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ON INJECTIVE BCI-ALGEBRAS

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ABSTRACT. In this paper, we show that BCI-algebras P and Q are injective if and only if its direct sum $P \oplus Q$ is injective. Moreover, we obtain the equivalent conditions for a p-semisimple BCI-algebra to be p-injective.

1. Introduction

K. Iséki introduced the notion of BCI-algebra which is a generalization of a BCK-algebra. W. A. Dudek ([5]) defined the concept of a medial BCI-algebra and studied various properties of it. C. S. Hoo ([8]) proved that BCI-algebra X is medial if and only if it is p-semisimple. S. S. Ahn, H. S. Kim and A. Dvurećenskij ([7]) discussed the Ker-Coker sequence in BCI-algebras. W. A. Dudek ([6]) showed that p-semisimple BCI-algebras are precisely medial quasigroups completely described via abelian groups. This means that any discussions on p-semisimple BCI-algebras can be derived easily from group theory ([6]). C. S. Hoo and P. V. R. Murty ([10]) and E. Y. Deeba and S. K. Goal ([4]) independently showed that $Hom(X)$ may not, in general, be a BCI-algebra for an arbitrary BCI-algebra X. In view of this result we can also see that $Hom(X, Y)$, the set of all homomorphisms of a BCI-algebra X into an arbitrary BCI-algebra Y may not be a BCI-algebra in general. However, E. Y. Deeba and S. K. Goal ([4]) proved that if X is a BCI-algebra and Y is a BCK-algebra, then $Hom(X, Y)$ is a BCK-algebra and hence a BCI-algebra. Y. Liu ([13]) showed that if X is a BCI-algebra and Y is a p-semisimple BCI-algebra

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then $Hom(X, Y)$ is a p-semisimple BCI-algebra. In [11] and [12], Y. B. Jun et al. investigated some properties of $Hom(-,-)$ as BCK/BCI-algebras. S. S. Ahn and H. S. Kim ([1]) defined a hom functor $Hom(\_,\_)$ in BCK/BCI-algebras and discussed the exactness of $Hom(\_,\_)$ in BCK/BCI-algebras, and obtained some properties of $Hom(\_,\_)$. S. S. Ahn and K. Bang ([2]) obtained that $Hom(P,\_)$ is an exact functor if P is a p-projective BCI-algebra. In this paper, we show that BCI-algebras P and Q are injective if and only if its direct sum $P \oplus Q$ is injective. Moreover, we obtain the equivalent conditions for a p-semisimple BCI-algebra to be p-injective.

2. Preliminaries

Recall that a *BCI-algebra* ([14]) is a non-empty set X with a binary operation " $*$ " and a constant 0 satisfying the axioms:

(1) $\{(x * y) * (x * z)\} * (z * y) = 0$,

(2) $\{x * (x * y)\} * y = 0$,

(3) $x * x = 0$,

(4) $x * y = 0$ and $y * x = 0$ imply that $x = y$,

for any $x, y, z \in X$. Furthermore, if satisfies (5) $0 * x = 0, \forall x \in X$, then the algebra is called a *BCK-algebra.* A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x * y = 0$. A mapping $f : X \to Y$ from a BCK/BCI-algebra X into a BCK/BCI-algebra Y is a called a *BCK/BCI-homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Define the trivial homomorphism 0 as $0(x) = 0$ for all $x \in X$. Denote by $Hom(X, Y)$ the set of all homomorphisms from a BCK/BCI-algebra X into a BCK/BCI-algebra Y

Let X and Y be BCI-algebras. A BCI-homomorphism $f : X \to Y$ is said to be *regular* if Imf is an ideal of Y. We call an ideal A of X *regular* in case A is a subalgebra of X. In the literature, Imf need not be an ideal of Y.

Let $f : A \to B$ and $g : B \to C$ be BCI-homomorphisms. The sequence $A \to B \to C$ is said to be *exact* at B if $Kerg = Imf$. A sequence $A_0 \to A_1 \to \cdots \to A_n \to A_{n+1}$ is said to be *exact* if it is exact at $A_1, \cdots, A_n$.

S. S. Ahn and H. S. Kim ([1]) defined a hom functor $Hom(\_,\_)$ in BCK/BCI-algebras and discussed the exactness of $Hom(\_,\_)$ in BCK/BCI-algebras. Let $\mathbb{C}$ and $\mathbb{D}$ be any categories of BCK/BCI-algebras. A *functor* from $\mathbb{C}$ to $\mathbb{D}$ is a triple $(\mathbb{C}, F, \mathbb{D})$, where F is a function from the class of BCK/BCI-homomorphisms of $\mathbb{C}$ to the class of BCK/BCI-homomorphisms of $\mathbb{D}$ (i.e., $F : Hom(\mathbb{C}) \to Hom(\mathbb{D})$) satisfying the following conditions:

(i) F preserves identities: if e is a $\mathbb{C}$-identity, then $F(e)$ is a $\mathbb{D}$-identity

(ii) F preserves composition: i.e., $F(f \circ g) = F(f) \circ F(g)$; i.e., whenever $dom(f) = cod(g)$, then $dom(F(f)) = cod(F(g))$ and the above equality holds.

Let $u : A' \to A$ and $v : B \to B'$ be BCI-homomorphisms, where A, B' are BCK-algebras. We define a mapping

$$Hom(u, v) : Hom(A, B) \to Hom(A', B')$$

by requiring that $f \in Hom(A, B)$ is to be mapped into $vfu \in Hom(A', B')$, where B is a BCK-algebra. Clearly, $Hom(u, v)$ is a BCI-homomorphism and if u, v are identity maps, then $Hom(u, v)$ is also an identity map.

Again if $u' : A'' \to A'$ and $v' : B' \to B''$ are BCI-homomorphisms, where A', B'' are BCK-algebras, then

$$Hom(uu', v'v) = Hom(u', v')Hom(u, v).$$

In fact, $Hom(\_,\_)$ is a functor. This functor is called a *hom functor*.

Note that $Hom(u, B) = Hom(u, 1_B)$ and $Hom(A, v) = Hom(1_A, v)$. Let X and Y be BCI-algebras and let $X \oplus Y = \{(x, y) \mid x \in X, y \in Y\}$. We define the operation " $*$ " on $X \oplus Y$ by

$$(x, y) * (x', y') := (x * x', y * y')$$

for all $(x, y), (x', y') \in X \oplus Y$. Then $(X \oplus Y, *, (0, 0))$ is a BCI-algebra, which is called the *direct sum* of X and Y (see [9]). If $u_1, u_2 : A' \to A$ and $v_1, v_2 : B \to B'$ are BCI-homomorphisms, where A, B' are BCK-algebras, then the mappings $u_1 \oplus u_2 : A' \oplus A' \to A \oplus A$ defined by $u_1 \oplus u_2(x_1, x_2) = (u_1(x_1), u_2(x_2))$ for any $(x_1, x_2) \in A' \oplus A'$, and $v_1 \oplus v_2 : B \oplus B \to B' \oplus B'$ defined by $v_1 \oplus v_2(x_1, x_2) = (v_1(x_1), v_2(x_2))$ for any $(x_1, x_2) \in B \oplus B$, are BCI-homomorphisms.

PROPOSITION 2.1. ([1]) *If $u_1, u_2 : A' \to A$ and $v_1, v_2 : B \to B'$ are BCI-homomorphisms, where A, B and B' are BCK-algebras, then*

$$Hom(u_1 \oplus u_2, B) = Hom(u_1, B) \oplus Hom(u_2, B)$$

and

$$Hom(A, v_1 \oplus v_2) = Hom(A, v_1) \oplus Hom(A, v_2)$$

i.e., a hom functor is additive.

LEMMA 2.2. ([2]) *Let $f : X \to Y$ be a homomorphism, where X is a BCI-algebra and Y is a p-semisimple BCI-algebra. Then Imf is an ideal of Y. Hence f is a regular homomorphism.*

THEOREM 2.3. ([1]) *If σ, π are regular BCI-homomorphisms in the exact sequence of BCI-algebras:*

$$M' \xrightarrow{\sigma} M \xrightarrow{\pi} M'' \to 0$$

then, for any BCK-algebra N, the sequence

$$0 \to Hom(M'', N) \xrightarrow{Hom(\pi, N)} Hom(M, N) \xrightarrow{Hom(\sigma, N)} Hom(M', N)$$

is exact.

Using Lemma 2.2 we know that if $f : X \to Y$ is a BCI-homomorphism from a BCI-algebra X into a p-semisimple BCI-algebra Y, then it is a regular BCI-homomorphism. Y. Liu ([13]) showed that if X is a BCI-algebra and Y is a p-semisimple BCI-algebra then $Hom(X, Y)$ is a p-semisimple BCI-algebra. With this concept S. S. Ahn and K. Bang ([2]) generalized Theorem 2.3 as follows:

THEOREM 2.3′. ([2]) *If $M' \xrightarrow{\sigma} M \xrightarrow{\pi} M'' \to 0$ is an exact sequence of* *3CI-algebras, where M and M'' are p-semisimple BCI-algebras, then, for any* *)-semisimple BCI-algebra N, the sequence*

$$0 \to Hom(M'', N) \xrightarrow{Hom(\pi, N)} Hom(M, N) \xrightarrow{Hom(\sigma, N)} Hom(M', N)$$

s exact.

3. Injective BCI-algebras

In this section, we show that $Hom(\_, J)$ is an exact functor if J is a p-njective BCI-algebra. Recall that a BCI-algebra X is called *injective* if for :very BCI-monomorphism $f : Y \to Z$ and every BCI-homomorphism ϕ : $\to X$ there exists a BCI-homomorphism $\mu : Z \to X$ satisfying $\mu f = \phi$, and ecall that a BCI-algebra X is said to be *p-injective* if for every p-semisimple 3CI-monomorphism $f : Y \to Z$ and every BCI-homomorphism $\phi : Y \to X$ here exists a BCI-homomorphism $\mu : Z \to X$ satisfying $\mu f = \phi$.

Let $A_1, A_2, \cdots, A_n$ be BCI-algebras and let $A_1 \oplus \cdots \oplus A_n := \{(a_1, \cdots, a_n) | a_i \in$ $A_i\}$. We define the operation "$\circledast$" on $A_1 \oplus \cdots \oplus A_n$ by $(a_1, \cdots, a_n) \circledast (b_1, \cdots, b_n) :=$ $(a_1 * b_1, \cdots, a_n * b_n)$ for all $(a_1, \cdots, a_n), (b_1, \cdots, b_n) \in A_1 \oplus \cdots \oplus A_n$. Then $(A_1 \oplus \cdots \oplus A_n; \circledast, (0, \cdots, 0))$ is a BCI-algebra, which is called the *direct sum* of $A_1, \cdots, A_n$. Clearly we have BCI-monomorphism $i_i : A_i \to A_1 \oplus \cdots \oplus A_n$ given by $i_i(a_i) := (0, \cdots, a_i, \cdots, 0)$ and BCI-epimorphism $\pi_i : A_1 \oplus \cdots \oplus A_n \to A_i$ given by

$\pi_i(a_1, \cdots, a_i, \cdots, a_n) := a_i$. Define a map $i_1\pi_1 \circledast i_2\pi_2 \circledast \cdots \circledast i_n\pi_n : A_1 \oplus \cdots \oplus$ $A_n \to A_1 \oplus \cdots \oplus A_n$ by $i_1\pi_1 \circledast i_2\pi_2 \circledast \cdots \circledast i_n\pi_n(a_1, \cdots, a_n) := i_1\pi_1(a_1, \cdots, a_n) \circledast$ $i_2\pi_2(a_1, \cdots, a_n) \circledast \cdots \circledast i_n\pi_n(a_1, \cdots, a_n)$. Then it is easy to show that $i_1\pi_1 \circledast$ $i_2\pi_2 \circledast \cdots \circledast i_n\pi_n$ is a BCI-homomorphism.

PROPOSITION 3.1. *Let $A, A_1, \cdots, A_n$ be BCI-algebras. Assume that for* *each $i = 1, \cdots, n$ there exist BCI-homomorphisms $\pi_i : A \to A_i$ and $i_i : A_i \to A$* *and $i_1\pi_1 \circledast i_2\pi_2 \circledast \cdots \circledast i_n\pi_n : A_1 \oplus \cdots \oplus A_n \to A_1 \oplus \cdots \oplus A_n$ which satisfies* *the following conditions:*

(i). $\pi_i i_i = 1_{A_i}$ *for* $i = 1, \cdots, n$;

(ii). $\pi_j i_i = 0$ *for* $i \neq j$;

(iii). $i_1\pi_1 \circledast i_2\pi_2 \circledast \cdots \circledast i_n\pi_n = 1_A$.

Then $A \cong A_1 \oplus \cdots \oplus A_n$.

Proof. Let $\pi_i : A \to A_i$ and $i_i : A_i \to A (i = 1, \cdots, n)$ be BCI-homomorphisms satisfying (i) $\sim$ (iii). Let $\widehat{\pi}_i : A_1 \oplus \cdots \oplus A_n \to A_i$ and $\widehat{\iota}_i : A_i \to A_1 \oplus \cdots \oplus A_n$ be the canonical projections and injections, respectively. If we define a map $\varphi : A_1 \oplus \cdots \oplus A_n \to A$ by $\varphi := i_1\widehat{\pi}_1 \circledast i_2\widehat{\pi}_2 \circledast \cdots \circledast i_n\widehat{\pi}_n$ and $\psi : A \to A_1 \oplus \cdots \oplus A_n$ by $\psi := \widehat{\iota}_1\pi_1 \circledast \widehat{\iota}_2\pi_2 \circledast \cdots \circledast \widehat{\iota}_n\pi_n$. Then

$$\begin{aligned}
\varphi\psi =&(i_1\widehat{\pi}_1 \circledast i_2\widehat{\pi}_2 \circledast \cdots \circledast i_n\widehat{\pi}_n)(\widehat{\iota}_1\pi_1 \circledast \widehat{\iota}_2\pi_2 \circledast \cdots \circledast \widehat{\iota}_n\pi_n) \\
=&i_1\widehat{\pi}_1\widehat{\iota}_1\pi_1 \circledast \cdots \circledast i_n\widehat{\pi}_n\widehat{\iota}_n\pi_n \\
=&i_1 1_{A_1}\pi_1 \circledast \cdots \circledast i_n 1_{A_n}\pi_n \\
=&i_1\pi_1 \circledast \cdots \circledast i_n\pi_n = 1_A.
\end{aligned}$$

Similarly $\psi\varphi = 1_{A_1 \oplus \cdots \oplus A_n}$. Therefore $A \cong A_1 \oplus \cdots \oplus A_n$, completing the proof. □

THEOREM 3.2. *Let P and Q be BCI-algebras. Then P and Q are injective if and only if $P \oplus Q$ is injective.*

Proof. Assume that P and Q are injective. Let $h : A \to B$ be any BCI-monomorphism and let $f : A \to P \oplus Q$ be a BCI-homomorphism.

$$\begin{array}{ccc}
A & \xrightarrow{h} & B \\
{\scriptstyle f}\downarrow & & \\
P \oplus Q & &
\end{array}$$

Since P and Q are injective, there exist BCI-homomorphisms $g_1 : B \to P$ and $g_2 : B \to Q$ such that $\pi_P f = g_1 h$ and $\pi_Q f = g_2 h$, where $\pi_P : P \oplus Q \to P$ and $\pi_Q : P \oplus Q \to Q$ are the canonical projection of BCI-algebras. Define a map $g : B \to P \oplus Q$ by $g(b) := (g_1(b), g_2(b))$ for any $b \in B$. Then g is a BCI-homomorphism. For any $a \in A$, we have

$$gh(a) = (g_1h(a), g_2h(a)) = (\pi_P f(a), \pi_Q f(a)) = f(a).$$

Hence $gh = f$, i.e., $P \oplus Q$ is injective.

Assume that $P \oplus Q$ is injective. Let $h : A \to B$ be a BCI-monomorphism and let $f_1 : A \to P$ and $f_2 : A \to Q$ be BCI-homomorphisms. Let $\pi_P : P \oplus Q \to P$ and $\pi_Q : P \oplus Q \to Q$ be the canonical projections of BCI-algebras and $i_P : P \to P \oplus Q$ and $i_Q : Q \to P \oplus Q$ be the canonical injections of BCI-algebras. Since $P \oplus Q$ is injective, there exist BCI-homomorphisms $g_i : B \to P \oplus Q (i = 1, 2)$ such that $g_1 h = i_P f_1$ and $g_2 h = i_Q f_2$. If we define $k_1 := \pi_P g_1$ and $k_2 := \pi_Q g_2$, then $k_1 : B \to P$ and $k_2 : B \to Q$ are BCI-homomorphisms satisfying $k_1 h = f_1$ and $k_2 h = f_2$, proving that P and Q are injective. □

COROLLARY 3.3. *Let P and Q be BCI-algebras. Then P and Q are p-injective if and only if $P \oplus Q$ is p-injective.*

Proof. Straightforward. □

THEOREM 3.4. *Every summand D of an injective BCI-algebra E is itself injective.*

Proof. Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle f}\downarrow & & \\ D & \xrightarrow{\lambda} & E \end{array}$$

where $\lambda : D \to E$ and $p : E \to D$ are the canonical injection and projection of BCI-algebras, respectively. Since E is injective, there is a BCI-homomorphism $g : B \to E$ with $g\alpha = \lambda f$. If we define $h : B \to D$ by $h = pg$, then $h\alpha = pg\alpha = p\lambda f = f$ since $p\lambda = 1_D$. Thus D is injective. □

THEOREM 3.5. *The following conditions on a p-semisimple BCI-algebra J are equivalent:*

(i). *J is p-injective;*

(ii). *if $\theta : A \to B$ is any p-semisimple BCI-monomorphism, then*

$$Hom(\theta, J) : Hom(B, J) \to Hom(A, J)$$

is an epimorphism of BCI*-algebras;*

(iii). *if* $0 \to A \xrightarrow{\theta} B \xrightarrow{\psi} C \to 0$ *is any short exact sequence of* p*-semisimple* BCI*-algebras, then* $0 \to Hom(C,J) \to Hom(B,J) \to Hom(A,J) \to 0$ *is an exact sequence of* BCI*-algebras.*

Proof. (i) $\Rightarrow$ (ii) Assume that J is p-injective. Let $\theta : A \to B$ be a p-semisimple BCI-monomorphism. Then the hom functor $Hom(\theta, J) : Hom(B,J) \to Hom(A,J)$ defined by $Hom(\theta,J)(g) = g\theta$ is a BCI-homomorphism. We claim that $Hom(\theta,J)$ is onto. For any $k \in Hom(A,J)$, since J is p-injective, there exists a BCI-homomorphism $g : B \to J$ such that $k = g\theta$. This means that $g \in Hom(B,J)$ and $Hom(\theta,J)(g) = g\theta = k$. Hence $Hom(\theta,J)$ is a BCI-epimorphism.

(ii) $\Rightarrow$ (i) Let $\theta : A \to B$ be a p-semisimple BCI-monomorphism and let $h : A \to J$ be a BCI-homomorphism. Since $Hom(\theta,J)$ is a BCI-epimorphism, there exists $g \in Hom(B,J)$ such that $h = Hom(\theta,J)(g)$. This means that $h = g\theta$, proving that J is p-injective.

(ii) $\Rightarrow$ (iii) It follows from Theorem 2.3$'$.

(iii) $\Rightarrow$ (ii) Let $\theta : A \to B$ be a BCI-monomorphism and let $C := B/\mathrm{Im}\theta$. Then $0 \to A \xrightarrow{\theta} B \xrightarrow{\pi} C = B/\mathrm{Im}\theta \to 0$ is an short exact sequence of BCI-algebras. By applying the condition (iii), $0 \to Hom(C,J) \xrightarrow{Hom(\pi,J)} Hom(B,J) \xrightarrow{Hom(\theta,J)} Hom(A,J) \to 0$ is an exact sequence of BCI-algebras, and hence $Hom(\theta,J) : Hom(B,J) \to Hom(A,J)$ is a BCI-epimorphism. □

The hom functor $Hom(\_, J)$ discussed in Theorem 3.5 is said to be *exact* if it satisfies the condition (iii) of Theorem 3.5. With this concept we conclude that *$Hom(\_, J)$ is exact if J is a p-injective BCI-algebra.*

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