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IRREDUCIBLE MODULES OVER THE E_6-TYPE LIE ALGEBRA

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ABSTRACT. Let $L := L(G)$ denote the classical modular Lie algebra of E_6-type over an algebraically closed field F of characteristic $p > 5$ associated with some simple and simply connected algebraic group G. After the prototypes of those for E_8-type in [4], we shall search for irreducible (=simple) L-modules in this paper.

§ 1. Introduction

We put L :=the modular classical E_6-type Lie algebra over an algebraically closed field F of characteristic $p \geq 7$ unless otherwise stated. Referring to chapter 5 in [4], we get the following facts without difficulty.

PROPOSITION(1.1). *Under the notations as above, we have* $[Q(\mathcal{U}(L)) : Q(\mathfrak{Z})] = p^{n-\ell} = p^{2m}$, *where* $\mathfrak{Z} := \mathfrak{Z}(\mathcal{U}(L))$, $\ell = rank\ L = 6$, *and* $n = dim\ L = 78$.

COROLLARY(1.2). *The universal enveloping algebra* $\mathcal{U}(L)$ *is a free* $\mathfrak{Z}$-*module of rank* $p^{2m} = p^{72}$.

As is known in [2], we may recollect the root system Φ of E_6 and its base $\Delta : \Phi = \{\varepsilon_i - \varepsilon_j$ for $i, j \leq 6$; $\frac{1}{2}(\sum_{i=1}^{6}(-1)^{k(i)}\varepsilon_i \pm \sqrt{2}\ \varepsilon_7)$ such that $k(i) = 0, 1$ and $\sum(-1)^{k(i)} = 0$; $\pm\sqrt{2}\ \varepsilon_7\}$ for orthonormal unit vectors $\varepsilon_i' s$ in the space $\mathbb{R}^7$ and $\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4 - \varepsilon_5, \varepsilon_5 - \varepsilon_6, \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \sqrt{2}\ \varepsilon_7)\}$. Let V be any finite dimensional irreducible L-module V; its associated irreducible representation $\rho : \mathcal{U}(L) \to End_F(V)$ is uniquely determined by $\mathfrak{Z}/\mathfrak{Z} \cap ker\rho$ up to isomorphisms. We also have relations

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$[Q(\mathcal{U}(L)) : Q(\mathfrak{Z})] = p^{2m}$ and $\rho(\mathcal{U}(L)) \subset M_m(F)$, the full matrix algebra of degree m [4]. Now let ξ_i for $i = 1, 2, \cdots, 78, 78+1$ be values in F corresponding to $x^p_{\alpha_1}, x^p_{-\alpha_1}, h^p_{\alpha_1} - h_{\alpha_1}, \cdots, x^p_{\alpha_6}, x^p_{-\alpha_6}, h^p_{\alpha_6} - h_{\alpha_6}, x^p_{\alpha_7}, x^p_{-\alpha_7}, \cdots, x^p_{\alpha_{36}}, x^p_{-\alpha_{36}}, s$ respectively which must satisfy $\mathrm{Irr}(s, \mathcal{O}(L))$ for the universal Casimir element s of L and for the p-center $\mathcal{O}(L)$ of L, where H is a standard Cartan subalgebra of L and $\{x_\alpha,\ \alpha \in \Phi\ ;\ h_{\alpha_i} \in H,\ 1 \leq i \leq 6\}$ is the standard Chevalley basis of L with a root system Φ. Recollect here that s has the form $s = \sum_{\alpha\in\Phi^+} a_\alpha x_\alpha x_{-\alpha} + \sum_{i=1}^{6} b_i h_{\alpha_i}$ ($b_i \neq 0$ in F)+ other terms of degree 2 in $\mathcal{U}(H)$. Suppose now that $\bar{\rho}$ is any left maximal ideal of $\mathcal{U}(L)$; then the annhilator $A(\mathcal{U}(L)/\bar{\rho}) := \{x \in \mathcal{U}(L) \mid x \cdot \mathcal{U}(L) \subset \bar{\rho}\}$ becomes the largest two sided ideal contained in $\bar{\rho}$, and hence it is just a maximal ideal $\mathfrak{M}_\chi$ of $\mathcal{U}(L)$ with an associated p-character χ to $\bar{\rho}$. It is easy to see that $A(\mathcal{U}(L)/\bar{\rho}) \cap \mathfrak{Z} = \mathfrak{Z}(x^p_{\alpha_1} - \xi_1) + \cdots + \mathfrak{Z}(x^p_{-\alpha_{36}} - \xi_{78}) + \mathfrak{Z}(s - \xi_{79})$ is equal to a maximal ideal $\overline{\mathfrak{M}}_\chi$ of $\mathfrak{Z} := \mathfrak{Z}(\mathcal{U}(L))$ for its associated p-character $\chi \in L^*$ and that $A(\mathcal{U}(L)/\bar{\rho})$ is the kernel of some irreducible representation ρ of L. On the other hand any maximal ideal $\mathfrak{M}_\chi$ of $\mathcal{U}(L)$ must contain some ideal $\{\sum_{i=1}^{78} \mathcal{U}(L)(\underset{\sim}{x_i}^p - \underset{\sim}{x_i}^{[p]} - \xi_i) + \mathcal{U}(L)(s - \xi_{79})\}$, where $\underset{\sim}{x_i}$ represents $x_{\alpha_1}, x_{-\alpha_1}, h_{\alpha_1}, \cdots, x_{\alpha_6}, x_{-\alpha_6}, h_{\alpha_6}, x_{\alpha_7}, x_{-\alpha_7}, \cdots, x_{\alpha_{36}}, x_{-\alpha_{36}}$ in order. In the next section 2, we shall deal with the dimensions of L-irreducible modules in connection with this. In section 3, we shall be concerned with the byproducts of Kac-Weisfeiler conjecture which is now clearly confirmed by A.Premet [5]. Finally in §4, some errata to the previous paper [3] shall be presented. By way of Jacobsion's density theorem, we will argue about the dimension of irreducible modules over the E_6-type Lie algebra.

PROPOSITION(2.1). *In the notations as above, an arbitrary nonrestricted simple L-module has its dimension* p^{36} *over* F.

Proof. Noting that E_6 has a sub-root system of E_8's, we shall imitate the proof for E_8 in chapter 7 in [4]. We shall do in several steps.

(I) Let $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and let $\xi_1 \neq 0$. For $1 \leq i \leq 72$, we set $B_i := b_{i1}(c_{i1} + h_{\varepsilon_1-\varepsilon_2}) + b_{i2}(c_{i2} + h_{\varepsilon_2-\varepsilon_3}) + b_{i3}(c_{i3} + h_{\varepsilon_3-\varepsilon_4}) + b_{i4}(c_{i4} + h_{\varepsilon_4-\varepsilon_5}) + b_{i5}(c_{i5} + h_{\varepsilon_5-\varepsilon_6}) + b_{i6}(c_{i6} + h_{\frac{1}{2}(-\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\sqrt{2}\ \varepsilon_7)})$, where we have chosen c_{ij}'$s \in F$ so that parentheses are invertible mod $\mathfrak{M}_\chi$, and we have chosen $(b_{i1}, \cdots, b_{i6}) \in F^6$ so that any 7 B_i's are linearly independent in $\mathbb{P}^6(F)$, the $B_i \cdot x_{\varepsilon_1-\varepsilon_2} \not\equiv x_{\varepsilon_1-\varepsilon_2} \cdot B_i$

(mod$\mathfrak{M}_\chi$), and the $\mathcal{B}$ below becomes an F-linearly independent set in $\mathcal{U}(L)$ if necessary. Here we select a basis candidate like $\mathcal{B} := \{(B_1 + A_{\varepsilon_1-\varepsilon_2})^{i_1} \otimes (B_2 + A_{-(\varepsilon_1-\varepsilon_2)})^{i_2} \otimes \cdots \otimes (B_{12} + A_{-\frac{1}{2}(-\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\sqrt{2}\ \varepsilon_7)})^{i_{12}} \otimes (\otimes_{j=13}^{72}(B_j + A_{\alpha_j})^{i_j})\}$ for $0 \le i_j \le p-1$, where we set

$$A_{\varepsilon_1-\varepsilon_2} = x_{\alpha_1} = x_{\varepsilon_1-\varepsilon_2},$$
$$A_{-(\varepsilon_1-\varepsilon_2)} = \{c_{\varepsilon_2-\varepsilon_1} + (h_{\varepsilon_1-\varepsilon_2}+1)^2 + 4x_{-(\varepsilon_1-\varepsilon_2)} \cdot x_{\varepsilon_1-\varepsilon_2}\}$$
$$\text{(with } c_{\varepsilon_2-\varepsilon_1} \text{ a constant)},$$
$$A_{-(\varepsilon_1-\varepsilon_k)} = x_{\varepsilon_3-\varepsilon_k}(c_{\varepsilon_k-\varepsilon_1} + x_{\varepsilon_1-\varepsilon_k} \cdot x_{-(\varepsilon_1-\varepsilon_k)} \pm x_{-(\varepsilon_k-\varepsilon_2)} \cdot x_{\varepsilon_k-\varepsilon_2})(k \neq 1,2,3),$$
$$A_{-(\varepsilon_1-\varepsilon_3)} = x_{-\varepsilon_2+\varepsilon_3}(c_{\varepsilon_3-\varepsilon_1} + x_{\varepsilon_1-\varepsilon_3} \cdot x_{-(\varepsilon_1-\varepsilon_3)} \pm x_{-(\varepsilon_3-\varepsilon_2)} \cdot x_{\varepsilon_3-\varepsilon_2}),$$
$$A_{\varepsilon_2-\varepsilon_k} = x_{-\varepsilon_3+\varepsilon_k}(c_{\varepsilon_2-\varepsilon_k} + x_{\varepsilon_k-\varepsilon_2} \cdot x_{-(\varepsilon_k-\varepsilon_2)} \pm x_{-(\varepsilon_1-\varepsilon_k)} \cdot x_{\varepsilon_1-\varepsilon_k})(k \neq 1,2,3),$$
$$A_{\varepsilon_2-\varepsilon_3} = x^2_{-\varepsilon_2+\varepsilon_3}(c_{\varepsilon_2-\varepsilon_3} + x_{\varepsilon_3-\varepsilon_2} \cdot x_{\varepsilon_2-\varepsilon_3} \pm x_{-(\varepsilon_1-\varepsilon_3)} \cdot x_{\varepsilon_1-\varepsilon_3}),$$
$$A_{\frac{1}{2}(\sum_{i=1}^6(-1)^{k(i)}\varepsilon_i \pm \sqrt{2}\ \varepsilon_7)} = x_{\frac{1}{2}(\sum_{i=1}^6(-1)^{k(i)}\varepsilon_i \pm \sqrt{2}\ \varepsilon_7)}$$

except that, for $\beta_{k(i)} = -\frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2 + \frac{1}{2}\sum_{i=3}^6(-1)^{k(i)}\varepsilon_i \pm \frac{\sqrt{2}}{2}\varepsilon_7$, we put $A_{\beta_{k(i)}} = x_{\frac{1}{2}\varepsilon_1 - \frac{1}{2}\varepsilon_2 - \frac{1}{2}\sum_{i=3}^6(-1)^{k(i)}\varepsilon_i \pm \frac{\sqrt{2}}{2}\varepsilon_7}(c_{\beta_{k(i)}} + x_{\beta_{k(i)}} \cdot x_{-\beta_{k(i)}} \pm x_{\gamma_{k(i)}} \cdot x_{-\gamma_{k(i)}})$ with $\gamma_{k(i)} = \frac{1}{2}\varepsilon_1 - \frac{1}{2}\varepsilon_2 - \frac{1}{2}\sum_{i=3}^6(-1)^{k(i)}\varepsilon_i \pm \frac{\sqrt{2}}{2}\varepsilon_7$, and for other roots $\alpha \in \Phi$ we set $A_\alpha = x_\alpha^2$ or x_α^3 accordingly as in proposition 5.2.1 in [4] so that each factor in the basis element of $\mathcal{B}$ may have a different weight from others with respect to H. Here $c'_\alpha s$ are chosen so that parentheses containing them are invertible and the signs $\pm$ are chosen so that they commute with x_{α_1}(and so with $x_{-\alpha_1}$). $\mathcal{B}$ will be shown to be a basis in $\mathcal{U}(L)/\mathfrak{M}_\chi$. Due to P-B-W theorem, $\mathcal{B}$ is a linearly independent set over F in $\mathcal{U}(L)$. Furthermore $\forall\alpha \in \Phi, A_\alpha \notin \mathfrak{M}_\chi$(see below for detail).

Suppose next that there is a dependence equation which is of least degree with respect to $h_{\alpha_j} \in H$ and the number of whose highest degree terms is also least. If it has an h-component, then conjugation of it by x_{α_1} gives a nontrivial dependence equation of lower degree than the given one, a contradiction. If it has not, it reduces to the following forms:

(i) $x_{\varepsilon_j-\varepsilon_2} \cdot C + C' \in \mathfrak{M}_\chi$,

(ii) $x_{\varepsilon_1-\varepsilon_j} \cdot C + C' \in \mathfrak{M}_\chi$,

(iii) $x_{\varepsilon_\ell-\varepsilon_k} \cdot C + C' \in \mathfrak{M}_\chi$; if $\ell, k \neq 1,2$,

(iv) $x_{\frac{1}{2}\sum_{i=1}^6(-1)^{k(i)}\varepsilon_i \pm \sqrt{2}\ \varepsilon_7} \cdot C + C' \in \mathfrak{M}_\chi$, where C and C' commute with x_{α_1}.

For the case (i), consider $x_{\varepsilon_2-\varepsilon_j} \cdot x_{\varepsilon_j-\varepsilon_2} \cdot C + x_{\varepsilon_2-\varepsilon_j} \cdot C' \in \mathfrak{M}_\chi$, from which we get $(h_{\varepsilon_2-\varepsilon_j} + x_{\varepsilon_j-\varepsilon_2} \cdot x_{\varepsilon_2-\varepsilon_j}) \cdot C + x_{\varepsilon_2-\varepsilon_j} \cdot C' \in \mathfrak{M}_\chi$. Applying $adx_{\varepsilon_1-\varepsilon_2}$, we get $x_{\varepsilon_1-\varepsilon_2} \cdot C \in \mathfrak{M}_\chi$ implying $C \in \mathfrak{M}_\chi$. So the first assumed dependence equation reduces to that of lower degree.

For the case (ii), considering $x_{\varepsilon_j-\varepsilon_1}(x_{\varepsilon_1-\varepsilon_j} \cdot C + C') \in \mathfrak{M}_\chi$ gives $C \in \mathfrak{M}_\chi$ similarly to (i). For (iii), we consider $x_{\varepsilon_2-\varepsilon_\ell}(x_{\varepsilon_\ell-\varepsilon_k} \cdot C + C') \in \mathfrak{M}_\chi$; so we get $(x_{\varepsilon_2-\varepsilon_k} + x_{\varepsilon_\ell-\varepsilon_k} \cdot x_{\varepsilon_2-\varepsilon_\ell}) \cdot C + x_{\varepsilon_2-\varepsilon_\ell} \cdot C' \in \mathfrak{M}_\chi$. Next applying $adx_{\varepsilon_1-\varepsilon_2}$ to the last relation yields $(x_{\varepsilon_1-\varepsilon_k} + x_{\varepsilon_\ell-\varepsilon_k} \cdot x_{\varepsilon_1-\varepsilon_\ell}) \cdot C + x_{\varepsilon_1-\varepsilon_\ell} \cdot C' \in \mathfrak{M}_\chi$, which again yields $x_{\varepsilon_1-\varepsilon_k} \cdot C \in \mathfrak{M}_\chi$. This gives $C \in \mathfrak{M}_\chi$ by the argument as above. Similarly we may show that (iv) gives $C \in \mathfrak{M}_\chi$. Repeating such processes gives rise to a contradiction $x_\alpha \in \mathfrak{M}_\chi$ for some $\alpha \in \Phi$.

(II) Suppose next that $\xi_1 = 0$ but $\xi_2 \neq 0$; we then see without difficulty that we have $dim_F(\mathcal{U}(L)/\mathfrak{M}_\chi) = p^{2m} = p^{72}$ in view of a Lie algebra automorphism sending $x_{-\alpha_1}$ to x_{α_1}.

(III) Suppose that $\xi_1 = \xi_2 = 0$ but $\xi_3 \neq 0$. In this case we set $\mathcal{B} := \{(B_1 + A_{\varepsilon_1-\varepsilon_2})^{i_1} \otimes (B_2 + A_{-(\varepsilon_1-\varepsilon_2)})^{i_2} \otimes \cdots \otimes (B_{12} + A_{-\frac{1}{2}(-\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\sqrt{2}\ \varepsilon_7)})^{i_{12}} \otimes (\otimes_{j=13}^{72}(B_j + A_{\alpha_j})^{i_j})\}$ for $0 \leq i_j \leq p-1$, where we designate

$$A_{\varepsilon_1-\varepsilon_2} = x^{p-1}_{\varepsilon_1-\varepsilon_2} - x_{-(\varepsilon_1-\varepsilon_2)} =: g_{\varepsilon_1-\varepsilon_2},$$

$$A_{-(\varepsilon_1-\varepsilon_2)} = \{c_{\varepsilon_2-\varepsilon_1} + (h_{\varepsilon_1-\varepsilon_2} + 1)^2 + 4x_{-(\varepsilon_1-\varepsilon_2)} \cdot x_{\varepsilon_1-\varepsilon_2}\}$$

$$\text{(with } c_{\varepsilon_2-\varepsilon_1} \text{ a constant)},$$

$$A_{-(\varepsilon_1-\varepsilon_k)} = x_{\varepsilon_3-\varepsilon_k}(c_{\varepsilon_k-\varepsilon_1} + x_{\varepsilon_1-\varepsilon_k} \cdot x_{-(\varepsilon_1-\varepsilon_k)} \pm x_{-(\varepsilon_k-\varepsilon_2)} \cdot x_{\varepsilon_k-\varepsilon_2})(k \neq 1,2,3),$$

$$A_{\varepsilon_1-\varepsilon_k} = x_{\varepsilon_k-\varepsilon_3}(c_{\varepsilon_1-\varepsilon_k} + x_{-\varepsilon_1+\varepsilon_k} \cdot x_{\varepsilon_1-\varepsilon_k} \pm x_{\varepsilon_k-\varepsilon_2} \cdot x_{\varepsilon_2-\varepsilon_k})(k \neq 1,2,3),$$

$$A_{-(\varepsilon_1-\varepsilon_3)} = x^2_{\varepsilon_4-\varepsilon_3}(c_{-(\varepsilon_1-\varepsilon_3)} + x_{\varepsilon_1-\varepsilon_3} \cdot x_{-(\varepsilon_1-\varepsilon_3)} \pm x_{-(\varepsilon_3-\varepsilon_2)} \cdot x_{-\varepsilon_2+\varepsilon_3}),$$

$$A_{\varepsilon_1-\varepsilon_3} = x^2_{\varepsilon_3-\varepsilon_4}(c_{\varepsilon_1-\varepsilon_3} + x_{\varepsilon_3-\varepsilon_1} \cdot x_{\varepsilon_1-\varepsilon_3} \pm x_{\varepsilon_3-\varepsilon_2} \cdot x_{\varepsilon_2-\varepsilon_3}),$$

$$A_{\varepsilon_2-\varepsilon_k} = x_{-\varepsilon_4+\varepsilon_k}(c_{\varepsilon_2-\varepsilon_k} + x_{-(\varepsilon_2-\varepsilon_k)} \cdot x_{\varepsilon_2-\varepsilon_k} \pm x_{-(\varepsilon_1-\varepsilon_k)} \cdot x_{\varepsilon_1-\varepsilon_k})(k \neq 1,2,3),$$

$$A_{\varepsilon_k-\varepsilon_2} = x_{\varepsilon_4-\varepsilon_k}(c_{\varepsilon_k-\varepsilon_2} + x_{\varepsilon_2-\varepsilon_k} \cdot x_{\varepsilon_k-\varepsilon_2} \pm x_{\varepsilon_1-\varepsilon_k} \cdot x_{\varepsilon_k-\varepsilon_1})(k \neq 1,2,3),$$

$$A_{\varepsilon_2-\varepsilon_3} = x^2_{\varepsilon_5-\varepsilon_3}(c_{\varepsilon_2-\varepsilon_3} + x_{-(\varepsilon_2-\varepsilon_3)} \cdot x_{\varepsilon_2-\varepsilon_3} \pm x_{-(\varepsilon_1-\varepsilon_3)} \cdot x_{\varepsilon_1-\varepsilon_3}),$$

$$A_{\varepsilon_3-\varepsilon_2} = x^2_{\varepsilon_3-\varepsilon_5}(c_{\varepsilon_3-\varepsilon_2} + x_{\varepsilon_2-\varepsilon_3} \cdot x_{\varepsilon_3-\varepsilon_2} \pm x_{\varepsilon_1-\varepsilon_3} \cdot x_{\varepsilon_3-\varepsilon_1}),$$

$$A_{\frac{1}{2}(\sum_{i=1}^{6}(-1)^{k(i)}\varepsilon_i \pm \sqrt{2}\ \varepsilon_7)} = x_{\frac{1}{2}(\sum_{i=1}^{6}(-1)^{k(i)}\varepsilon_i \pm \sqrt{2}\ \varepsilon_7)}$$

except that for $\beta_{k(i)} = -\frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2 + \frac{1}{2}\sum_{i=3}^{6}(-1)^{k(i)}\varepsilon_i \pm \frac{\sqrt{2}}{2}\varepsilon_7$ and $\beta'_{k(i)} = \frac{1}{2}\varepsilon_1 - \frac{1}{2}\varepsilon_2 + \frac{1}{2}\sum_{i=3}^{6}(-1)^{k(i)}\varepsilon_i \pm \frac{\sqrt{2}}{2}\varepsilon_7$, we put

$$A_{\beta_{k(i)}} = x^{1 or 2}_{\frac{1}{2}\varepsilon_1+\frac{1}{2}\varepsilon_2+\frac{1}{2}\sum_{i=3}^{6}(-1)^{k'(i)}\varepsilon_i\pm\frac{\sqrt{2}}{2}\varepsilon_7}(c_{\beta_{k(i)}} + x_{\beta_{k(i)}} \cdot x_{-\beta_{k(i)}} \pm x_{\gamma_{k(i)}} \cdot x_{-\gamma_{k(i)}})$$

and

$$A_{\beta'_{k(i)}} = x^{1 or 2}_{-\frac{1}{2}\varepsilon_1-\frac{1}{2}\varepsilon_2+\frac{1}{2}\sum_{i=3}^{6}(-1)^{k''(i)}\varepsilon_i\pm\frac{\sqrt{2}}{2}\varepsilon_7}(c_{\beta'_{k(i)}} + x_{\beta'_{k(i)}} \cdot x_{-\beta'_{k(i)}} \pm x_{\gamma'_{k(i)}} \cdot x_{-\gamma'_{k(i)}})$$

respectively appropriately as in proposition 2.1, and for other roots $\alpha \in \Phi$ we set $A_\alpha = x_\alpha^2$ or x_α^3 suitably as in (I). We may verify without difficulty that $\mathcal{U}(L)/\mathfrak{M}_\chi$ has $\mathcal{B}$ as its F-basis. Its proof almost goes in parallel with that of (I).

(IV) Since we have no short root and since any root is sent to any other root by the Weyl group, the remaining cases are also verified automatically.

(V) Since $dim_F\mathcal{U}(L)/\mathfrak{M}_\chi) = p^{72}$, we get our conclusion by virtue of Jacobson's density theorem. □

§3. Kac-Weisfeiler conjecture

The well-known Kac-Weisfeiler conjecture was proved by A.Premet around 1995, but the dimensions of the coadjoint orbits are still unknown. In the present situation we can get the following fact.

PROPOSITION(3.1). *For $p \geq 7$, E_6 has no subregular point. In other words, we have $S(L, p, \chi) = \emptyset$ for $p \geq 7$, $\chi \neq 0$, and $L = E_6$-type.*

Proof. Proposition 2.1 ensures our assertion in view of §3-3 in [4]. □

LEMMA(3.2). *We can obtain Irr$(s, \mathcal{O}(L))$ by expanding out both sides of the relation:*

$$N_{Q(3)}^{Q(3)(h_{\alpha_1},\cdots,h_{\alpha_6})}\Big(\sum_{\alpha\in\Phi^+} a_\alpha x_\alpha x_{-\alpha}\Big)$$
$$= N_{Q(3)}^{Q(3)(h_{\alpha_1},\cdots,h_{\alpha_6})}\{s - (b_1 h_{\alpha_1} + \cdots + b_6 h_{\alpha_6} + \sum_{i,j} a_{ij} h_{\alpha_i} h_{\alpha_j})\},$$

where $\sum_{i,j} a_{ij} h_{\alpha_i} h_{\alpha_j}$ indicates the part of other terms of degree 2 of s in $\mathcal{U}(H)$.

Proof. See proposition 3.2.2 in [4]. □

COROLLARY(3.3). *For any $\chi \in L^*$, where have at most p^6-nonisomorphic irreducible L-modules.*

Proof. By virtue of [1], we have exactly p^6-nonisomorphic irreducible restricted L-modules. Since $[Q(\mathcal{U}(L)) : Q(\mathfrak{Z})] = p^{72}$, $\mathcal{U}(L)/\mathfrak{M}_\chi$ for $\chi \in L^*\backslash\{0\}$ can be expressed as $\sum_{i=1}^{72} \mathfrak{Z}\cdot(x_i + \mathfrak{M}_\chi)/\mathfrak{M}_\chi$ for basis elements $x_i \in \mathcal{B}$. So theorem 3.1 gives a unique maximal ideal $\sum \mathcal{U}(L)(x_{\alpha_i}^p - x_{\alpha_i}^{[p]} - \xi_i) + \mathcal{U}(L)(s - (s_\chi)_j)$ for each root $(s_\chi)_j$ of $\mathrm{Irr}(s, \mathcal{O}(L))$ associated with χ. Hence we obtain the conclusion by virtue of the above lemma 3.2. □

REMARK(3.4). Using various facts up to now, we can conclude that the dimension of the coadjoint orbit of any nonzero $\chi \in L^*\backslash\{0\}$ becomes 2×36 if p is good. V.G.Kac confirms that we have an irreducible representation ρ_χ of dimension $p^{2m'}$ if the dimension of the coadjoint orbit of χ is $2m' \leq 2m = 72$. So we can assert that the dimension of the coadjoint orbits is $2m = 72$ in view of Premet's proof of Kac-Weisfeiler conjecture.

§4. Errata to a previous paper

In [3], we presented the root system Φ of E_7 incorrectly, thereby gave an incorrect proof of its proposition 2.1. So we correct some part of [3] and give the result in this section.

PROPOSITION(4.1). *Let Φ be the root system of E_7-type. Then Φ has a base $\Delta = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \cdots, \varepsilon_7 - \varepsilon_8, \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8)\}$ and $\Phi = \{\varepsilon_i - \varepsilon_j, 1 \leq i \neq j \leq 8\ ;\ \frac{1}{2}(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_8)(\text{four pluses})\}$ and any nonrestricted simple module for E_7 has its dimension p^{63} for $p \geq 7$.*

Proof. For Φ and Δ, refer to §6.7 in [2]. Since E_7 has a sub-root system of $E_8's$, we have only to imitate the proof for E_8 in chapter 7 in [4]. But we may also imitate proposition (2.1)'s proof because of similar reason for E_6. Several steps are in order.

(I) If $\alpha_1 = \varepsilon_2 - \varepsilon_3$ and let $\xi_1 \neq 0$, then we set $B_i := b_{i1}(c_{i1} + h_{\varepsilon_2 - \varepsilon_3}) + b_{i2}(c_{i2} + h_{\varepsilon_3 - \varepsilon_4}) + \cdots + b_{i6}(c_{i6} + h_{\varepsilon_7 - \varepsilon_8}) + b_{i7}(c_{i7} + h_{\frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8)})$, where we have chosen $c'_{ij}s \in F$ so that parentheses are invertible $\bmod \mathfrak{M}_\chi$, and we have chosen $(b_{i1}, \cdots, b_{i7}) \in F^7$ so that any 8 $B'_i s$ are linearly independent in $\mathbb{P}^7(F)$, the $B_i \cdot x_{\varepsilon_2-\varepsilon_3} \not\equiv x_{\varepsilon_2-\varepsilon_3} \cdot B_i \pmod{\mathfrak{M}_\chi}$, and the $\mathcal{B}$ below becomes an F-linearly independent set in $\mathcal{U}(L)$ if necessary. Here we pick out a basis candidate like $\mathcal{B} := \{(B_1 + A_{\varepsilon_2-\varepsilon_3})^{i_1} \otimes (B_2 + A_{-(\varepsilon_2-\varepsilon_3)})^{i_2} \otimes \cdots \otimes (B_{14} + A_{-\frac{1}{2}(-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7+\varepsilon_8)}) \otimes (\otimes_{j=15}^{126}(B_j + A_{\alpha_j})^{i_j})\}$ for $0 \leq i_j \leq p-1$, where we designate

$$
\begin{aligned}
&A_{\varepsilon_2-\varepsilon_3} = x_{\alpha_1} = x_{\varepsilon_2-\varepsilon_3},\\
&A_{-(\varepsilon_2-\varepsilon_3)} = \{c_{\varepsilon_3-\varepsilon_2} + (h_{\varepsilon_2-\varepsilon_3}+1)^2 + 4x_{-(\varepsilon_2-\varepsilon_3)} \cdot x_{\varepsilon_2-\varepsilon_3}\}\\
&\qquad\qquad\text{(with } c_{\varepsilon_3-\varepsilon_2} \text{ a constant)},\\
&A_{-(\varepsilon_2-\varepsilon_k)} = x_{\varepsilon_1-\varepsilon_k}(c_{\varepsilon_k-\varepsilon_2} + x_{\varepsilon_2-\varepsilon_k} \cdot x_{-(\varepsilon_2-\varepsilon_k)} \pm x_{-(\varepsilon_k-\varepsilon_3)} \cdot x_{\varepsilon_k-\varepsilon_3})(k \neq 1,2,3),\\
&A_{-(\varepsilon_2-\varepsilon_1)} = x_{-\varepsilon_3+\varepsilon_1}(c_{\varepsilon_1-\varepsilon_2} + x_{\varepsilon_2-\varepsilon_1} \cdot x_{-(\varepsilon_2-\varepsilon_1)} \pm x_{-(\varepsilon_1-\varepsilon_3)} \cdot x_{\varepsilon_1-\varepsilon_3}),\\
&A_{\varepsilon_3-\varepsilon_1} = x^2_{-\varepsilon_3+\varepsilon_1}(c_{\varepsilon_3-\varepsilon_1} + x_{\varepsilon_1-\varepsilon_3} \cdot x_{\varepsilon_3-\varepsilon_1} \pm x_{-(\varepsilon_2-\varepsilon_1)} \cdot x_{\varepsilon_2-\varepsilon_1}),\\
&A_{\varepsilon_3-\varepsilon_k} = x_{-\varepsilon_1+\varepsilon_k}(c_{\varepsilon_3-\varepsilon_k} + x_{\varepsilon_k-\varepsilon_3} \cdot x_{\varepsilon_3-\varepsilon_k} \pm x_{-(\varepsilon_2-\varepsilon_k)} \cdot x_{\varepsilon_2-\varepsilon_k})(k \neq 1,2,3),\\
&A_{\beta_{k(i)}} = x_{\frac{1}{2}\varepsilon_2-\frac{1}{2}\varepsilon_3+\frac{1}{2}\sum_{i=4}^{8}(-1)^{k(i)}\varepsilon_i \pm \frac{1}{2}\varepsilon_1}(c_{\beta_{k(i)}} + x_{\beta_{k(i)}} \cdot x_{-\beta_{k(i)}} \pm x_{\gamma_{k(i)}} \cdot x_{-\gamma_{k(i)}})
\end{aligned}
$$

for $\beta_{k(i)} = -\frac{1}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3 + \frac{1}{2}\sum_{i=4}^{8}(-1)^{k(i)}\varepsilon_i \pm \frac{1}{2}\varepsilon_1$ and $\gamma_{k(i)} = \frac{1}{2}\varepsilon_2 - \frac{1}{2}\varepsilon_3 + \frac{1}{2}\sum_{i=4}^{8}(-1)^{k(i)}\varepsilon_i \pm \frac{1}{2}\varepsilon_1$, and for other roots $\alpha \in \Phi$ we designate $A_\alpha = x_\alpha^2$ or x_α^3 or even x_α accordingly as in proposition 2.1. Similarly we may show $\mathcal{B}$ is an F-basis of $\mathcal{U}(L)/\mathfrak{M}_\chi$ as before.

(II) Next if $\xi_1 = 0$ but $\xi_2 \neq 0$, then we get $dim_F(\mathcal{U}(L)/\mathfrak{M}_\chi) = p^{2m} = p^{126}$ by a Lie algebra automorphism sending $x_{-\alpha_1}$ to x_{α_1}.

(III) Suppose that $\xi_1 = \xi_2 = 0$ but $\xi_3 \neq 0$. In this case we put $\mathcal{B} := \{(B_1 + A_{\varepsilon_2-\varepsilon_3})^{i_1} \otimes (B_2 + A_{-(\varepsilon_2-\varepsilon_3)})^{i_2} \otimes \cdots \otimes (B_{14} + A_{-\frac{1}{2}(-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7+\varepsilon_8)}) \otimes$

$(\otimes_{j=15}^{126}(B_j + A_{\alpha_j})^{i_j})\}$ for $0 \le i_j \le p-1$, where we designate

$$A_{\varepsilon_2-\varepsilon_3} = x_{\varepsilon_2-\varepsilon_3}^{p-1} - x_{-(\varepsilon_2-\varepsilon_3)} =: g_{\alpha_1} = g_{\varepsilon_2-\varepsilon_3},$$

$$A_{-(\varepsilon_2-\varepsilon_3)} = \{c_{\varepsilon_3-\varepsilon_2} + (h_{\varepsilon_2-\varepsilon_3}+1)^2 + 4x_{-(\varepsilon_2-\varepsilon_3)} \cdot x_{\varepsilon_2-\varepsilon_3}\}$$

$$\text{(with } c_{\varepsilon_3-\varepsilon_2} \text{ a constant)},$$

$$A_{-(\varepsilon_2-\varepsilon_k)} = x_{\varepsilon_1-\varepsilon_k}(c_{\varepsilon_k-\varepsilon_2} + x_{\varepsilon_2-\varepsilon_k} \cdot x_{-(\varepsilon_2-\varepsilon_k)} \pm x_{-(\varepsilon_k-\varepsilon_3)} \cdot x_{\varepsilon_k-\varepsilon_3})(k \ne 1,2,3),$$

$$A_{\varepsilon_2-\varepsilon_k} = x_{\varepsilon_k-\varepsilon_1}(c_{\varepsilon_2-\varepsilon_k} + x_{-(\varepsilon_2-\varepsilon_k)} \cdot x_{\varepsilon_2-\varepsilon_k} \pm x_{\varepsilon_k-\varepsilon_3} \cdot x_{\varepsilon_3-\varepsilon_k})(k \ne 1,2,3),$$

$$A_{\varepsilon_1-\varepsilon_2} = x_{\varepsilon_1-\varepsilon_4}(c_{\varepsilon_1-\varepsilon_2} + x_{\varepsilon_1-\varepsilon_2} \cdot x_{\varepsilon_2-\varepsilon_1} \pm x_{\varepsilon_1-\varepsilon_3} \cdot x_{\varepsilon_3-\varepsilon_1}),$$

$$A_{\varepsilon_2-\varepsilon_1} = x_{\varepsilon_4-\varepsilon_1}(c_{\varepsilon_2-\varepsilon_1} + x_{\varepsilon_2-\varepsilon_1} \cdot x_{\varepsilon_1-\varepsilon_2} \pm x_{\varepsilon_3-\varepsilon_1} \cdot x_{\varepsilon_1-\varepsilon_3}),$$

$$A_{\varepsilon_3-\varepsilon_k} = x_{\varepsilon_5-\varepsilon_k}(c_{\varepsilon_3-\varepsilon_k} + x_{\varepsilon_3-\varepsilon_k} \cdot x_{\varepsilon_k-\varepsilon_3} \pm x_{\varepsilon_2-\varepsilon_k} \cdot x_{\varepsilon_k-\varepsilon_2})(k \ne 2,3,5),$$

$$A_{\varepsilon_k-\varepsilon_3} = x_{\varepsilon_k-\varepsilon_5}(c_{\varepsilon_k-\varepsilon_3} + x_{\varepsilon_k-\varepsilon_3} \cdot x_{\varepsilon_3-\varepsilon_k} \pm x_{\varepsilon_k-\varepsilon_2} \cdot x_{\varepsilon_2-\varepsilon_k})(k \ne 2,3,5),$$

$$A_{\varepsilon_5-\varepsilon_3} = x_{\varepsilon_5-\varepsilon_1}^2(c_{\varepsilon_5-\varepsilon_3} + x_{\varepsilon_5-\varepsilon_3} \cdot x_{\varepsilon_3-\varepsilon_5} \pm x_{\varepsilon_5-\varepsilon_2} \cdot x_{\varepsilon_2-\varepsilon_5}),$$

$$A_{\varepsilon_3-\varepsilon_5} = x_{\varepsilon_1-\varepsilon_5}^2(c_{\varepsilon_3-\varepsilon_5} + x_{\varepsilon_3-\varepsilon_5} \cdot x_{\varepsilon_5-\varepsilon_3} \pm x_{\varepsilon_2-\varepsilon_5} \cdot x_{\varepsilon_5-\varepsilon_2}),$$

and for other roots α we put A_α similarly as in (III) of the proof in proposition 2.1. Similarly we can show $\mathcal{B}$ is an F-basis of $\mathcal{U}(L)/\mathfrak{M}_\chi$ as before.

(IV) Since there is only one root length, the remaining cases are easily verified automatically.

(V) Since $dim_F(\mathcal{U}(L)/\mathfrak{M}_\chi) = p^{126}$, we have our conclusion by dint of Jacobson's density theorem. □

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