollary 3.10. *Let $\mathscr{T}$ be a class of subsets of a set X consisting of four sets, i.e., $\mathscr{T} = \{X, \emptyset, A, B\}$, where A and B are nonempty distinct proper subsets of X.*

(i) *If $\{A, B\}$ is a partition of X, then $\mathscr{T}^s = \mathscr{T} = \mathscr{T}^\alpha$.*
(ii) *If either $A \subset B$ or $B \subset A$, then $\mathscr{T} \subset \mathscr{T}^s = \mathscr{T}^\alpha$.*

Proof. Straightforward. □

4. Applications of semi-open sets

Definition 4.1. Let A be a subset of a topological space X. A point $x \in X$ is said to be *semi-limit point* (resp. *α-limit point*) of A if it satisfies the following assertion:

$$(\forall G \in \mathscr{T}^s(\text{ resp. } \mathscr{T}^\alpha))\,(x \in G \Rightarrow G \cap (A \setminus \{x\}) \neq \emptyset).$$

The set of all semi-limit points (resp. α-limit points) of A is called the *semi-derived set* (resp. *α-derived set*) of A and is denoted by $D_s(A)$ (resp. $D_\alpha(A)$).

Note that for a subset A of X, a point $x \in X$ is not a semi-limit point (resp. α-limit point) of A if and only if there exists a semi-open set (resp. α-open set) G in X such that

$$x \in G \quad \text{and} \quad G \cap (A \setminus \{x\}) = \emptyset$$

or, equivalently,

$$x \in G \quad \text{and} \quad G \cap A = \emptyset \quad \text{or} \quad G \cap A = \{x\}$$

or, equivalently,

$$x \in G \quad \text{and} \quad G \cap A \subset \{x\}.$$

Example 4.2. (1) Let $X = \{a, b, c\}$ with topology $\mathscr{T} = \{X, \emptyset, \{a\}\}$. Then we have the followings:

(i) $\mathscr{T}^s = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\} = \mathscr{T}^\alpha$.

(ii) If $A = \{c\}$, then $D(A) = \{b\}$, $D_\alpha(A) = \emptyset$, and $D_s(A) = \emptyset$.

(iii) If $B = \{a\}$ and $C = \{b, c\}$, then $D_s(B) = \{b, c\}$, $D_s(C) = \emptyset$ and $D_s(B \cup C) = \{b, c\}$.

(2) Let $X = \{a, b, c\}$ with topology $\mathscr{T} = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then we have the followings:

(i) $\mathscr{T}^\alpha = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} = \mathscr{T}$.

(ii) $\mathscr{T}^s = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$.

(iii) If $A = \{c\}$, then $D(A) = \emptyset$, $D_\alpha(A) = \emptyset$, and $D_s(A) = \emptyset$.

(iv) If $B = \{a\}$ and $C = \{b, c\}$, then $D_s(B) = \emptyset = D_s(C)$, $D_s(B \cup C) = \{c\}$, $D_\alpha(B) = \{c\} = D_\alpha(C)$ and $D_\alpha(B \cup C) = \{c\}$.

Theorem 4.3. *Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be topologies on X such that $\mathscr{T}_1^s \subset \mathscr{T}_2^s$. For any subset A of X, every semi-limit point of A with respect to $\mathscr{T}_2$ is a semi-limit point of A with respect to $\mathscr{T}_1$.*

Proof. Let x be a semi-limit point of A with respect to $\mathscr{T}_2$. Then $G \cap (A \setminus \{x\}) \neq \emptyset$ for every $G \in \mathscr{T}_2^s$ such that $x \in G$. But $\mathscr{T}_1^s \subset \mathscr{T}_2^s$, so, in particular, $G \cap (A \setminus \{x\}) \neq \emptyset$ for every $G \in \mathscr{T}_1^s$ such that $x \in G$. Hence x is a semi-limit point of A with respect to $\mathscr{T}_1$. □

The following example shows that the converse of Theorem 4.3 is not true in general.

Example 4.4. Let

$$\mathscr{T}_1 = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$$

and

$$\mathscr{T}_2 = \{X, \emptyset, \{c\}, \{b, c\}, \{b, c, d\}\}$$

be topologies on $X = \{a, b, c, d\}$. Then

$$\mathscr{T}_1^s = \{X, \emptyset, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\},$$

$$\mathscr{T}_2^s = \{X, \emptyset, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, \{a, b, c\}\}.$$

Let $A = \{a, c\}$ be a subset of X. Then semi-limit points of A with respect to $\mathscr{T}_1$ are b and d; and semi-limit points of A with respect to $\mathscr{T}_2$ are a, b and d.

Lemma 4.5. *If $\{A_i \mid i \in \Lambda\}$ is a family of semi-open sets in X, then $\bigcup_{i \in \Lambda} A_i$ is a semi-open set in X where Λ is any index set.*

Proof. Straightforward. □

Theorem 4.6. *For any subsets A and B of X, the following assertions are valid:*

(1) $D_s(A) \subset D_\alpha(A) \subset D(A)$.
(2) *If* $A \subset B$, *then* $D_s(A) \subset D_s(B)$.
(3) $D_s(A) \cup D_s(B) \subset D_s(A \cup B)$ *and* $D_s(A \cap B) \subset D_s(A) \cap D_s(B)$.
(4) $D_s(D_s(A)) \setminus A \subset D_s(A)$.
(5) $D_s(A \cup D_s(A)) \subset A \cup D_s(A)$.

Proof. (1) It suffices to observe that every α-open set is semi-open, and every open set is an α-open set.

(2) Assume that $A \subset B$ and let $x \in D_s(A)$. Then $G \cap (A \setminus \{x\}) \neq \emptyset$ for all semi-open set G containing x. It follows from $A \subset B$ that $G \cap (B \setminus \{x\}) \neq \emptyset$ so that $x \in D_s(B)$.

(3) Straightforward by (2).

(4) Let $x \in D_s(D_s(A)) \setminus A$ and let $G \in \mathscr{T}^s$ with $x \in G$. Then $G \cap (D_s(A) \setminus \{x\}) \neq \emptyset$. Let $y \in G \cap (D_s(A) \setminus \{x\})$. Then $y \in G$ and $y \in D_s(A)$, and so $G \cap (A \setminus \{y\}) \neq \emptyset$. If we take $z \in G \cap (A \setminus \{y\})$, then $z \neq x$ because $x \notin A$. Hence $G \cap (A \setminus \{x\}) \neq \emptyset$, and therefore $x \in D_s(A)$.

(5) Let $x \in D_s(A \cup D_s(A))$. If $x \in A$, then the result is obvious. Suppose $x \notin A$. Then $G \cap ((A \cup D_s(A)) \setminus \{x\}) \neq \emptyset$ for all $G \in \mathscr{T}^s$ with $x \in G$. Hence $(G \cap A) \setminus \{x\} \neq \emptyset$ or $G \cap (D_s(A) \setminus \{x\}) \neq \emptyset$. The first case implies $x \in D_s(A)$. If $G \cap (D_s(A) \setminus \{x\}) \neq \emptyset$, then it follows similarly from (4) that $(G \cap A) \setminus \{x\} \neq \emptyset$ so that $x \in D_s(A)$. Therefore (5) is valid. □

In Theorem 4.6, the reverse inclusions are not valid as seen in the following examples.

Example 4.7. Consider a topology $\mathscr{T} = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ on a set $X = \{a, b, c\}$. Then $\mathscr{T}^\alpha = \mathscr{T}$ and

$$\mathscr{T}^s = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

Then $D_s(\{a\}) = \emptyset$ and $D_\alpha(\{a\}) = \{c\}$. For two subsets $A = \{a\}$ and $B = \{b, c\}$ of X, we get $D_s(A) \cup D_s(B) = \emptyset$ and $D_s(A \cup B) = \{c\}$.

Example 4.8. Let $\mathscr{T}$ be the topology on $X = \{a, b, c, d, e\}$ which is given in Example 3.2. Then $D_\alpha(\{b\}) = \emptyset$ and $D(\{b\}) = \{e\}$. For two subsets $A = \{b\}$ and $B = \{c, d, e\}$ of X, we have $D_s(A) = \emptyset$ and $D_s(B) = \{c, d\}$. For two subsets $C = \{a, c\}$ and $D = \{a, d\}$ of X, we get $D_s(C) \cap D_s(D) = \{b, e\}$ and $D_s(C \cap D) = \emptyset$. For a subset $G = \{c, d, e\}$ of X, we obtain $D_s(G) = \{c, d\}$ and so $D_s(D_s(G)) \setminus G = \emptyset$. Also, $D_s(G \cup D_s(G)) = \{c, d\}$ and $G \cup D_s(G) = \{c, d, e\}$.

Theorem 4.9. *Let A be a subset of X and $x \in X$. Then the following are equivalent:*

(i) $(\forall G \in \mathscr{T}^s)\ (x \in G \Rightarrow A \cap G \neq \emptyset)$.

(ii) $x \in \mathrm{Cl}_s(A)$.

Proof. (i) $\Rightarrow$ (ii). If $x \notin \mathrm{Cl}_s(A)$, then there exists a semi-closed set F such that $A \subset F$ and $x \notin F$. Hence $X \setminus F$ is a semi-open set in X containing x, and $A \cap (X \setminus F) \subset A \cap (X \setminus A) = \emptyset$. This is a contradiction, and hence the result is valid.

(ii) $\Rightarrow$ (i). Straightforward. □

Corollary 4.10. *For any subset A of X, we have $D_s(A) \subset \mathrm{Cl}_s(A)$.*

Proof. Straightforward. □

Theorem 4.11. *For any subset A of X, $\mathrm{Cl}_s(A) = A \cup D_s(A)$.*

Proof. Let $x \in \mathrm{Cl}_s(A)$. Assume that $x \notin A$ and let $G \in \mathscr{T}^s$ with $x \in G$. Then $(G \cap A) \setminus \{x\} \neq \emptyset$ by Theorem 4.9 and so $x \in D_s(A)$. Hence $\mathrm{Cl}_s(A) \subset A \cup D_s(A)$. The reverse inclusion is by $A \subset \mathrm{Cl}_s(A)$ and Corollary 4.10. □

Theorem 4.12. *Let A and B be subsets of X. If $A \in \mathscr{T}^s$ and $\mathscr{T}^s$ is a topology on X, then $A \cap \mathrm{Cl}_s(B) \subset \mathrm{Cl}_s(A \cap B)$.*

Proof. Let $x \in A \cap \mathrm{Cl}_s(B)$. Then $x \in A$ and $x \in \mathrm{Cl}_s(B) = B \cup D_s(B)$. If $x \in B$, then $x \in A \cap B \subset \mathrm{Cl}_s(A \cap B)$. If $x \notin B$, then $x \in D_s(B)$ and so $G \cap B \neq \emptyset$ for all semi-open set G containing x. Since $A \in \mathscr{T}^s$, $G \cap A$ is a semi-open set containing x. Hence $G \cap (A \cap B) = (G \cap A) \cap B \neq \emptyset$ and consequently $x \in D_s(A \cap B) \subset \mathrm{Cl}_s(A \cap B)$. Therefore $A \cap \mathrm{Cl}_s(B) \subset \mathrm{Cl}_s(A \cap B)$. □

The equality in Theorem 4.12 may not be true as seen in the following example.

Example 4.13. Let $\mathscr{T} = \{X, \emptyset, \{c\}, \{b, c\}, \{b, c, d\}\}$ be a topology on $X = \{a, b, c, d\}$. Then

$$\mathscr{T}^s = \{X, \emptyset, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, \{a, b, c\}\}$$

is a topology on X. Let $A = \{b, c\}$ and $B = \{c, d\}$ be subsets of X. Then $\mathrm{Cl}_s(B) = X = \mathrm{Cl}_s(A \cap B)$ and $A \cap \mathrm{Cl}_s(B) = \{b, c\}$.

In Theorem 4.12, the condition that *$\mathscr{T}^s$ is a topology on X* can't be omitted. In fact, if we consider a topology $\mathscr{T} = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ on $X = \{a, b, c, d\}$, then

$$\mathscr{T}^s = \{X, \emptyset, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}$$

is not a topology on X. For two subsets $A = \{c, d\}$ and $B = \{a, c\}$ of X, we have $A \cap \mathrm{Cl}_s(B) = \{c, d\}$ and $\mathrm{Cl}_s(A \cap B) = \{c\}$.

Lemma 4.14. *A subset A of X is semi-open if and only if there exists an open set H in X such that $H \subset A \subset \mathrm{Cl}(H)$.*

Proof. If A is semi-open, then $\mathrm{Int}(A) \subset A \subset \mathrm{Cl}(\mathrm{Int}(A))$. Since $\mathrm{Int}(A)$ is an open set, the result is valid when we take $H = \mathrm{Int}(A)$. Conversely, assume that there exists an open set H in X such that $H \subset A \subset \mathrm{Cl}(H)$. Then

$$A \subset \mathrm{Cl}(H) = \mathrm{Cl}(\mathrm{Int}(H)) \subset \mathrm{Cl}(\mathrm{Int}(A)),$$

and so A is semi-open. □

Theorem 4.15. *The intersection of an open set and a semi-open set is a semi-open set.*

Proof. Let A be an open set in X and B a semi-open set in X. Then there exists an open set G in X such that $G \subset B \subset \mathrm{Cl}(G)$. It follows that

$$A \cap G \subset A \cap B \subset A \cap \mathrm{Cl}(G) \subset \mathrm{Cl}(A \cap G).$$

Now since $A \cap G$ is open, we know that $A \cap B$ is semi-open by Lemma 4.14. □

Theorem 4.16. *Let A and B be subsets of X. If A is semi-closed, then $\mathrm{Cl}_s(A \cap B) \subset A \cap \mathrm{Cl}_s(B)$.*

Proof. If A is semi-closed, then $A = \mathrm{Cl}_s(A)$ and so

$$\mathrm{Cl}_s(A \cap B) \subset \mathrm{Cl}_s(A) \cap \mathrm{Cl}_s(B) = A \cap \mathrm{Cl}_s(B).$$

□

Theorem 4.17. *If A is a subset of a discrete topological space X, then $D_s(A) = \emptyset$.*

Proof. Let x be any element of X. Recall that every subset of X is open, and so semi-open. In particular, the singleton set $G := \{x\}$ is semi-open. But $x \in G$ and $G \cap A = \{x\} \cap A \subset \{x\}$. Hence x is not a semi-limit point of A, and so $D_s(A) = \emptyset$. □

Theorem 4.18. *If A is a subset of an indiscrete topological space X, then*

$$D_s(A) = \begin{cases} \emptyset & \text{if } A = \emptyset, \\ X \setminus A & \text{if } A \text{ is a singleton set,} \\ X & \text{otherwise,} \end{cases}$$

Proof. Straightforward. □

Theorem 4.19. *For any subset A of X, we have*

A is semi-closed if and only if $D_s(A) \subset A$.

Proof. ($\Rightarrow$) Let $x \notin A$, i.e., $x \in X \setminus A$. Since $X \setminus A$ is semi-open, x is not a semi-limit point of A, i.e., $x \notin D_s(A)$, because $X \setminus A \cap (A \setminus \{x\}) = \emptyset$. Hence $D_s(A) \subset A$.

($\Leftarrow$) It is obvious. □

Corollary 4.20. *Let A be a subset of X. If F is a semi-closed superset of A, then $D_s(A) \subset F$.*

Proof. Straightforward. □

Theorem 4.21. *Let A be a subset of X and $x \in X$. Then*

$$x \in D_s(A) \Longleftrightarrow x \in D_s(A \setminus \{x\}).$$

Proof. ($\Rightarrow$) Let $x \in X$ be a semi-limit point of A. Then $G \cap (A \setminus \{x\}) \neq \emptyset$ for all semi-open set G in X containing x. It follows that

$$G \cap ((A \setminus \{x\}) \setminus \{x\}) = G \cap (A \setminus \{x\}) \neq \emptyset$$

so that x is a semi-limit point of $A \setminus \{x\}$.

($\Leftarrow$) It is by Theorem 4.6(ii). □

Definition 4.22. [2] Let A be a subset of a topological space X. A point $x \in X$ is called an *α-interior point* of A if there exists an α-open set G containing x such that $G \subset A$. The set of all α-interior points of A is called the *α-interior* of A and is denoted by $\mathrm{Int}_\alpha(A)$.

Based on the above definition, we give the notion of a semi-interior point.

Definition 4.23. Let A be a subset of a topological space X. A point $x \in X$ is called a *semi-interior point* of A if there exists a semi-open set G such that $x \in G \subset A$. The set of all semi-interior points of A is called the *semi-interior* of A and is denoted by $\mathrm{Int}_s(A)$.

Example 4.24. Let $\mathscr{T}$ be the topology on $X = \{a, b, c, d, e\}$ which is given in Example 3.2. For a subset $A = \{a, b, c, e\}$ of X, we have $\mathrm{Int}_\alpha(A) = \{a\}$ and $\mathrm{Int}_s(A) = \{a, b, e\}$.

Theorem 4.25. *Let A be a subset of X. Then every α-interior point of A is a semi-interior point of A, i.e.,* $\operatorname{Int}_\alpha(A) \subset \operatorname{Int}_s(A)$.

Proof. Let $x \in \operatorname{Int}_\alpha(A)$. Then there exists $G \in \mathscr{T}^\alpha$ such that $x \in G \subset A$. Since $\mathscr{T}^\alpha \subset \mathscr{T}^s$, it follows that $x \in G \subset A$ for some $G \in \mathscr{T}^s$ so that $x \in \operatorname{Int}_s(A)$. This completes the proof. □

Note that, in Example 4.24, b is a semi-interior point of A which is not an α-interior point of A. Thus the converse of Theorem 4.25 is not true in general.

Proposition 4.26. *For subsets A and B of X, the following assertions are valid.*

(1) $\operatorname{Int}_s(A)$ *is the union of all semi-open subsets of* A;
(2) A *is semi-open if and only if* $A = \operatorname{Int}_s(A)$;
(3) $\operatorname{Int}_s(\operatorname{Int}_s(A)) = \operatorname{Int}_s(A)$;
(4) $\operatorname{Int}_s(A) = A \setminus D_s(X \setminus A)$.
(5) $X \setminus \operatorname{Int}_s(A) = \operatorname{Cl}_s(X \setminus A)$.
(6) $X \setminus \operatorname{Cl}_s(A) = \operatorname{Int}_s(X \setminus A)$.
(7) $A \subset B \Rightarrow \operatorname{Int}_s(A) \subset \operatorname{Int}_s(B)$.
(8) $\operatorname{Int}_s(A) \cup \operatorname{Int}_s(B) \subset \operatorname{Int}_s(A \cup B)$.
(9) $\operatorname{Int}_s(A \cap B) \subset \operatorname{Int}_s(A) \cap \operatorname{Int}_s(B)$.

Proof. (1) Let $\{G_i\}$ be a class of semi-open subsets of A. If $x \in \operatorname{Int}_s(A)$, then there exists $G_j \in \mathscr{T}^s$ such that $x \in G_j \subset A$. Thus $x \in G_j \subset \bigcup_i G_i$, and so $\operatorname{Int}_s(A) \subset \bigcup_i G_i$. Now let $x \in \bigcup_i G_i$. Then $x \in G_i$ for some $G_i \in \mathscr{T}^s$. Since $G_i \subset A$, we have $x \in \operatorname{Int}_s(A)$. Therefore $\bigcup_i G_i \subset \operatorname{Int}_s(A)$.

(2) Straightforward.

(3) Since $\operatorname{Int}_s(A)$ is semi-open, it follows from (2) that $\operatorname{Int}_s(\operatorname{Int}_s(A)) = \operatorname{Int}_s(A)$.

(4) If $x \in \operatorname{Int}_s(A)$, then $x \in A$. Note that

$$\operatorname{Int}_s(A) \cap ((X \setminus A) \setminus \{x\}) = (\operatorname{Int}_s(A) \setminus \{x\}) \cap (X \setminus A) \subset A \cap (X \setminus A) = \emptyset,$$

which shows that $x \notin D_s(X \setminus A)$, and thus $x \in A \setminus D_s(X \setminus A)$. Therefore $\mathrm{Int}_s(A) \subset A \setminus D_s(X \setminus A)$. Now assume that $x \in A$ and $x \notin D_s(X \setminus A)$. Then there exists $G \in \mathscr{T}^s$ such that $x \in G$ and $((X \setminus A) \cap G) \setminus \{x\} = \emptyset$. It follows that $x \in G \subset A$ so that $x \in \mathrm{Int}_s(A)$. Hence $A \setminus D_s(X \setminus A) \subset \mathrm{Int}_s(A)$.

(5) Using Theorem 4.11, we have

$$X \setminus \mathrm{Int}_s(A) = X \setminus (A \setminus D_s(X \setminus A)) = (X \setminus A) \cup D_s(X \setminus A) = \mathrm{Cl}_s(X \setminus A).$$

(6) and (7) Straightforward.

(8) and (9) They are by (7). □

The converse of (7) in Proposition 4.26 is not true in general as seen in the following example.

Example 4.27. Consider a topology

$$\mathscr{T} = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$$

on $X = \{a, b, c, d\}$. Then $\mathscr{T}^s = \mathscr{T}$. Let $A = \{b, d\}$ and $B = \{a, b, c\}$ be subsets of X. Then $\mathrm{Int}_s(A) = \{b\} \subset \{b, c\} = \mathrm{Int}_s(B)$.

Definition 4.28. [2] For any subset A of X, the set

$$b_\alpha(A) := A \setminus \mathrm{Int}_\alpha(A)$$

is called the *α-border* of A, and the set

$$\mathrm{Fr}_\alpha(A) := \mathrm{Cl}_\alpha(A) \setminus \mathrm{Int}_\alpha(A)$$

is called the *α-frontier* of A.

Definition 4.29. For any subset A of X, the set

$$b_s(A) := A \setminus \mathrm{Int}_s(A)$$

is called the *semi-border* of A, and the set

$$\mathrm{Fr}_s(A) := \mathrm{Cl}_s(A) \setminus \mathrm{Int}_s(A)$$

is called the *semi-frontier* of A.

Note that if A is a semi-closed subset of X, then the semi-border of A is equal to the semi-frontier of A.

Example 4.30. Let $\mathscr{T}$ be the topology on $X = \{a, b, c, d, e\}$ which is given in Example 3.2. For a subset $A = \{a, b, c, e\}$ of X, we have $b_\alpha(A) = \{b, c, e\}$, $b_s(A) = \{c\}$, $\mathrm{Fr}_\alpha(A) = \{b, c, d, e\}$ and $\mathrm{Fr}_s(A) = \{c, d\}$.

Proposition 4.31. *For a subset A of X, the following statements hold:*

(1) $b_s(A) \subset b_\alpha(A)$.
(2) $A = \mathrm{Int}_s(A) \cup b_s(A)$.
(3) $\mathrm{Int}_s(A) \cap b_s(A) = \emptyset$.
(4) *A is a semi-open set if and only if $b_s(A) = \emptyset$.*
(5) $b_s(\mathrm{Int}_s(A)) = \emptyset$.
(6) $\mathrm{Int}_s(b_s(A)) = \emptyset$.
(7) $b_s(b_s(A)) = b_s(A)$.
(8) $b_s(A) = A \cap \mathrm{Cl}_s(X \setminus A)$.
(9) $b_s(A) = A \cap D_s(X \setminus A)$.

Proof. (1) Obviously $b_s(A) \subset b_\alpha(A)$ since $\mathrm{Int}_\alpha(A) \subset \mathrm{Int}_s(A)$.

(2) and (3) Straightforward.

(4) Note from Proposition 4.26(2) that

$$A \text{ is semi-open} \;\Leftrightarrow\; A = \mathrm{Int}_s(A) \;\Leftrightarrow\; b_s(A) = A \setminus \mathrm{Int}_s(A) = \emptyset.$$

(5) Since $\mathrm{Int}_s(A)$ is semi-open, it follows from (4) that $b_s(\mathrm{Int}_s(A)) = \emptyset$.

(6) If $x \in \mathrm{Int}_s(b_s(A))$, then there exists $G \in \mathscr{T}^s$ such that $x \in G \subset b_s(A) \subset A$, which implies that $x \in \mathrm{Int}_s(A)$. Thus $x \in \mathrm{Int}_s(A) \cap b_s(A) = \emptyset$. This is a contradiction, and so $\mathrm{Int}_s(b_s(A)) = \emptyset$.

(7) Using (6), we have

$$b_s(b_s(A)) = b_s(A) \setminus \mathrm{Int}_s(b_s(A)) = b_s(A).$$

(8) Using Proposition 4.26(6), we get

$$b_s(A) = A \setminus \mathrm{Int}_s(A) = A \setminus (X \setminus \mathrm{Cl}_s(X \setminus A)) = A \cap \mathrm{Cl}_s(X \setminus A).$$

(9) Using Proposition 4.26(4), we obtain

$$b_s(A) = A \setminus \mathrm{Int}_s(A) = A \setminus (A \setminus D_s(X \setminus A)) = A \cap D_s(X \setminus A).$$

This completes the proof. □

In Example 4.30 we can see that the reverse inclusion of Proposition 4.31(1) may not be true.

Combining Theorem 4.6(1) and [2, Theorem 2.4], we have the following assertion.

Lemma 4.32. *For a subset A of X, we have* $\mathrm{Cl}_s(A) \subset \mathrm{Cl}_\alpha(A)$.

Theorem 4.33. *For a subset A of X, the following assertions are valid:*

(1) $\mathrm{Fr}_s(A) \subset \mathrm{Fr}_\alpha(A)$.
(2) $\mathrm{Cl}_s(A) = \mathrm{Int}_s(A) \cup \mathrm{Fr}_s(A)$.
(3) $\mathrm{Int}_s(A) \cap \mathrm{Fr}_s(A) = \emptyset$.
(4) $b_s(A) \subset \mathrm{Fr}_s(A)$.
(5) $\mathrm{Fr}_s(A) = b_s(A) \cup (D_s(A) \setminus \mathrm{Int}_s(A))$.
(6) *A is a semi-open set if and only if* $\mathrm{Fr}_s(A) = b_s(X \setminus A)$.
(7) $\mathrm{Fr}_s(A) = \mathrm{Cl}_s(A) \cap \mathrm{Cl}_s(X \setminus A)$.
(8) $\mathrm{Fr}_s(A) = \mathrm{Fr}_s(X \setminus A)$.
(9) $\mathrm{Fr}_s(A)$ *is semi-closed.*
(10) $\mathrm{Fr}_s(\mathrm{Fr}_s(A)) \subset \mathrm{Fr}_s(A)$.
(11) $\mathrm{Fr}_s(\mathrm{Int}_s(A)) \subset \mathrm{Fr}_s(A)$.
(12) $\mathrm{Fr}_s(\mathrm{Cl}_s(A)) \subset \mathrm{Fr}_s(A)$.
(13) $\mathrm{Int}_s(A) = A \setminus \mathrm{Fr}_s(A)$.

Proof. (1) Using Lemma 4.32 and Theorem 4.25 we have

$$\mathrm{Fr}_s(A) = \mathrm{Cl}_s(A) \setminus \mathrm{Int}_s(A) \subset \mathrm{Cl}_\alpha(A) \setminus \mathrm{Int}_\alpha(A) = \mathrm{Fr}_\alpha(A).$$

(2) Straightforward.

(3) $\text{Int}_s(A) \cap \text{Fr}_s(A) = \text{Int}_s(A) \cap (\text{Cl}_s(A) \setminus \text{Int}_s(A)) = \emptyset$.

(4) Since $A \subset \text{Cl}_s(A)$, we get

$$b_s(A) = A \setminus \text{Int}_s(A) \subset \text{Cl}_s(A) \setminus \text{Int}_s(A) = \text{Fr}_s(A).$$

(5) Using Theorem 4.11, we get

$$\begin{aligned} \text{Fr}_s(A) &= \text{Cl}_s(A) \setminus \text{Int}_s(A) = (A \cup D_s(A)) \cap (X \setminus \text{Int}_s(A)) \\ &= (A \setminus \text{Int}_s(A)) \cup (D_s(A) \setminus \text{Int}_s(A)) \\ &= b_s(A) \cup (D_s(A) \setminus \text{Int}_s(A)). \end{aligned}$$

(6) Assume that A is semi-open. Then

$$\text{Fr}_s(A) = \text{Cl}_s(A) \setminus \text{Int}_s(A) = (A \cup D_s(A)) \setminus A = D_s(A) \setminus A = b_s(X \setminus A)$$

by Theorem 4.11, Proposition 4.26(2) and Proposition 4.31(9). Conversely suppose that $\text{Fr}_s(A) = b_s(X \setminus A)$. Then

$$\emptyset = \text{Fr}_s(A) \setminus b_s(X \setminus A) = (\text{Cl}_s(A) \setminus \text{Int}_s(A)) \setminus ((X \setminus A) \setminus \text{Int}_s(X \setminus A)) = A \setminus \text{Int}_s(A)$$

by Proposition 4.26(5), and so $A \subset \text{Int}_s(A)$. Since $\text{Int}_s(A) \subset A$, it follows that $\text{Int}_s(A) = A$ so from Proposition 4.26(2) that A is semi-open.

(7) Using Proposition 4.26(5), it can be easily proved.

(8) This is by (7).

(9) We have

$$\begin{aligned} \text{Cl}_s(\text{Fr}_s(A)) &= \text{Cl}_s(\text{Cl}_s(A) \setminus \text{Int}_s(A)) \\ &= \text{Cl}_s(\text{Cl}_s(A) \cap \text{Cl}_s(X \setminus A)) \\ &\subset \text{Cl}_s(\text{Cl}_s(A)) \cap \text{Cl}_s(\text{Cl}_s(X \setminus A)) \\ &= \text{Cl}_s(A) \cap \text{Cl}_s(X \setminus A) \\ &= \text{Fr}_s(A), \end{aligned}$$

and so $\text{Fr}_s(A)$ is semi-closed.

(10) Using (7), Proposition 4.26(6) and Proposition 4.26(8), we have

$$\begin{aligned}
&\mathrm{Fr}_s(\mathrm{Fr}_s(A)) = \mathrm{Fr}_s(\mathrm{Cl}_s(A) \cap \mathrm{Cl}_s(X \setminus A)) \\
&= \mathrm{Cl}_s(\mathrm{Cl}_s(A) \cap \mathrm{Cl}_s(X \setminus A)) \cap \mathrm{Cl}_s(X \setminus (\mathrm{Cl}_s(A) \cap \mathrm{Cl}_s(X \setminus A))) \\
&= \mathrm{Cl}_s(\mathrm{Cl}_s(A) \cap \mathrm{Cl}_s(X \setminus A)) \cap \mathrm{Cl}_s(\mathrm{Int}_s(X \setminus A) \cup \mathrm{Int}_s(A)) \\
&\subset \mathrm{Cl}_s(\mathrm{Cl}_s(A) \cap \mathrm{Cl}_s(X \setminus A)) \cap \mathrm{Cl}_s(\mathrm{Int}_s((X \setminus A) \cup A)) \\
&= \mathrm{Cl}_s(\mathrm{Cl}_s(A) \cap \mathrm{Cl}_s(X \setminus A)) \\
&\subset \mathrm{Cl}_s(A) \cap \mathrm{Cl}_s(X \setminus A) \\
&= \mathrm{Fr}_s(A).
\end{aligned}$$

(11) Applying Proposition 4.26(3), we get

$$\begin{aligned}
\mathrm{Fr}_s(\mathrm{Int}_s(A)) &= \mathrm{Cl}_s(\mathrm{Int}_s(A)) \setminus \mathrm{Int}_s(\mathrm{Int}_s(A)) \\
&= \mathrm{Cl}_s(\mathrm{Int}_s(A)) \setminus \mathrm{Int}_s(A) \\
&\subset \mathrm{Cl}_s(A) \setminus \mathrm{Int}_s(A) = \mathrm{Fr}_s(A).
\end{aligned}$$

(12) We obtain

$$\begin{aligned}
\mathrm{Fr}_s(\mathrm{Cl}_s(A)) &= \mathrm{Cl}_s(\mathrm{Cl}_s(A)) \setminus \mathrm{Int}_s(\mathrm{Cl}_s(A)) \\
&\subset \mathrm{Cl}_s(A) \setminus \mathrm{Int}_s(A) = \mathrm{Fr}_s(A).
\end{aligned}$$

(13) Using (7) and Proposition 4.26(6) we have

$$\begin{aligned}
A \setminus \mathrm{Fr}_s(A) &= A \setminus (\mathrm{Cl}_s(A) \cap \mathrm{Cl}_s(X \setminus A)) \\
&= A \cap (\mathrm{Int}_s(X \setminus A) \cup \mathrm{Int}_s(A)) \\
&= (A \cap \mathrm{Int}_s(X \setminus A)) \cup \mathrm{Int}_s(A) \\
&= (A \cap (X \setminus \mathrm{Cl}_s(A))) \cup \mathrm{Int}_s(A) \\
&= (A \setminus \mathrm{Cl}_s(A)) \cup \mathrm{Int}_s(A) \\
&= \mathrm{Int}_s(A).
\end{aligned}$$

This completes the proof. □

The converses of (1) and (4) of Theorem 4.33 are not true in general as seen in Example 4.30.

Definition 4.34. [2] For a subset A of X, $\mathrm{Ext}_\alpha(A) = \mathrm{Int}_\alpha(X \setminus A)$ is said to be an *α-exterior* of A.

Definition 4.35. For a subset A of X, the semi-interior of $X \setminus A$ is called the *semi-exterior* of A, and is denoted by $\text{Ext}_s(A)$, that is,

$$\text{Ext}_s(A) = \text{Int}_s(X \setminus A).$$

Theorem 4.36. *For subsets A and B of X, the following assertions are valid.*

(1) $\text{Ext}_\alpha(A) \subset \text{Ext}_s(A)$.
(2) $\text{Ext}_s(A)$ *is semi-open.*
(3) $\text{Ext}_s(A) = X \setminus \text{Cl}_s(A)$.
(4) $\text{Ext}_s(\text{Ext}_s(A)) = \text{Int}_s(\text{Cl}_s(A)) \supset \text{Int}_s(A)$.
(5) $A \subset B \Rightarrow \text{Ext}_s(B) \subset \text{Ext}_s(A)$.
(6) $\text{Ext}_s(A \cup B) \subset \text{Ext}_s(A) \cap \text{Ext}_s(B)$.
(7) $\text{Ext}_s(A \cap B) \supset \text{Ext}_s(A) \cup \text{Ext}_s(B)$.
(8) $\text{Ext}_s(X) = \emptyset$, $\text{Ext}_s(\emptyset) = X$.
(9) $\text{Ext}_s(A) = \text{Ext}_s(X \setminus \text{Ext}_s(A))$.
(10) $X = \text{Int}_s(A) \cup \text{Ext}_s(A) \cup \text{Fr}_s(A)$.

Proof. (1) Using Theorem 4.25, we have

$$\text{Ext}_\alpha(A) = \text{Int}_\alpha(X \setminus A) \subset \text{Int}_s(X \setminus A) = \text{Ext}_s(A).$$

(2) and (3) are straightforward.

(4) Applying Proposition 4.26(5), we get

$$\begin{aligned}\text{Ext}_s(\text{Ext}_s(A)) &= \text{Ext}_s(\text{Int}_s(X \setminus A)) \\ &= \text{Int}_s(X \setminus \text{Int}_s(X \setminus A)) \\ &= \text{Int}_s(\text{Cl}_s(A)) \supset \text{Int}_s(A).\end{aligned}$$

(5) Assume that $A \subset B$. Then

$$\text{Ext}_s(B) = \text{Int}_s(X \setminus B) \subset \text{Int}_s(X \setminus A) = \text{Ext}_s(A)$$

by using Proposition 4.26(7).

(6) Applying Proposition 4.26(9), we get

$$\begin{aligned}\mathrm{Ext}_s(A\cup B) &= \mathrm{Int}_s(X\setminus(A\cup B))\\ &= \mathrm{Int}_s((X\setminus A)\cap(X\setminus B))\\ &\subset \mathrm{Int}_s(X\setminus A)\cap\mathrm{Int}_s(X\setminus B)\\ &= \mathrm{Ext}_s(A)\cap\mathrm{Ext}_s(B).\end{aligned}$$

(7) Using Proposition 4.26(8), we obtain

$$\begin{aligned}\mathrm{Ext}_s(A\cap B) &= \mathrm{Int}_s(X\setminus(A\cap B))\\ &= \mathrm{Int}_s((X\setminus A)\cup(X\setminus B))\\ &\supset \mathrm{Int}_s(X\setminus A)\cup\mathrm{Int}_s(X\setminus B)\\ &= \mathrm{Ext}_s(A)\cup\mathrm{Ext}_s(B).\end{aligned}$$

(8) Straightforward.

(9) We have

$$\mathrm{Ext}_s(X\setminus\mathrm{Ext}_s(A))=\mathrm{Ext}_s(X\setminus\mathrm{Int}_s(X\setminus A))=\mathrm{Int}_s(X\setminus A)=\mathrm{Ext}_s(A).$$

(10) Straightforward. □

In Example 3.2, take $A=\{c,d\}$. Then $\mathrm{Ext}_\alpha(A)=\{a\}$ and $\mathrm{Ext}_s(A)=\{a,b,e\}$. Thus the reverse inclusion of Theorem 4.36(1) is not valid. In Example 4.27, let $B=\{a,c\}$ and $A=\{d\}$. Then $\mathrm{Ext}_s(B)=\{b\}\subset\{b,c\}=\mathrm{Ext}_s(A)$. This shows that the converse of (5) in Theorem 4.36 is not valid. In Example 3.2, let $A=\{b,c,d\}$ and $B=\{a,b\}$. Then $\mathrm{Ext}_s(A\cup B)=\mathrm{Int}_s(\{e\})=\emptyset$, $\mathrm{Ext}_s(A)=\mathrm{Int}_s(\{a,e\})=\{a,e\}$, and $\mathrm{Ext}_s(B)=\mathrm{Int}_s(\{c,d,e\})=\{c,d,e\}$. Thus $\mathrm{Ext}_s(A\cup B)\neq\mathrm{Ext}_s(A)\cap\mathrm{Ext}_s(B)$, which shows that the equality in Theorem 4.36(6) is not valid. In Example 4.27, let $A=\{b,c,d\}$ and $B=\{a,c\}$. Then $\mathrm{Ext}_s(A\cap B)=\mathrm{Int}_s(\{a,b,d\})=\{a,b,d\}$, $\mathrm{Ext}_s(A)=\mathrm{Int}_s(\{a\})=\emptyset$, and $\mathrm{Ext}_s(B)=\mathrm{Int}_s(\{b,d\})=\{b\}$. Hence $\mathrm{Ext}_s(A)\cup\mathrm{Ext}_s(B)=\{b\}\neq\mathrm{Ext}_s(A\cap B)$, which shows that the equality in Theorem 4.36(7) is not valid.

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