Honam Mathematical J. **29** (2007), No. 2, pp. 299–305

CARATHÉODORY FINITELY COMPACTNESS OF THE BOUNDED ATTRACTING BASIN OF THE ORIGIN

Sung-Hee Park

Abstract. We prove that the bounded attracting basin of the origin for a complex homogeneous polynomial of degree larger than two is Carathéodory finitely compact.

1. Introduction

For a bounded pseudoconvex balanced domain $D = D_h \subset \mathbb{C}^n$ with Minkowski function h, it is a well known problem to characterize the Carathéodory, the Bergman or the Kobayashi completeness of D via properties of h: see e.g. [10], [5]. But this problem does not yet understand completely. In 1983 T. Barth proved that if $D = D_h$ is a bounded pseudoconvex balanced domain which is Kobayashi complete, then its Minkowski function h is necessarily continuous (see [1]). In 1991 M. Jarnicki and P. Pflug proved that for $n \geq 3$, there is a bounded balanced domain of holomorphy in $\mathbb{C}^n$ with continuous Minkowski function that is not Kobayashi complete (see [4], [9]). In 2000 M. Jarnicki, P. Pflug, and W. Zwonek showed that any bounded pseudoconvex balanced domain is Bergman complete (see [6]).

In this paper, we shall recall the definition and some results of attracting basin $\mathcal{A}_F$ of the origin for a non-degenerate complex homogeneous polynomial F of degree ≥ 2, an example of balanced domains, which

Received May. 10, 2007. Accepted Jun. 8, 2007.
2000 Mathematics Subject Classification: 32F45, 32A07.
Key words and phrases: balanced domain, Carathéodory finitely compact, basin of attraction.

was already studied by T. Ueda [11]. We shall give a characterization of hyperbolicity of $\mathcal{A}_F$ and prove that if $\mathcal{A}_F$ is bounded, then it is also Carathéodory finitely compact.

2. Preliminaries

Before we state the result of the paper, let us recall the notation and definitions we shall need.

Let $G \subset \mathbb{C}^n$, $\Omega \subset \mathbb{C}^m$ be domains. Denote by $\mathcal{O}(G, \Omega)$ the set of all holomorphic maps from G to Ω with the compact open topology, $\mathcal{O}(G) := \mathcal{O}(G, \mathbb{C})$, and by $\mathcal{H}^\infty(G)$ the family of all bounded holomorphic functions on G.

Let us recall that G is pseudoconvex iff it is $\mathcal{O}(G)$-*convex*, that is, the set

$$\widehat{K}^{\mathcal{O}(G)} := \left\{ z \in G : |f(z)| \leq \sup_{w \in K} |f(w)|, \, \forall f \in \mathcal{O}(G) \right\}$$

is compact for every compact set $K \subset G$. We put $\mathcal{H}_p^\infty(G) := \{f \in \mathcal{H}^\infty(G) : f(p) = 0\}$ and say that G is $\mathcal{H}_p^\infty(G)$-*convex* whenever

$$\widehat{K}^{\mathcal{H}_p^\infty(G)} := \left\{ z \in G : |f(z)| \leq \sup_{w \in K} |f(w)|, \, \forall f \in \mathcal{H}^\infty(G), \, f(p) = 0 \right\}$$

is compact for every compact set $K \subset G$.

By E we denote the open unit disc in the complex plane $\mathbb{C}$, i.e., $E = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. A set $D \subset \mathbb{C}^n$ is called *balanced* whenever $\bar{E} \cdot D = D$, i.e., $\lambda z \in D$ for any $z \in D$ and $\lambda \in \bar{E}$. For a balanced domain $D \subset \mathbb{C}^n$ there exists a unique function $h \equiv h_D$ on $\mathbb{C}^n$ such that:

- h is upper semicontinuous on $\mathbb{C}^n$,
- h is absolutely homogeneous on $\mathbb{C}^n$, i.e. $h(\lambda z) = |\lambda| h(z)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$,
- $D = \{z \in \mathbb{C}^n : h(z) < 1\} =: D_h$.

Namely, such a function $h = h_D : \mathbb{C}^n \longrightarrow [0, \infty)$ is called the *Minkowski function of* D, which is designated by

$$h_D(z) := \inf \left\{ \alpha > 0 : \frac{z}{\alpha} \in D \right\}, \qquad z \in \mathbb{C}^n.$$

Note that $D = D_h$ does not contain a complex line through o iff $h^{-1}(0) = \{\mathrm{o}\}$, where $\mathrm{o} \equiv \mathrm{o}_n := (0, \cdots, 0) \in \mathbb{C}^n$. It is known that $D = D_h$ is pseudocovex iff $\log h \in \mathcal{PSH}(\mathbb{C}^n)$ iff $h \in \mathcal{PSH}(\mathbb{C}^n)$, where $\mathcal{PSH}(\mathbb{C}^n)$ is the set of all plurisubharmonic functions on $\mathbb{C}^n$ (cf. [3]).

We say that a domain G is:

- *Carathéodory finitely compact* if any Carathéodory ball with finite radius is relatively compact (in the Euclidean topology) inside G;
- *taut* if $\mathcal{O}(E, G)$ is a *normal family*;
- *hyperconvex* if there exists a continuous *bounded plurisubharmonic* exhaustion function on G.

Recall that the following implications are well known:

$$\text{Carathéodory finitely compact} \Longrightarrow \text{hyperconvex}$$
$$\Longrightarrow \text{taut} \Longrightarrow \text{pseudoconvex}.$$

In particular, a balanced domain $D = D_h \subset \mathbb{C}^n$ is hyperconvex iff it is taut iff it is bounded and $h \in (\mathcal{C} \cap \mathcal{PSH})(\mathbb{C}^n)$. These properties can be found in e.g. [1], [7], [4], [9].

Now let us recall some known results that will be needed in the sequel.

Theorem 2.1. ([2]) *Any bounded domain $G \subset \mathbb{C}^n$ which is $\mathcal{H}_p^\infty(G)$-convex for some $p \in G$ is hyperconvex.*

Theorem 2.2. ([8]) *A bounded balanced domain $D = D_h \subset \mathbb{C}^n$ is Carathéodory finitely compact iff it is $\mathcal{H}_{\mathrm{o}}^\infty(D)$-convex and $h \in \mathcal{C}(\mathbb{C}^n)$.*

3. Main results

In this section, unless otherwise stated, we shall use the following notations: Let $F = (F_1, \cdots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a homogeneous complex polynomial of degree $d \geq 2$, i.e. $F(\lambda z_1, \cdots, \lambda z_n) = \lambda^d F(z_1, \cdots, z_n)$ for any $\lambda \in \mathbb{C}$, $(z_1, \cdots, z_n) \in \mathbb{C}^n$. Put $F^1 := F$ and for any $m \geq 2$ we write $F^m := F \circ F^{m-1}$. Now we define a set

$$\mathcal{A}_F := \left\{ z \in \mathbb{C}^n : \lim_{m \to \infty} F^m(z) = o \right\}.$$

We call it by the *basin of attraction at the origin* (or the *attracting basin at the origin*)

Now we will represent some basic properties of the attracting basin $\mathcal{A}_F$ at o with respect to a homogeneous complex polynomial F. From the definition of $\mathcal{A}_F$, it follows immediately that $\mathcal{A}_F$ is a balanced domain in $\mathbb{C}^n$. From now on, we will denote *the Minkowski function* of $\mathcal{A}_F$ by h_F.

Proposition 3.1. ([11]) *We have $h_F(F(z)) = [h_F(z)]^d$ for any $z \in \mathbb{C}^n$. If $0 < r < (2M)^{-1/(d-1)}$, where $M := \sup\{\|F(z)\| : \|z\| = 1\}$, then we have:*

(a) $\|F(z)\| \leq \frac{1}{2}\|z\|$ whenever $\|z\| < r$.

(b) $\mathcal{A}_F$ is pseudoconvex, i.e. $\log h_F \in \mathcal{PSH}(\mathbb{C}^n)$.

The following result gives a characterization of the Minkowski function h_F of A_F via the homogeneous complex polynomial F of degree larger than two.

Proposition 3.2. ([11]) *The function Φ defined by*

$$\Phi(z) := \lim_{j \to \infty} d^{-j} \log \|F^j(z)\| \qquad (z \in \mathbb{C}^n)$$

is plurisubharmonic on $\mathbb{C}^n$ and $\Phi(z) = \log h_F(z)$ for any $z \in \mathbb{C}^n$.

Next, it turns out that there is the following full characterization of hyperbolic basin of attraction at the origin, namely:

Lemma 3.3. *The following conditions are equivalent:*

(a) F is non-degenerate, *i.e.*, $F^{-1}(o) = \{o\}$.

(b) $\liminf_{\|z\|\to\infty} \|F(z)\|/\|z\|^d > 0$.

(c) $\mathcal{A}_F$ is bounded.

(d) $\mathcal{A}_F$ does not contain complex lines through the origin.

(e) $h_F^{-1}(0) = \{o\}$.

Thus, all notions of hyperbolicity announced in Remark 7.1.1 of [5] coincide in the class of attracting basin at the origin.

Proof. Note that the implications (c) $\Longrightarrow$ (d) $\Longrightarrow$ (e) are obvious.

(a) $\Longrightarrow$ (b): If F is nondegenerate, then

$$\inf\left\{\frac{\|F(z)\|}{\|z\|^d} : z \in \mathbb{C}^n,\, z \neq o\right\} = \inf\left\{\|F(z)\| : z \in \mathbb{C}^n,\, \|z\| = 1\right\} > 0,$$

so we obtain (b).

(b) $\Longrightarrow$ (c): Put $m := \liminf_{\|z\|\to\infty} \|F(z)\|/\|z\|^d > 0$. For any $\varepsilon \in (0, m/2)$, there exists $R > (4/m)^{\frac{1}{d-1}}$ such that $\|F(z)\|/\|z\|^d > m - \varepsilon > m/2$ for any $\|z\| > R$. That is, for any $\|z\| > R$ one has

$$\|F(z)\| \geq \frac{m}{2}\|z\|^d = 2\|z\|\left(\frac{m}{4}\|z\|^{d-1}\right) > 2\|z\|\left(\frac{m}{4}R^{d-1}\right) \geq 2\|z\|,$$

which implies that $\{z \in \mathbb{C}^n : \|z\| > R\} \cap \mathcal{A}_F = \emptyset$. Thus $\mathcal{A}_F$ is bounded in $\mathbb{C}^n$.

(e) $\Longrightarrow$ (a): Suppose the contrary. Let $z_0 \in F^{-1}(o) \setminus \{o\}$. Then one has $0 = h_F(F(z_0)) = [h_F(z_0)]^d$, so $z_0 \in h_F^{-1}(0)\setminus\{o\}$, i.e. $h_F^{-1}(0) \neq \{o\}$.□

Recall that

Theorem 3.4. ([11]) *If F is non-degenerate, then $h_F \in \mathcal{C}(\mathbb{C}^n)$, so $\mathcal{A}_F$ is hyperconvex (also taut).*

Using Theorem 2.1, we are also able to reprove the previous result. For this we need the following fact.

Theorem 3.5. *If F is non-degnerate, then the balanced domain $\mathcal{A} \equiv \mathcal{A}_F$ is $\mathcal{H}_o^\infty(\mathcal{A})$-convex.*

Proof. Note that $\mathcal{A}$ is bounded by Lemma 3.3. Now, assume that $\mathcal{A}$ is not $\mathcal{H}_{\circ}^{\infty}(\mathcal{A})$-convex. Then there is a compact set $K \subset \mathcal{A}$ such that $\widehat{K}^{\mathcal{H}_{\circ}^{\infty}(\mathcal{A})} \cap \partial A \neq \varnothing$, that is, there are a sequence $(z^{\nu})_{\nu \geq 1}$ in $\widehat{K}^{\mathcal{H}_{\circ}^{\infty}(\mathcal{A})}$ and a point $z^* \in \partial\mathcal{A}$ such that $z^{\nu} \to z^*$ as $\nu \to \infty$. Let $j \geq 1$. Observe that $F^j(\mathcal{A}) \subset \mathcal{A}$ and $F^j(\mathrm{o}) = \mathrm{o}$. If $j \geq 1$ and we write $F^j =: (F_{j,1}, \cdots, F_{j,n})$, one has $F_{j,\mu} \in \mathcal{H}_{\circ}^{\infty}(\mathcal{A})$ for any $\mu = 1, \cdots, n$, and also

$$|F_{j,\mu}(z^{\nu})| \leq \sup_{w \in K} |F_{j,\mu}(w)| \leq \sup_{w \in K} \|F^j(w)\|, \qquad \mu = 1, \cdots, n.$$

Let $0 < r < (2M)^{-1/(d-1)}$, where $M := \sup\{\|F(z)\| : \|z\| = 1\}$. By (a) of Proposition 3.1, we can take a number $j(K) \in \mathbb{N}$ such that for any $\nu \in \mathbb{N}$ and $\mu \in \{1, \cdots, 2\}$ one has

$$|F_{j,\mu}(z^{\nu})| < \frac{r}{2\sqrt{n}}, \qquad j \geq j(K).$$

Therefore, for every $j \geq j(K)$ we have

$$\begin{aligned} \|F^j(z^*)\| &= \left\| F^j\left(\lim_{\nu \to \infty} z^{\nu}\right) \right\| \\ &= \lim_{\nu \to \infty} \|F^j(z^{\nu})\| = \lim_{\nu \to \infty} \left(\sum_{\mu=1}^{n} |F_{j,\mu}(z^{\nu})|^2 \right)^{1/2} < r. \end{aligned}$$

Using again (a) of Proposition 3.1, we have that $\lim_{j \to \infty} \|F^j(z^*)\| = 0$, which is a contradiction to the fact that $z^* \in \partial\mathcal{A}$. □

Combining Theorem 3.5 and Theorem 2.1 we get

Corollary 3.6. *If F is non-degenerate, then $\mathcal{A}_F$ is a bounded hyperconvex domain, in particular, h_F is continuous on $\mathbb{C}^n$.*

Moreover, in the view of Theorem 2.2, we obtain

Corollary 3.7. *The bounded basin of attraction of the origin for a complex homogeneous polynomial of degree ≥ 2 is Carathéodory finitely complete.*

References

[1] T. J. Barth, *The Kobayashi indicatrix at the center of a circular domain*, Proc. Amer. Math. Sco. **88** (1983) 527-530

[2] G. Berg, *Hyperconvexity and the Carathéodory metric*, Arch. Math. **32** (1979) 189-191

[3] P. Jakóbczak & M. Jarnicki, *Lectures on holomorphic functions of several complex variables*, PS File at 'http://www.im.uj.edu.pl/~jarnicki/mjp.htm', 2001.

[4] M. Jarnicki & P. Pflug, *A counterexample for Kobayashi completeness of balanced domains*, Proc. Amer. Math. Soc. **112** (1991), 973-978.

[5] M. Jarnicki & P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, de Gruyter Expositions in Mathematics 9, Walter de Gruyter 1993.

[6] M. Jarnicki, P. Pflug, & W. Zwonek *On Bergman completeness of non-hyperconvex domains*, Univ. Iag. Acta Math. **38** (2000), 169-184.

[7] N. Kerzman & J.-P. Rosay, *Fonctions plurisousharmoniques d'exhasution bornées et domaines taut*, Math. Ann. **257** (1981), 171-184.

[8] V. Z. Khristov, *A sufficient condition for Carathéodory finite compactness of bounded complete circular domain in* $\mathbb{C}^n$, C. R. Acad. Bulg. Sci. **42** (3) (1989), 9-11.

[9] S.-H. Park, *Tautness and Kobayashi hyperbolciity*, doctor theses, University of Oldenburg (2003).

[10] P. Pflug, *Invariant metrics and completeness*, J. Korean Math. Soc. **37** (2) (2000), 269-284.

[11] T. Ueda, *Fatou sets in complex dynamics on projective spaces*, J. Math. Soc. Japan **46** (1994), 545-555.

Department of Mathematical Education,
Jeonju University,
Jeonju, Jeonbuk 560-759, Republic of Korea
E-mail: `wshpark@jj.ac.kr`