

## INTERVAL-VALUED FUZZY SEMI-PREOPEN SETS AND INTERVAL-VALUED FUZZY SEMI-PRECONTINUOUS MAPPINGS

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**Abstract.** We introduce the notions of interval-valued fuzzy semi-preopen sets (mappings), interval-valued fuzzy semi-pre interior and interval-valued fuzzy semi-pre-continuous mappings by using the notion of interval-valued fuzzy sets. We also investigate related properties and characterize interval-valued fuzzy semi-preopen sets (mappings) and interval-valued fuzzy semi-precontinuous mappings.

### 1. Introduction

Since Zadeh introduced the concept of fuzzy sets [19], several researchers have been concerned about the generalization of the notion of fuzzy sets such as fuzzy set of type  $n$  [20], intuitionistic fuzzy sets [1] and interval-valued fuzzy sets [6]. The concept of interval-valued fuzzy sets was introduced by Gorzalczany [6], and recently there has been progress in the study of such sets by several researchers (see [4], [11], [12], [13], [14], [15], [18]). Azad [2] introduced fuzzy semiopen (semiclosed) sets and fuzzy regular open (closed) sets, and then considered generalizations of semicontinuous mapping, semiopen mapping, semiclosed mapping, almost continuous mapping and weakly continuous mapping in fuzzy settings. In [12], the topology of interval-valued fuzzy sets is defined, and some of its properties were discussed by Mondal and Samanta, and then

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Mondal et al. [11] studied the connectedness in the topology of interval-valued fuzzy sets. Using the concept of interval-valued fuzzy (IVF) sets, Jun et al. [9] introduced the notion of IVF semiopen (semiclosed) sets, IVF preopen (preclosed) sets and IVF  $\alpha$ -open ( $\alpha$ -closed) sets, and then they investigated relationships between IVF semiopen (semiclosed) sets, IVF preopen (preclosed) sets and IVF  $\alpha$ -open ( $\alpha$ -closed) sets. They also introduced the notion of IVF open mappings, IVF preopen mappings, IVF semiopen mappings and IVF  $\alpha$ -open mappings, and then they provided relationships between IVF open mappings, IVF preopen mappings, IVF semiopen mappings and IVF  $\alpha$ -open mappings. In [7], Jun et al. introduced the notions of IVF strongly semiopen (semiclosed) mappings and IVF strongly semi-continuous mappings, and investigated several properties. They discussed the following items:

- The characterization of an IVF strongly semiopen (semiclosed) mapping.
- The characterization of an IVF strongly semicontinuous mapping.

In this paper, we introduce the notions of IVF semi-preopen sets (mappings), IVF semi-pre interior and IVF semi-precontinuous mappings, and investigate related properties. We characterize IVF semi-preopen sets (mappings) and IVF semi-precontinuous mappings.

## 2. Preliminaries

Let  $D[0, 1]$  be the set of all closed subintervals of the unit interval  $[0, 1]$ . The elements of  $D[0, 1]$  are generally denoted by capital letters  $M, N, \dots$ , and note that  $M = [M^L, M^U]$ , where  $M^L$  and  $M^U$  are the lower and the upper end points respectively. Especially, we denote  $\mathbf{0} = [0, 0]$ ,  $\mathbf{1} = [1, 1]$ , and  $\mathbf{a} = [a, a]$  for every  $a \in (0, 1)$ . We also note that

- (i)  $(\forall M, N \in D[0, 1]) (M = N \Leftrightarrow M^L = N^L, M^U = N^U)$ .
- (ii)  $(\forall M, N \in D[0, 1]) (M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$ .

For every  $M \in D[0, 1]$ , the *complement* of  $M$ , denoted by  $M^c$ , is defined by  $M^c = 1 - M = [1 - M^U, 1 - M^L]$ .

Let  $X$  be a nonempty set. A function  $\mathcal{A} : X \rightarrow D[0, 1]$  is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in  $X$ . For each  $x \in X$ ,  $\mathcal{A}(x)$  is a closed interval whose lower and upper end points are denoted by  $\mathcal{A}(x)^L$  and  $\mathcal{A}(x)^U$ , respectively. For any  $[a, b] \in D[0, 1]$ , the IVF set whose value is the interval  $[a, b]$  for all  $x \in X$  is denoted by  $\widetilde{[a, b]}$ . In particular, for any  $a \in [0, 1]$ , the IVF set whose value is  $\mathbf{a} = [a, a]$  for all  $x \in X$  is denoted by simply  $\tilde{a}$ . For a point  $p \in X$  and for  $[a, b] \in D[0, 1]$  with  $b > 0$ , the IVF set which takes the value  $[a, b]$  at  $p$  and  $\mathbf{0}$  elsewhere in  $X$  is called an *interval-valued fuzzy point* (briefly, an *IVF point*) and is denoted by  $[a, b]_p$ . In particular, if  $b = a$ , then it is also denoted by  $a_p$ . The set of all IVF sets in  $X$  are denoted by  $IVF(X)$ . An IVF point  $M_x$ , where  $M \in D[0, 1]$ , is said to *belong* to an IVF set  $\mathcal{A}$  in  $X$ , denoted by  $M_x \tilde{\in} \mathcal{A}$ , if  $\mathcal{A}(x)^L \geq M^L$  and  $\mathcal{A}(x)^U \geq M^U$ . It can be easily shown that  $\mathcal{A} = \cup \{M_x \mid M_x \tilde{\in} \mathcal{A}\}$  (see [12]).

Let  $f : X \rightarrow Y$  be a mapping and let  $\mathcal{A}$  be an IVF set in  $X$ . Then the *image* of  $\mathcal{A}$  under  $f$ , denoted by  $f(\mathcal{A})$ , is defined as follows:

$$[f(\mathcal{A})(y)]^L = \begin{cases} \sup_{y=f(x)} [\mathcal{A}(x)]^L, & f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$[f(\mathcal{A})(y)]^U = \begin{cases} \sup_{y=f(x)} [\mathcal{A}(x)]^U, & f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

for all  $y \in Y$ . Let  $\mathcal{B}$  be an IVF set in  $Y$ . Then the *inverse image* of  $\mathcal{B}$  under  $f$ , denoted by  $f^{-1}(\mathcal{B})$ , is defined as follows:

$$(\forall x \in X) ([f^{-1}(\mathcal{B})(x)]^L = [\mathcal{B}(f(x))]^L, [f^{-1}(\mathcal{B})(x)]^U = [\mathcal{B}(f(x))]^U).$$

**Definition 2.1.** [12] A family  $\tau$  of IVF sets in  $X$  is called an *interval-valued fuzzy topology* (briefly, *IVF topology*) for  $X$  if it satisfies:

- (i)  $\tilde{0}, \tilde{1} \in \tau$ ,
- (ii)  $\mathcal{A}, \mathcal{B} \in \tau \Rightarrow \mathcal{A} \cap \mathcal{B} \in \tau$ ,

$$(iii) \mathcal{A}_i \in \tau, i \in \Delta \Rightarrow \bigcup_{i \in \Delta} \mathcal{A}_i \in \tau.$$

Every member of  $\tau$  is called an *IVF open set*. An IVF set  $\mathcal{A}$  in  $X$  is called an *IVF closed set* if the complement of  $\mathcal{A}$  is an IVF open set, that is,  $\mathcal{A}^c \in \tau$ . Moreover,  $(X, \tau)$  is called an *interval-valued fuzzy topological space* (briefly, *IVF topological space*).

**Definition 2.2.** [12] Let  $(X, \tau)$  and  $(Y, \kappa)$  be IVF topological spaces. A function  $f : X \rightarrow Y$  is said to be *continuous* if  $f^{-1}(\mathcal{A}) \in \tau$  for all  $\mathcal{A} \in \kappa$ .

For an IVF set  $\mathcal{A}$  in an IVF topological space  $(X, \tau)$ , the *closure*  $\text{cl}(\mathcal{A})$  and the *interior*  $\text{int}(\mathcal{A})$  of  $\mathcal{A}$  are defined, respectively, as

$$\text{cl}(\mathcal{A}) = \bigcap \{ \mathcal{B} \in \text{IVF}(X) \mid \mathcal{B} \text{ is IVF closed and } \mathcal{A} \subseteq \mathcal{B} \},$$

$$\text{int}(\mathcal{A}) = \bigcup \{ \mathcal{B} \in \text{IVF}(X) \mid \mathcal{B} \text{ is IVF open and } \mathcal{B} \subseteq \mathcal{A} \}.$$

Note that  $\text{int}(\mathcal{A})$  is the largest IVF open set which is contained in  $\mathcal{A}$ , and that  $\mathcal{A}$  is IVF open if and only if  $\mathcal{A} = \text{int}(\mathcal{A})$ .

### 3. IVF semi-preopen sets

In the follow, let  $(X, \tau)$  denote an IVF topological space unless otherwise specified.

**Definition 3.1.** [9] An IVF set  $\mathcal{A}$  in  $(X, \tau)$  is called an *IVF semiopen set* of  $(X, \tau)$  if it satisfies:

$$(\exists \mathcal{B} \in \tau) (\mathcal{B} \subseteq \mathcal{A} \subseteq \text{cl}(\mathcal{B}));$$

and an *IVF semiclosed set* of  $(X, \tau)$  if it satisfies:

$$(\exists \mathcal{B}^c \in \tau) (\text{int}(\mathcal{B}) \subseteq \mathcal{A} \subseteq \mathcal{B}).$$

Denote by  $\text{IVFSO}(X)$  (resp.  $\text{IVFSC}(X)$ ) the set of all IVF semiopen (resp. semiclosed) sets of  $(X, \tau)$ .

**Definition 3.2.** [9] An IVF set  $\mathcal{A}$  in  $(X, \tau)$  is called an *IVF preopen set* of  $(X, \tau)$  if  $\mathcal{A} \subseteq \text{int}(\text{cl}(\mathcal{A}))$ ; and is called an *IVF preclosed set* of  $(X, \tau)$  if  $\text{cl}(\text{int}(\mathcal{A})) \subseteq \mathcal{A}$ .

Denote by  $IVFPO(X)$  (resp.  $IVFPC(X)$ ) the set of all IVF preopen (resp. preclosed) sets of  $(X, \tau)$ .

**Remark 3.3.** [9] It is obvious that every IVF open (closed) set is an IVF preopen (preclosed) set, but the converse may not be true, and that an IVF preopen set need not be an IVF semiopen set, and vice versa.

**Definition 3.4.** An IVF set  $\mathcal{A}$  in  $(X, \tau)$  is said to be

- *IVF semi-preopen* if there exists an IVF preopen set  $\mathcal{B}$  such that  $\mathcal{B} \subseteq \mathcal{A} \subseteq \text{cl}(\mathcal{B})$ ,
- *IVF semi-preclosed* if there exists an IVF preclosed set  $\mathcal{C}$  such that  $\text{int}(\mathcal{C}) \subseteq \mathcal{A} \subseteq \mathcal{C}$ .

Denote by  $IVFSPO(X)$  (resp.  $IVFSPC(X)$ ) the set of all IVF semi-preopen (resp. IVF semi-preclosed) sets in  $(X, \tau)$ .

**Example 3.5.** Let  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  be IVF sets in  $X = \{a, b, c\}$  defined by

$$\mathcal{A}(a) = [0.8, 0.9], \quad \mathcal{A}(b) = [0.7, 0.8], \quad \mathcal{A}(c) = [0.6, 0.7],$$

$$\mathcal{B}(a) = [0.7, 0.8], \quad \mathcal{B}(b) = [0.6, 0.7], \quad \mathcal{B}(c) = [0.5, 0.6],$$

$$\mathcal{C}(a) = [0.6, 0.7], \quad \mathcal{C}(b) = [0.5, 0.6], \quad \mathcal{C}(c) = [0.5, 0.6].$$

Then  $\tau = \{\tilde{0}, \mathcal{A}, \mathcal{B}, \tilde{1}\}$  is an IVF topology on  $X$  and  $\mathcal{C}$  is an IVF preopen set in  $(X, \tau)$ . Let  $\mathcal{D}$  be an IVF set in  $X$  given by

$$\mathcal{D}(a) = [0.7, 0.8], \quad \mathcal{D}(b) = [0.5, 0.6], \quad \mathcal{D}(c) = [0.5, 0.6].$$

Then  $\mathcal{C} \subseteq \mathcal{D} \subseteq \text{cl}(\mathcal{C})$ , and so  $\mathcal{D}$  is an IVF semi-preopen set in  $(X, \tau)$ .

**Theorem 3.6.** Let  $\mathcal{A}$  be an IVF set in  $(X, \tau)$ . Then  $\mathcal{A}$  is IVF semi-preopen if and only if  $\mathcal{A}^c$  is IVF semi-preclosed.

*Proof.* Straightforward. □

**Theorem 3.7.** (i) Every IVF semiopen (resp. IVF semiclosed) set is IVF semi-preopen (resp. IVF semi-preclosed).

(ii) Every IVF preopen (resp. IVF preclosed) set is IVF semi-preopen (resp. IVF semi-preclosed).

*Proof.* Straightforward. □

The converse of Theorem 3.7 need not be true as seen in the following example.

**Example 3.8.** (1) Let  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  be IVF sets in  $X = \{a, b\}$  defined by

$$\mathcal{A}(a) = [0.3, 0.35], \quad \mathcal{A}(b) = [0.4, 0.45],$$

$$\mathcal{B}(a) = [0.7, 0.75], \quad \mathcal{B}(b) = [0.8, 0.85].$$

Consider an IVF topology  $\tau = \{\tilde{0}, \mathcal{A}, \tilde{1}\}$  for  $X$ . Then

(i)  $\mathcal{B}$  is IVF semi-preopen which is not IVF semiopen.

(ii)  $\mathcal{B}^c$  is IVF semi-preclosed which is not IVF semiclosed.

(2) Let  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  and  $\mathcal{D}$  be IVF sets in  $X = \{a, b\}$  defined by

$$\mathcal{A}(a) = [0.6, 0.7], \quad \mathcal{A}(b) = [0.6, 0.8],$$

$$\mathcal{B}(a) = [0.1, 0.2], \quad \mathcal{B}(b) = [0.2, 0.3],$$

$$\mathcal{C}(a) = [0.7, 0.8], \quad \mathcal{C}(b) = [0.65, 0.8],$$

$$\mathcal{D}(a) = [0.5, 0.6], \quad \mathcal{D}(b) = [0.5, 0.55].$$

Consider an IVF topology  $\tau = \{\tilde{0}, \mathcal{A}, \mathcal{B}, \tilde{1}\}$  for  $X$ . Then

(i)  $\mathcal{C}$  is IVF semi-preopen which is not IVF preopen.

(ii)  $\mathcal{C}^c$  is IVF semi-preclosed which is not IVF pre-closed.

**Theorem 3.9.** (i) Any union of IVF semi-preopen sets is IVF semi-preopen.

(ii) Any intersection of IVF semi-preclosed sets is IVF semi-preclosed.

*Proof.* Straightforward. □

**Remark 3.10.** The intersection of two IVF semi-preopen sets need not be IVF semi-preopen as seen in the following example.

**Example 3.11.** Let  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{D}$  and  $\mathcal{E}$  be IVF sets in  $I = [0, 1]$  defined by

$$\mathcal{A}(x) = \begin{cases} \mathbf{1}, & 0 \leq x < 1, \\ \mathbf{0.9}, & x = 1, \end{cases}$$

$$\mathcal{B}(x) = \begin{cases} \mathbf{0}, & 0 \leq x \leq \frac{1}{2}, \\ [0.2, 0.3], & \frac{1}{2} < x \leq 1, \end{cases}$$

$$\mathcal{C}(x) = \begin{cases} \mathbf{0}, & 0 \leq x \leq \frac{1}{2}, \\ [0.6, 0.7], & \frac{1}{2} < x \leq 1, \end{cases}$$

$$\mathcal{D}(x) = \begin{cases} [0.5, 0.6], & 0 \leq x \leq \frac{1}{2}, \\ \mathbf{0}, & \frac{1}{2} < x < 1, \\ \mathbf{0.05}, & x = 1, \end{cases}$$

$$\mathcal{E}(x) = \begin{cases} [0.2, 0.3], & 0 \leq x \leq \frac{1}{2}, \\ \mathbf{0}, & \frac{1}{2} < x \leq 1. \end{cases}$$

The collection  $\tau = \{\tilde{0}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \cup \mathcal{E}, \tilde{1}\}$  is an IVF topology for  $I$ . Then  $\mathcal{C}$  and  $\mathcal{D}$  are IVF semi-preopen. But  $\mathcal{C} \cap \mathcal{D}$  is not IVF semi-preopen.

The following example shows that the intersection of an IVF open set and an IVF semi-preopen set may not be an IVF semi-preopen set.

**Example 3.12.** Let  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{D}$  and  $\mathcal{E}$  be IVF sets in  $I = [0, 1]$  described in Example 3.11. Then  $\tau = \{\tilde{0}, \tilde{1}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{E}, \mathcal{B} \cup \mathcal{E}, \mathcal{C} \cup \mathcal{E}\}$  is an IVF topology for  $X$ . We can easily check that  $\mathcal{D}$  is IVF semi-preopen, but the intersection  $\mathcal{C} \cap \mathcal{D}$  is not IVF semi-preopen.

**Theorem 3.13.** Let  $\mathcal{A}$  be an IVF set in  $(X, \tau)$ . Then  $\mathcal{A}$  is IVF semi-preopen if and only if for each IVF point  $[a, b]_p \tilde{\in} \mathcal{A}$ , there exists an IVF semi-preopen set  $\mathcal{B}$  such that  $[a, b]_p \tilde{\in} \mathcal{B} \subseteq \mathcal{A}$ .

*Proof.* Necessity is obvious. Conversely, we have

$$\mathcal{A} = \bigcup_{[a,b]_p \in \tilde{\mathcal{A}}} [a,b]_p \subseteq \bigcup_{[a,b]_p \in \tilde{\mathcal{A}}} \mathcal{B} \subseteq \mathcal{A},$$

and so  $\mathcal{A} = \bigcup_{[a,b]_p \in \tilde{\mathcal{A}}} \mathcal{B}$ , which is IVF semi-preopen.  $\square$

**Theorem 3.14.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be IVF sets in  $(X, \tau)$ .

- (1) If  $\mathcal{A} \subseteq \mathcal{B} \subseteq \text{cl}(\mathcal{A})$  and  $\mathcal{A}$  is IVF semi-preopen, then  $\mathcal{B}$  is IVF semi-preopen.
- (2) If  $\text{int}(\mathcal{A}) \subseteq \mathcal{B} \subseteq \mathcal{A}$  and  $\mathcal{A}$  is IVF semi-preclosed, then  $\mathcal{B}$  is IVF semi-preclosed.

*Proof.* (1) Let  $\mathcal{C}$  be an IVF preopen set in  $X$  such that  $\mathcal{C} \subseteq \mathcal{A} \subseteq \text{cl}(\mathcal{C})$ . It follows from hypothesis that  $\mathcal{C} \subseteq \mathcal{A} \subseteq \mathcal{B} \subseteq \text{cl}(\mathcal{A}) \subseteq \text{cl}(\mathcal{C})$  so that  $\mathcal{B}$  is IVF semi-preopen.

(2) This result is proved by a similar method to (1).  $\square$

Consider an IVF set  $\mathcal{A}$  in  $(X, \tau)$  that satisfies  $\mathcal{A} \subseteq \text{cl}(\text{int}(\text{cl}(\mathcal{A})))$ . The following example shows that if  $\mathcal{A}$  is not an IVF semi-closed, then it is not an IVF semi-preopen in general.

**Example 3.15.** Consider an IVF topology  $\tau = \{\tilde{0}, \mathcal{A}, \mathcal{B}, \tilde{1}\}$  on  $X = \{a, b, c, d\}$ , where  $\mathcal{A}$  and  $\mathcal{B}$  are defined as follows:  $\mathcal{A}(a) = \mathcal{A}(b) = \mathcal{A}(c) = \mathcal{A}(d) = [0.2, 0.3]$  and

$$\mathcal{B}(a) = [0.2, 0.3], \mathcal{B}(b) = [0.4, 0.5], \mathcal{B}(c) = [0.5, 0.6], \mathcal{B}(d) = [0.7, 0.8].$$

Let  $\mathcal{C}$  be an IVF set in  $(X, \tau)$  defined by

$$\mathcal{C}(a) = [0.2, 0.3], \mathcal{C}(b) = [0.4, 0.7], \mathcal{C}(c) = [0.2, 0.3], \mathcal{C}(d) = [0.2, 0.3].$$

Then  $\mathcal{C}$  is not IVF semi-closed, but satisfies  $\mathcal{C} \subseteq \text{cl}(\text{int}(\text{cl}(\mathcal{C})))$ . If  $\mathcal{C}$  is IVF semi-preopen, then there exists IVF pre-open set  $\mathcal{D}$  such that  $\mathcal{D} \subseteq \mathcal{C} \subseteq \text{cl}(\mathcal{D})$ . It follows that  $\text{cl}(\mathcal{D}) \subseteq \text{cl}(\mathcal{C}) = \mathcal{A}^c \subseteq \text{cl}(\mathcal{D})$  so that  $\text{cl}(\mathcal{D}) = \mathcal{A}^c$ . Hence  $\mathcal{D} \subseteq \text{int}(\text{cl}(\mathcal{D})) = \text{int}(\mathcal{A}^c) = \mathcal{B}$ , and so  $\mathcal{A}^c = \text{cl}(\mathcal{D}) \subseteq \text{cl}(\mathcal{B} \cap \mathcal{C}) = \mathcal{B}^c$ . This is a contradiction. Therefore  $\mathcal{C}$  is not IVF semi-preopen.



Now assume that an IVF semi-closed set  $\mathcal{A}$  in  $(X, \tau)$  does not satisfy the inclusion  $\mathcal{A} \subseteq \text{cl}(\text{int}(\text{cl}(\mathcal{A})))$ . Then  $\mathcal{A}$  is not an IVF semi-preopen set as seen in the following example.

**Example 3.16.** Consider an IVF topology  $\tau = \{\tilde{0}, \mathcal{A}, \mathcal{B}, \tilde{1}\}$  on  $X = \{a, b, c\}$ , where  $\mathcal{A}$  and  $\mathcal{B}$  are defined as follows:

$$\mathcal{A}(a) = [0.1, 0.2], \mathcal{A}(b) = [0.2, 0.3], \mathcal{A}(c) = [0.1, 0.3],$$

and

$$\mathcal{B}(a) = [0.2, 0.3], \mathcal{B}(b) = [0.3, 0.4], \mathcal{B}(c) = [0.2, 0.3].$$

Let  $\mathcal{C}$  be an IVF set in  $(X, \tau)$  defined by

$$\mathcal{C}(a) = [0.8, 0.9], \mathcal{C}(b) = [0.65, 0.7], \mathcal{C}(c) = [0.7, 0.85].$$

Then  $\mathcal{C}$  is IVF semi-closed, but does not satisfy the inclusion  $\mathcal{C} \subseteq \text{cl}(\text{int}(\text{cl}(\mathcal{C})))$ . By the similar method to Example 3.15, we know that  $\mathcal{C}$  is an IVF semi-preopen set.

**Lemma 3.17.** *Every IVF semi-preopen set  $\mathcal{A}$  in  $(X, \tau)$  satisfies the inclusion  $\mathcal{A} \subseteq \text{cl}(\text{int}(\text{cl}(\mathcal{A})))$ .*

*Proof.* Straightforward. □

**Lemma 3.18.** [9] *Let  $\mathcal{A}$  be an IVF set in  $(X, \tau)$ . Then*

- (i)  $\mathcal{A} \in \text{IVFSO}(X) \Leftrightarrow \mathcal{A} \subseteq \text{cl}(\text{int}(\mathcal{A}))$ ,
- (ii)  $\mathcal{A} \in \text{IVFSC}(X) \Leftrightarrow \text{int}(\text{cl}(\mathcal{A})) \subseteq \mathcal{A}$ .

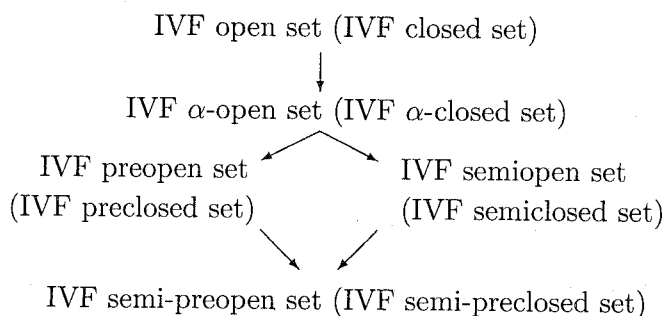
**Theorem 3.19.** *Let  $\mathcal{A}$  be an IVF semi-closed set in  $(X, \tau)$  such that  $\mathcal{A} \subseteq \text{cl}(\text{int}(\text{cl}(\mathcal{A})))$ . Then  $\mathcal{A}$  is an IVF semi-preopen set in  $(X, \tau)$ .*

*Proof.* If we take  $\mathcal{B} = \text{int}(\text{cl}(\mathcal{A}))$ , then  $\mathcal{B}$  is IVF open and hence IVF pre-open. Since  $\mathcal{A}$  is IVF semi-closed,

$$\mathcal{B} = \text{int}(\text{cl}(\mathcal{A})) \subseteq \mathcal{A} \subseteq \text{cl}(\text{int}(\text{cl}(\mathcal{A}))) = \text{cl}(\mathcal{B})$$

by Lemma 3.18 and hypothesis. Hence  $\mathcal{A}$  is an IVF semi-preopen set in  $(X, \tau)$ . □

We have the following diagram described relations between an IVF open set (an IVF closed set), an IVF  $\alpha$ -open set (an IVF  $\alpha$ -closed set), an IVF semiopen set (an IVF semiclosed set), an IVF preopen set (an IVF preclosed set) and an IVF semi-preopen set (an IVF semi-preclosed set) in which reverse implications do not valid (see Theorem 3.7, Example 3.8 and [9, Example 5.3]).



**Definition 3.20.** For an IVF set  $\mathcal{A}$  in an IVF topological space  $(X, \tau)$ , the *IVF semi-pre interior* (respectively, *IVF semi-pre closure*) of  $\mathcal{A}$  is defined to be the sets

- $\text{int}_{sp}(\mathcal{A}) := \cup\{\mathcal{B} \in IVFSPO(X) \mid \mathcal{B} \subseteq \mathcal{A}\},$
- $\text{cl}_{sp}(\mathcal{A}) := \cap\{\mathcal{B} \in IVFSPC(X) \mid \mathcal{A} \subseteq \mathcal{B}\}.$

**Example 3.21.** Consider an IVF topology  $\tau = \{\tilde{0}, \mathcal{A}, \mathcal{B}, \bar{1}\}$  for  $I$  where  $\mathcal{A}$  and  $\mathcal{B}$  are IVF sets in  $I = [0, 1]$  defined by

$$\begin{aligned}
 \mathcal{A}(x) &= \begin{cases} [\frac{1}{8}, \frac{3}{16}], & 0 \leq x \leq \frac{1}{2}, \\ [\frac{3}{16}, \frac{1}{4}], & \frac{1}{2} < x \leq 1, \end{cases} \\
 \mathcal{B}(x) &= \begin{cases} [\frac{3}{8}, \frac{7}{16}], & 0 \leq x \leq \frac{1}{2}, \\ [\frac{5}{16}, \frac{3}{8}], & \frac{1}{2} < x \leq 1, \end{cases}
 \end{aligned}$$

Then  $IVFSPO(I) = \{\mathcal{C} \in IVF(I) \mid \tilde{0} \subseteq \mathcal{C} \subseteq \mathcal{B}^c\} \cup \{\mathcal{C} \in IVF(I) \mid \mathcal{C} \not\subseteq \mathcal{A}^c\}$ . Take an IVF set  $\mathcal{E}$  in  $I$  given by  $\mathcal{E}(x) = [\frac{7}{16}x + \frac{1}{4}, \frac{7}{16}x + \frac{5}{16}]$  for all  $x \in I$ . Then  $\text{int}_{sp}(\mathcal{E}) = \mathcal{F}$  and  $\text{cl}_{sp}(\mathcal{E}) = \mathcal{G}$ , where

$$\mathcal{F}(x) = \begin{cases} [\frac{7}{16}x + \frac{1}{4}, \frac{7}{16}x + \frac{5}{16}], & 0 \leq x \leq \frac{6}{7}, \\ [\frac{5}{8}, \frac{11}{16}], & \frac{6}{7} \leq x \leq 1, \end{cases}$$

$$\mathcal{G}(x) = \begin{cases} [\frac{3}{8}, \frac{7}{16}], & 0 \leq x \leq \frac{2}{7}, \\ [\frac{7}{16}x + \frac{1}{4}, \frac{7}{16}x + \frac{5}{16}], & \frac{2}{7} \leq x \leq 1. \end{cases}$$

**Proposition 3.22.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be IVF sets in an IVF topological space  $(X, \tau)$ . Then the following assertions are valid.

- (i)  $\text{int}(\mathcal{A}) \subseteq \text{int}_\alpha(\mathcal{A}) \subseteq \text{int}_s(\mathcal{A}) \subseteq \text{int}_{sp}(\mathcal{A}) \subseteq \mathcal{A} \subseteq \text{cl}_{sp}(\mathcal{A}) \subseteq \text{cl}_s(\mathcal{A}) \subseteq \text{cl}_\alpha(\mathcal{A}) \subseteq \text{cl}(\mathcal{A})$ ,
- (ii)  $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \text{cl}_{sp}(\mathcal{A}) \subseteq \text{cl}_{sp}(\mathcal{B}), \text{int}_{sp}(\mathcal{A}) \subseteq \text{int}_{sp}(\mathcal{B})$ ,
- (iii)  $\text{int}_{sp}(\tilde{0}) = \tilde{0} = \text{cl}_{sp}(\tilde{0}), \text{int}_{sp}(\tilde{1}) = \tilde{1} = \text{cl}_{sp}(\tilde{1})$ ,
- (iv)  $\text{int}_{sp}(\text{int}_{sp}(\mathcal{A})) = \text{int}_{sp}(\mathcal{A}), \text{cl}_{sp}(\text{cl}_{sp}(\mathcal{A})) = \text{cl}_{sp}(\mathcal{A})$ ,
- (v)  $\mathcal{A} \in \text{IVFSPO}(X) \Leftrightarrow \text{int}_{sp}(\mathcal{A}) = \mathcal{A}$ ,
- (vi)  $\mathcal{A} \in \text{IVFSPC}(X) \Leftrightarrow \text{cl}_{sp}(\mathcal{A}) = \mathcal{A}$ ,
- (vii)  $\text{int}_{sp}(\mathcal{A} \cap \mathcal{B}) \subseteq \text{int}_{sp}(\mathcal{A}) \cap \text{int}_{sp}(\mathcal{B})$ ,
- (viii)  $\text{cl}_{sp}(\mathcal{A} \cup \mathcal{B}) \supseteq \text{cl}_{sp}(\mathcal{A}) \cup \text{cl}_{sp}(\mathcal{B})$ .

*Proof.* Straightforward. □

**Proposition 3.23.** If  $\mathcal{A}$  is an IVF set in an IVF topological space  $(X, \tau)$ , then  $(\text{int}_{sp}(\mathcal{A}))^c = \text{cl}_{sp}(\mathcal{A}^c)$  and  $(\text{cl}_{sp}(\mathcal{A}))^c = \text{int}_{sp}(\mathcal{A}^c)$ .

*Proof.* Since  $\text{int}_{sp}(\mathcal{A}) \subseteq \mathcal{A}$ , we have

$$\mathcal{A}^c \subseteq (\text{int}_{sp}(\mathcal{A}))^c \in \text{IVFSPC}(X).$$

Thus  $\text{cl}_{sp}(\mathcal{A}^c) \subseteq \text{cl}_{sp}((\text{int}_{sp}(\mathcal{A}))^c) = (\text{int}_{sp}(\mathcal{A}))^c$ . Now since  $\mathcal{A}^c \subseteq \text{cl}_{sp}(\mathcal{A}^c)$ , it follows that  $(\text{cl}_{sp}(\mathcal{A}^c))^c \subseteq \mathcal{A}$  so that

$$(\text{cl}_{sp}(\mathcal{A}^c))^c = \text{int}_{sp}((\text{cl}_{sp}(\mathcal{A}^c))^c) \subseteq \text{int}_{sp}(\mathcal{A}).$$

Hence  $(\text{int}_{sp}(\mathcal{A}))^c \subseteq \text{cl}_{sp}(\mathcal{A}^c)$ . Therefore the first assertion is valid. Similarly we have the second result. □

**Definition 3.24.** Let  $\mathcal{A}$  be an IVF set in  $X$  and let  $M_x$  be an IVF point of  $X$ . Then  $\mathcal{A}$  is called a *semi-preneighborhood* of  $M_x$  if there exists  $\mathcal{B} \in \text{IVFSPO}(X)$  such that  $M_x \tilde{\in} \mathcal{B} \subseteq \mathcal{A}$ .

**Theorem 3.25.** *If an IVF set  $\mathcal{A}$  in  $X$  is IVF preopen, then for each IVF point  $M_x \tilde{\in} \mathcal{A}$ ,  $\mathcal{A}$  is a semi-preneighborhood of  $M_x$ .*

*Proof.* Straightforward. □

The converse of Theorem 3.25 may not be true as seen in the following example.

**Example 3.26.** Let  $\tau = \{\tilde{0}, \mathcal{A}, \tilde{1}\}$  be an IVF topology for  $I$  where  $\mathcal{A}$  is defined by

$$\mathcal{A}(x) = \begin{cases} [\frac{1}{2}x, \frac{1}{2}x + \frac{1}{8}], & 0 \leq x \leq \frac{1}{2}, \\ [\frac{1}{2} - \frac{1}{2}x, \frac{5}{8} - \frac{1}{2}x], & \frac{1}{2} \leq x \leq 1 \end{cases}$$

for all  $x \in I$ . Let  $\mathcal{G}$  be an IVF set in  $I$  such that  $\mathcal{A} \subseteq \mathcal{G} \subseteq \mathcal{A}^c$ . Then  $\mathcal{G} \in IVFSPO(I)$  and  $M_x \tilde{\in} \mathcal{G} \subseteq \mathcal{G}$ . Hence  $\mathcal{G}$  is a semi-preneighborhood of  $M_x$ , but  $\mathcal{G}$  is not an IVF preopen set since  $\mathcal{G} \not\subseteq \mathcal{A} = \text{int}(\text{cl}(\mathcal{G}))$ .

#### 4. IVF semi-preopen mappings

**Definition 4.1.** Let  $(X, \tau)$  and  $(Y, \kappa)$  be IVF topological spaces. A mapping  $f : X \rightarrow Y$  is said to be

- *interval-valued fuzzy semi-preopen* (briefly, *IVF semi-preopen*) if  $f(\mathcal{A})$  is an IVF semi-preopen set in  $Y$  for each IVF open set  $\mathcal{A}$  in  $X$ .
- *interval-valued fuzzy semi-preclosed* (briefly, *IVF semi-preclosed*) if  $f(\mathcal{A})$  is an IVF semi-preclosed set in  $Y$  for each IVF closed set  $\mathcal{A}$  in  $X$ .

**Example 4.2.** (1) Consider IVF topologies  $\tau = \{\tilde{0}, \mathcal{A}, \tilde{1}\}$  and  $\kappa = \{\tilde{0}, \mathcal{B}, \mathcal{C}, \tilde{1}\}$  for  $I$  where  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are IVF sets in  $I = [0, 1]$  defined by

$$\mathcal{A}(x) = \begin{cases} [\frac{11}{20}, \frac{3}{5}], & 0 \leq x \leq \frac{1}{2}, \\ [\frac{9}{10}, \frac{19}{20}], & \frac{1}{2} < x \leq 1, \end{cases}$$

$$\mathcal{B}(x) = \begin{cases} [\frac{1}{5}x + \frac{1}{4}, \frac{1}{5}x + \frac{7}{20}], & 0 \leq x \leq \frac{1}{2}, \\ [-\frac{2}{5}x + \frac{11}{20}, -\frac{2}{5}x + \frac{13}{20}], & \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$\mathcal{C}(x) = \begin{cases} [\frac{2}{5}x + \frac{1}{20}, \frac{2}{5}x + \frac{1}{10}], & 0 \leq x \leq \frac{1}{2}, \\ [-\frac{2}{5}x + \frac{9}{20}, -\frac{2}{5}x + \frac{1}{2}], & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then a function  $f : (I, \tau) \rightarrow (I, \kappa)$  given by  $f(x) = x$  for all  $x \in I$  is an IVF semi-preopen mapping, which is not IVF semiopen.

(2) Let  $\tau = \{\tilde{0}, \mathcal{A}, \tilde{1}\}$  and  $\kappa = \{\tilde{0}, \mathcal{B}, \mathcal{C}, \tilde{1}\}$  be IVF topologies for  $I$  where

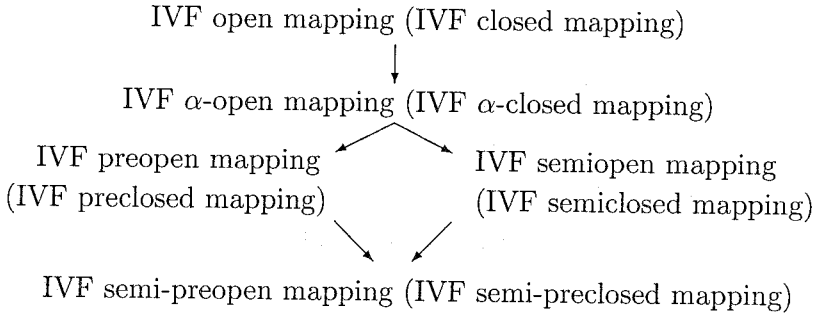
$$\mathcal{A}(x) = \begin{cases} [\frac{2}{5}, \frac{9}{20}], & 0 \leq x \leq \frac{1}{2}, \\ [\frac{1}{20}, \frac{1}{10}], & \frac{1}{2} < x \leq 1, \end{cases}$$

and  $\mathcal{B}$  and  $\mathcal{C}$  are IVF sets in  $I$  described in (1). Then a function  $g : (I, \tau) \rightarrow (I, \kappa)$  defined by  $g(x) = x$  for all  $x \in I$  is an IVF semi-preclosed mapping which is not IVF semiclosed.

(3) Let  $f : (I, \tau) \rightarrow (I, \kappa)$  be a mapping given by  $f(x) = x$  for all  $x \in I$ , where  $\tau = \{\tilde{0}, \mathcal{A}, \tilde{1}\}$ ,  $\kappa = \{\tilde{0}, \mathcal{B}, \tilde{1}\}$ ,  $\mathcal{A}(x) = [-\frac{3}{10}x + \frac{2}{5}, -\frac{3}{10}x + \frac{1}{2}]$  and  $\mathcal{B}(x) = [\frac{1}{4}x + \frac{1}{5}, \frac{1}{4}x + \frac{3}{10}]$  for all  $x \in I$ . Then  $f$  is an IVF semi-preopen mapping. But it is neither IVF preopen nor IVF semiopen.

Obviously, every IVF semiopen (respectively, IVF preopen) mapping is an IVF semi-preopen mapping and every IVF semiclosed (respectively, IVF preclosed) mapping is an IVF semi-preclosed mapping. But the converse may not be true as seen in the above example.

We have the following diagram described relations between an IVF open mapping (an IVF closed mapping), an IVF  $\alpha$ -open mapping (an IVF  $\alpha$ -closed mapping), an IVF semiopen mapping (an IVF semiclosed mapping), an IVF preopen mapping (an IVF preclosed mapping) and an IVF semi-preopen mapping (an IVF semi-preclosed mapping) in which reverse implications do not valid.



**Lemma 4.3.** [12, Theorem 2] *Let  $f : X \rightarrow Y$  be a mapping. Then*

- (i)  $(\forall \mathcal{B} \in IVF(Y)) (f^{-1}(\mathcal{B}^c) = [f^{-1}(\mathcal{B})]^c)$ .
- (ii)  $(\forall \mathcal{A} \in IVF(X)) ([f(\mathcal{A})]^c \subseteq f(\mathcal{A}^c))$ .
- (iii)  $(\forall \mathcal{B}_1, \mathcal{B}_2 \in IVF(Y)) (\mathcal{B}_1 \subseteq \mathcal{B}_2 \Rightarrow f^{-1}(\mathcal{B}_1) \subseteq f^{-1}(\mathcal{B}_2))$ .
- (iv)  $(\forall \mathcal{A}_1, \mathcal{A}_2 \in IVF(X)) (\mathcal{A}_1 \subseteq \mathcal{A}_2 \Rightarrow f(\mathcal{A}_1) \subseteq f(\mathcal{A}_2))$ .
- (v)  $(\forall \mathcal{B} \in IVF(Y)) (f(f^{-1}(\mathcal{B})) \subseteq \mathcal{B})$ .
- (vi)  $(\forall \mathcal{A} \in IVF(X)) (\mathcal{A} \subseteq f^{-1}(f(\mathcal{A})))$ .

**Theorem 4.4.** *A mapping  $f : X \rightarrow Y$  is IVF semi-preopen (respectively, IVF semi-preclosed) if and only if for every IVF set  $\mathcal{B}$  in  $Y$  and an IVF closed (respectively, IVF open) set  $\mathcal{A}$  in  $X$  containing  $f^{-1}(\mathcal{B})$ , there exists an IVF semi-preclosed (respectively, IVF semi-preopen) set  $\mathcal{C}$  in  $Y$  such that  $\mathcal{B} \subseteq \mathcal{C}$  and  $f^{-1}(\mathcal{C}) \subseteq \mathcal{A}$ .*

*Proof.* Let  $f : X \rightarrow Y$  be IVF semi-preopen,  $\mathcal{B} \in IVF(Y)$  and  $\mathcal{A}$  an IVF closed set in  $X$  containing  $f^{-1}(\mathcal{B})$ . Then  $f(\mathcal{A}^c) \in IVFSPO(Y)$ , and so  $\mathcal{C} := (f(\mathcal{A}^c))^c \in IVFSPC(Y)$ . Since  $f^{-1}(\mathcal{B}) \subseteq \mathcal{A}$ , we have  $\mathcal{A}^c \subseteq (f^{-1}(\mathcal{B}))^c = f^{-1}(\mathcal{B}^c)$ . It follows that  $f(\mathcal{A}^c) \subseteq f(f^{-1}(\mathcal{B}^c)) \subseteq \mathcal{B}^c$  so that  $\mathcal{B} \subseteq (f(\mathcal{A}^c))^c = \mathcal{C}$ . Finally,  $f^{-1}(\mathcal{C}) = f^{-1}((f(\mathcal{A}^c))^c) = (f^{-1}(f(\mathcal{A}^c)))^c \subseteq \mathcal{A}$ . Conversely, let  $\mathcal{A}$  be an IVF open set in  $X$  and take  $\mathcal{B} = (f(\mathcal{A}))^c$ . Then

$$f^{-1}(\mathcal{B}) = f^{-1}((f(\mathcal{A}))^c) = (f^{-1}(f(\mathcal{A})))^c \subseteq \mathcal{A}^c$$

and  $\mathcal{A}^c$  is an IVF closed set in  $X$ . By assumption, there exists an IVF semi-preclosed set  $\mathcal{C}$  in  $Y$  such that  $\mathcal{B} \subseteq \mathcal{C}$  and  $f^{-1}(\mathcal{C}) \subseteq \mathcal{A}^c$ . Thus

$(f(\mathcal{A}))^c = \mathcal{B} \subseteq \mathcal{C}$ , and so  $\mathcal{C}^c \subseteq f(\mathcal{A})$ . Since  $f^{-1}(\mathcal{C}) \subseteq \mathcal{A}^c$ , it follows that  $\mathcal{A} \subseteq (f^{-1}(\mathcal{C}))^c = f^{-1}(\mathcal{C}^c)$  so that  $f(\mathcal{A}) \subseteq f(f^{-1}(\mathcal{C}^c)) \subseteq \mathcal{C}^c$ . Hence  $f(\mathcal{A}) = \mathcal{C}^c$  which is an IVF semi-preopen set. Therefore  $f$  is an IVF semi-preopen mapping.  $\square$

**Theorem 4.5.** *A mapping  $f : X \rightarrow Y$  is IVF semi-preopen if and only if it satisfies the following assertion:*

$$(1) \quad (\forall \mathcal{A} \in IVF(X)) (f(\text{int}(\mathcal{A})) \subseteq \text{int}_{sp}(f(\mathcal{A}))).$$

*Proof.* Assume that  $f$  is an IVF semi-preopen mapping. Let  $\mathcal{A} \in IVF(X)$ . Then  $f(\text{int}(\mathcal{A})) \in IVFSPO(Y)$ , and so

$$f(\text{int}(\mathcal{A})) = \text{int}_{sp}(f(\text{int}(\mathcal{A}))) \subseteq \text{int}_{sp}(f(\mathcal{A}))$$

by Proposition 3.22(v). Now assume that (1) is valid. Let  $\mathcal{A}$  be an IVF open set in  $X$ . Then  $\mathcal{A} = \text{int}(\mathcal{A})$ , and so  $f(\mathcal{A}) = f(\text{int}(\mathcal{A})) \subseteq \text{int}_{sp}(f(\mathcal{A}))$ . Since  $\text{int}_{sp}(f(\mathcal{A})) \subseteq f(\mathcal{A})$  in general,  $f(\mathcal{A}) = \text{int}_{sp}(f(\mathcal{A}))$ . It follows from Proposition 3.22(v) that  $f(\mathcal{A}) \in IVFSPO(Y)$  so that  $f$  is an IVF semi-preopen mapping.  $\square$

**Theorem 4.6.** *A mapping  $f : X \rightarrow Y$  is IVF semi-preopen if and only if it satisfies the following assertion:*

$$(2) \quad (\forall \mathcal{B} \in IVF(Y)) (\text{int}(f^{-1}(\mathcal{B})) \subseteq f^{-1}(\text{int}_{sp}(\mathcal{B}))).$$

*Proof.* Assume that  $f : X \rightarrow Y$  is IVF semi-preopen. Let  $\mathcal{B} \in IVF(Y)$ . Since  $f^{-1}(\mathcal{B}) \in IVF(X)$ ,  $\text{int}(f^{-1}(\mathcal{B}))$  is an IVF open set in  $X$ . Since  $f$  is IVF semi-preopen, it follows that  $f(\text{int}(f^{-1}(\mathcal{B}))) \in IVFSPO(Y)$  so that

$$f(\text{int}(f^{-1}(\mathcal{B}))) = \text{int}_{sp}(f(\text{int}(f^{-1}(\mathcal{B})))) \subseteq \text{int}_{sp}(f(f^{-1}(\mathcal{B}))) \subseteq \text{int}_{sp}(\mathcal{B}).$$

Hence  $\text{int}(f^{-1}(\mathcal{B})) \subseteq f^{-1}(\text{int}_{sp}(\mathcal{B}))$ . Conversely, suppose (2) is valid. Let  $\mathcal{A}$  be an IVF open set in  $X$ . Then  $f(\mathcal{A}) \in IVF(Y)$ , and so  $\text{int}(\mathcal{A}) \subseteq \text{int}(f^{-1}(f(\mathcal{A}))) \subseteq f^{-1}(\text{int}_{sp}(f(\mathcal{A})))$ . It follows that

$$f(\mathcal{A}) = f(\text{int}(\mathcal{A})) \subseteq f(f^{-1}(\text{int}_{sp}(f(\mathcal{A})))) \subseteq \text{int}_{sp}(f(\mathcal{A}))$$

so that  $f(\mathcal{A}) = \text{int}_{sp}(f(\mathcal{A})) \in IVFSPO(Y)$ . Therefore  $f$  is an IVF semi-preopen mapping.  $\square$

**Theorem 4.7.** *A mapping  $f : X \rightarrow Y$  is IVF semi-preclosed if and only if it satisfies the following assertion:*

$$(3) \quad (\forall \mathcal{A} \in IVF(X)) (\text{cl}_{sp}(f(\mathcal{A})) \subseteq f(\text{cl}(\mathcal{A}))).$$

*Proof.* Let  $\mathcal{A} \in IVF(X)$ . Then  $\text{cl}(\mathcal{A})$  is an IVF closed set in  $X$ , and so  $f(\text{cl}(\mathcal{A})) \in IVFSPC(Y)$ . Hence  $\text{cl}_{sp}(f(\mathcal{A})) \subseteq \text{cl}_{sp}(f(\text{cl}(\mathcal{A}))) = f(\text{cl}(\mathcal{A}))$ . Now assume that (3) is valid. Let  $\mathcal{A}$  be an IVF closed set in  $X$ . Then  $\text{cl}_{sp}(f(\mathcal{A})) \subseteq f(\text{cl}(\mathcal{A})) = f(\mathcal{A})$ . Since  $f(\mathcal{A}) \subseteq \text{cl}_{sp}(f(\mathcal{A}))$ , it follows that  $f(\mathcal{A}) = \text{cl}_{sp}(f(\mathcal{A})) \in IVFSPC(Y)$ . Hence  $f$  is an IVF semi-preclosed mapping.  $\square$

**Theorem 4.8.** *A mapping  $f : X \rightarrow Y$  is IVF semi-preopen if and only if it satisfies the following assertion:*

$$(\forall \mathcal{B} \in IVF(Y)) (f^{-1}(\text{cl}_{sp}(\mathcal{B}^c)) \subseteq \text{cl}(f^{-1}(\mathcal{B}^c))).$$

*Proof.* This result is proved by a similar method to the proof of Theorem 4.6.  $\square$

**Theorem 4.9.** *If  $f : X \rightarrow Y$  is IVF open (respectively, IVF closed) and  $g : Y \rightarrow Z$  is IVF semi-preopen (respectively, IVF semi-preclosed), then  $g \circ f$  is IVF semi-preopen (respectively, IVF semi-preclosed).*

*Proof.* Straightforward.  $\square$

**Theorem 4.10.** *Let  $f : X \rightarrow Y$  be IVF continuous and onto. For every mapping  $g : Y \rightarrow Z$ , if  $g \circ f$  is IVF semi-preclosed, then  $g$  is IVF semi-preclosed.*

*Proof.* Let  $\mathcal{B}$  be an IVF closed set in  $Y$ . Since  $f$  is IVF continuous,  $f^{-1}(\mathcal{B})$  is IVF closed in  $X$ . Since  $f$  is onto,

$$g(\mathcal{B}) = g(f(f^{-1}(\mathcal{B}))) = (g \circ f)(f^{-1}(\mathcal{B}))$$



which is IVF semi-preclosed in  $Z$  because  $g \circ f$  is IVF semi-preclosed by assumption. Hence  $g$  is an IVF semi-preclosed mapping.  $\square$

## 5. IVF semi-precontinuous mappings

**Definition 5.1.** Let  $(X, \tau)$  and  $(Y, \kappa)$  be IVF topological spaces. A mapping  $f : X \rightarrow Y$  is said to be *interval-valued fuzzy semi-precontinuous* (briefly, *IVF semi-precontinuous*) if  $f^{-1}(\mathcal{B})$  is an IVF semi-preopen set in  $X$  for every IVF open set  $\mathcal{B}$  in  $Y$ .

**Definition 5.2.** Let  $(X, \tau)$  and  $(Y, \kappa)$  be IVF topological spaces. A mapping  $f : X \rightarrow Y$  is said to be (i) *interval-valued fuzzy precontinuous* (briefly, *IVF precontinuous*) if  $f^{-1}(\mathcal{B})$  is an IVF preopen set in  $X$  for every IVF open set  $\mathcal{B}$  in  $Y$ , and (ii) *interval-valued fuzzy semicontinuous* (briefly, *IVF semicontinuous*) if  $f^{-1}(\mathcal{B})$  is an IVF semiopen set in  $X$  for every IVF open set  $\mathcal{B}$  in  $Y$ .

Obviously, every IVF precontinuous (resp. IVF semicontinuous) mapping is an IVF semi-precontinuous mapping, but the converse is not true in general as seen in the following example.

**Example 5.3.** Let  $\mathcal{A}$  be an IVF set in  $X = \{a, b\}$  and let  $\mathcal{B}$  and  $\mathcal{C}$  be IVF sets in  $Y = \{x, y\}$  defined as follows, respectively.

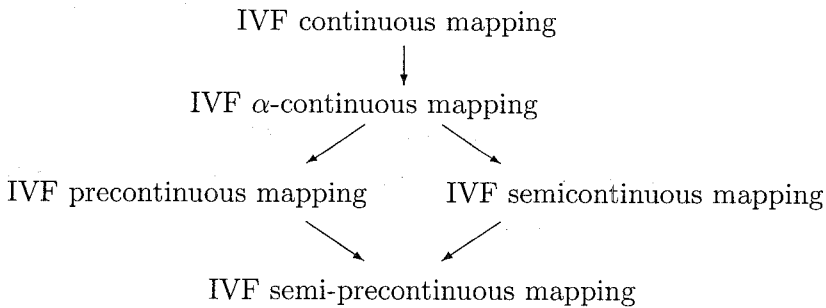
$$\mathcal{A}(a) = [0.3, 0.5], \mathcal{A}(b) = [0.4, 0.45],$$

$$\mathcal{B}(x) = [0.7, 0.77], \mathcal{B}(y) = [0.8, 0.88], \mathcal{C}(x) = [0.3, 0.5], \mathcal{C}(y) = [0.5, 0.58].$$

Let  $\tau = \{\tilde{0}, \tilde{1}, \mathcal{A}\}$ ,  $\kappa_1 = \{\tilde{0}, \tilde{1}, \mathcal{B}\}$  and  $\kappa_2 = \{\tilde{0}, \tilde{1}, \mathcal{C}\}$ . Then the mapping  $f : (X, \tau) \rightarrow (Y, \kappa_1)$  defined by  $f(a) = x$  and  $f(b) = y$  is IVF semi-precontinuous, but not IVF semi-continuous and the mapping  $g : (X, \tau) \rightarrow (Y, \kappa_2)$  defined by  $g(a) = x$  and  $g(b) = y$  is IVF semi-precontinuous but not IVF precontinuous.

**Example 5.4.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be IVF sets in a set  $X = \{a, b, c\}$  given by  $\mathcal{A}(a) = [0.1, 0.2]$ ,  $\mathcal{A}(b) = [0.2, 0.3]$ ,  $\mathcal{A}(c) = [0.2, 0.25]$ ,  $\mathcal{B}(a) = [0.55, 0.6]$ ,  $\mathcal{B}(b) = [0.55, 0.65]$ ,  $\mathcal{B}(c) = [0.6, 0.7]$ , and let  $\mathcal{C}$  be an IVF set in a set  $Y = \{x, y, z\}$  defined by  $\mathcal{C}(x) = [0.2, 0.3]$ ,  $\mathcal{C}(y) = [0.6, 0.7]$ , and  $\mathcal{C}(z) = [0.7, 0.75]$ . Then  $\tau = \{\tilde{0}, \tilde{1}, \mathcal{A}, \mathcal{B}\}$  and  $\kappa = \{\tilde{0}, \tilde{1}, \mathcal{C}\}$  are IVF topologies on  $X$  and  $Y$ , respectively. It is easy to prove that a mapping  $f : X \rightarrow Y$  defined by  $f(a) = x$ ,  $f(b) = y$  and  $f(c) = z$  is an IVF semi-precontinuous mapping which is neither IVF semicontinuous nor IVF precontinuous, since  $f^{-1}(\mathcal{C}) \not\subseteq \text{cl}(\text{int}(f^{-1}(\mathcal{C})))$  and  $f^{-1}(\mathcal{C}) \not\subseteq \text{int}(\text{cl}(f^{-1}(\mathcal{C})))$ .

We have the following diagram described relations between an IVF continuous mapping, an IVF  $\alpha$ -continuous mapping, an IVF semi-continuous mapping, an IVF precontinuous mapping and an IVF semi-precontinuous mapping in which reverse implications do not valid.



**Lemma 5.5.** Let  $f : X \rightarrow Y$  be an IVF  $\alpha$ -open mapping. Then

- (i)  $(\forall \mathcal{B} \in \text{IVF}(Y)) (f^{-1}(\text{cl}(\text{int}(\text{cl}(\mathcal{B})))) \subseteq \text{cl}(f^{-1}(\mathcal{B})))$ .
- (ii)  $(\forall \mathcal{C} \in \text{IVFPO}(Y)) (f^{-1}(\text{cl}(\mathcal{C})) \subseteq \text{cl}(f^{-1}(\mathcal{C})))$ .
- (iii)  $(\forall \mathcal{C} \in \text{IVFSO}(Y)) (f^{-1}(\text{cl}(\mathcal{C})) \subseteq \text{cl}(f^{-1}(\mathcal{C})))$ .
- (iv) If  $f$  is IVF precontinuous, then

$$(\forall \mathcal{B} \in \text{IVFPO}(Y)) (f^{-1}(\mathcal{B}) \in \text{IVFPO}(X)).$$

*Proof.* (i) Let  $\mathcal{B} \in \text{IVF}(Y)$ . Then  $\text{cl}(f^{-1}(\mathcal{B}))$  is IVF closed and  $f^{-1}(\mathcal{B}) \subseteq \text{cl}(f^{-1}(\mathcal{B}))$ . Since  $f$  is an IVF  $\alpha$ -open mapping, it follows

from [7, Theorem 3.12] that there exists an IVF  $\alpha$ -closed set  $\mathcal{D}$  in  $Y$  such that  $\mathcal{B} \subseteq \mathcal{D}$  and  $f^{-1}(\mathcal{D}) \subseteq \text{cl}(f^{-1}(\mathcal{B}))$ . Thus

$$f^{-1}(\text{cl}(\text{int}(\text{cl}(\mathcal{B})))) \subseteq f^{-1}(\text{cl}(\text{int}(\text{cl}(\mathcal{D})))) \subseteq f^{-1}(\mathcal{D}) \subseteq \text{cl}(f^{-1}(\mathcal{B})).$$

(ii) Let  $\mathcal{C} \in \text{IVFPO}(Y)$ . Then  $\mathcal{C} \subseteq \text{int}(\text{cl}(\mathcal{C}))$ , and so  $\text{cl}(\mathcal{C}) \subseteq \text{cl}(\text{int}(\text{cl}(\mathcal{C})))$ . It follows from (i) that

$$f^{-1}(\text{cl}(\mathcal{C})) \subseteq f^{-1}(\text{cl}(\text{int}(\text{cl}(\mathcal{C})))) \subseteq \text{cl}(f^{-1}(\mathcal{C})).$$

(iii) This result is proved by a similar method to the proof of (ii).

(iv) Assume that  $f$  is IVF precontinuous and let  $\mathcal{B} \in \text{IVFPO}(Y)$ . Then  $\mathcal{B} \subseteq \text{int}(\text{cl}(\mathcal{B}))$ , and so

$$f^{-1}(\mathcal{B}) \subseteq f^{-1}(\text{int}(\text{cl}(\mathcal{B}))) \subseteq \text{int}(\text{cl}(f^{-1}(\text{int}(\text{cl}(\mathcal{B})))) \subseteq \text{int}(\text{cl}(f^{-1}(\text{cl}(\mathcal{B}))))).$$

Since  $f$  is an IVF  $\alpha$ -open mapping, it follows from (ii) that

$$f^{-1}(\mathcal{B}) \subseteq \text{int}(\text{cl}(f^{-1}(\text{cl}(\mathcal{B})))) \subseteq \text{int}(\text{cl}(\text{cl}(f^{-1}(\mathcal{B})))) = \text{int}(\text{cl}(f^{-1}(\mathcal{B})))$$

so that  $f^{-1}(\mathcal{B}) \in \text{IVFPO}(X)$ .  $\square$

**Theorem 5.6.** *Let  $f : X \rightarrow Y$  be an IVF  $\alpha$ -open mapping.*

- (i) *If  $f$  is IVF precontinuous, then  $f^{-1}(\mathcal{B}) \in \text{IVFSPO}(X)$  for every  $\mathcal{B} \in \text{IVFSPO}(Y)$ .*
- (ii) *If  $f$  is IVF semi-precontinuous, then  $f^{-1}(\mathcal{B}) \in \text{IVFSPO}(X)$  for every  $\mathcal{B} \in \text{IVFSO}(Y)$ .*

*Proof.* (i) Let  $\mathcal{B} \in \text{IVFSPO}(Y)$ . Then there exists  $\mathcal{C} \in \text{IVFPO}(Y)$  such that  $\mathcal{C} \subseteq \mathcal{B} \subseteq \text{cl}(\mathcal{C})$ . It follows from (ii) and (iv) of Lemma 5.5 that

$$f^{-1}(\mathcal{C}) \subseteq f^{-1}(\mathcal{B}) \subseteq f^{-1}(\text{cl}(\mathcal{C})) \subseteq \text{cl}(f^{-1}(\mathcal{C}))$$

so that  $f^{-1}(\mathcal{B}) \in \text{IVFSPO}(X)$ .

(ii) Let  $\mathcal{B} \in \text{IVFSO}(Y)$ . Then  $\text{int}(\mathcal{B}) \subseteq \mathcal{B} \subseteq \text{cl}(\text{int}(\mathcal{B}))$ , and so

$$f^{-1}(\text{int}(\mathcal{B})) \subseteq f^{-1}(\mathcal{B}) \subseteq f^{-1}(\text{cl}(\text{int}(\mathcal{B}))) \subseteq \text{cl}(f^{-1}(\text{int}(\mathcal{B}))),$$

where the last inclusion is by Lemma 5.5(ii). Since  $f$  is IVF semi-precontinuous and since  $\text{int}(\mathcal{B}) \in \text{IVFO}(Y)$ , we have  $f^{-1}(\text{int}(\mathcal{B})) \in$

$IVFSPO(X)$ . Using Theorem 3.14, we conclude that  $f^{-1}(\mathcal{B}) \in IVFSPO(X)$ .  $\square$

**Theorem 5.7.** *Let  $f : X \rightarrow Y$  be a mapping from an IVF topological space  $X$  to an IVF topological space  $Y$ . The the following statements are equivalent:*

- (i)  $f$  is IVF semi-precontinuous.
- (ii)  $f^{-1}(\mathcal{B}) \in IVFSPC(X)$  for every IVF closed set  $\mathcal{B}$  in  $Y$ .
- (iii) For every IVF point  $M_x$  in  $X$  and every IVF open set  $\mathcal{B}$  in  $Y$  such that  $f(M_x) \tilde{\in} \mathcal{B}$  there exists  $\mathcal{A} \in IVFSPO(X)$  such that  $M_x \tilde{\in} \mathcal{A}$  and  $f(\mathcal{A}) \subseteq \mathcal{B}$ .
- (iv) For every IVF point  $M_x$  in  $X$  and every neighborhood  $\mathcal{B}$  of  $f(M_x)$ ,  $f^{-1}(\mathcal{B})$  is a semi-preneighborhood of  $M_x$ .

*Proof.* (i)  $\Rightarrow$  (ii). Let  $\mathcal{B} \in IVFC(Y)$ . Then  $\mathcal{B}^c \in IVFO(Y)$ , and so  $(f^{-1}(\mathcal{B}))^c = f^{-1}(\mathcal{B}^c) \in IVFSPO(X)$  since  $f$  is IVF semi-precontinuous. Hence  $f^{-1}(\mathcal{B}) \in IVFSPC(X)$ .

(ii)  $\Rightarrow$  (i). Let  $\mathcal{B} \in IVFO(Y)$ . Then  $\mathcal{B}^c \in IVFC(Y)$ . It follows from (ii) that  $(f^{-1}(\mathcal{B}))^c = f^{-1}(\mathcal{B}^c) \in IVFSPC(X)$  so that  $f^{-1}(\mathcal{B}) \in IVFSPO(X)$ . Therefore  $f$  is IVF semi-precontinuous.

(i)  $\Rightarrow$  (iii). Let  $M_x$  be an IVF point in  $X$  and  $\mathcal{B}$  be an IVF open set in  $Y$  such that  $f(M_x) \tilde{\in} \mathcal{B}$ . Then  $M_x \tilde{\in} f^{-1}(f(M_x)) \subseteq f^{-1}(\mathcal{B})$ . Putting  $\mathcal{A} = f^{-1}(\mathcal{B})$ , then  $M_x \tilde{\in} f^{-1}(\mathcal{B}) = \mathcal{A}$  and  $f(\mathcal{A}) = f(f^{-1}(\mathcal{B})) \subseteq \mathcal{B}$ . Since  $f$  is IVF semi-precontinuous, obviously  $\mathcal{A} = f^{-1}(\mathcal{B}) \in IVFSPO(X)$ . Thus (iii) is valid.

(iii)  $\Rightarrow$  (i). Let  $\mathcal{B} \in IVFO(X)$  and let  $M_x$  be an IVF point in  $X$  such that  $M_x \tilde{\in} f^{-1}(\mathcal{B})$ . Then  $f(M_x) \tilde{\in} \mathcal{B}$ , which implies from (iii) that there exists  $\mathcal{A} \in IVFSPO(X)$  such that  $M_x \tilde{\in} \mathcal{A}$  and  $f(\mathcal{A}) \subseteq \mathcal{B}$ . It follows that  $M_x \tilde{\in} \mathcal{A} \subseteq f^{-1}(f(\mathcal{A})) \subseteq f^{-1}(\mathcal{B})$  so from Theorem 3.13 that  $f$  is IVF semi-precontinuous.

(iii)  $\Rightarrow$  (iv). Let  $M_x$  be an IVF point in  $X$  and  $\mathcal{B}$  is a neighborhood of  $f(M_x)$ . Then there exists  $\mathcal{C} \in IVFO(Y)$  such that  $f(M_x) \tilde{\in} \mathcal{C} \subseteq \mathcal{B}$ . It

follows from (iii) that there exists  $\mathcal{A} \in IVFSPO(X)$  such that  $M_x \tilde{\in} \mathcal{A}$  and  $f(\mathcal{A}) \subseteq \mathcal{C} \subseteq \mathcal{B}$ . Thus

$$M_x \tilde{\in} \mathcal{A} \subseteq f^{-1}(f(\mathcal{A})) \subseteq f^{-1}(\mathcal{C}) \subseteq f^{-1}(\mathcal{B}),$$

and so  $f^{-1}(\mathcal{B})$  is a semi-preneighborhood  $M_x$ .

(iv)  $\Rightarrow$  (iii). Let  $M_x$  be an IVF point in  $X$  and  $\mathcal{B} \in IVFO(Y)$  such that  $f(M_x) \tilde{\in} \mathcal{B}$ . Then clearly  $\mathcal{B}$  is a neighborhood of  $f(M_x)$ . By (iv), there exists  $\mathcal{D} \in IVFSPO(X)$  such that  $M_x \tilde{\in} \mathcal{D} \subseteq f^{-1}(\mathcal{B})$ . Then

$$f(M_x) \tilde{\in} f(\mathcal{D}) \subseteq f(f^{-1}(\mathcal{B})) \subseteq \mathcal{B},$$

and thus (iii) is valid. □

**Theorem 5.8.** *Let  $X, Y$  and  $Z$  be IVF topological spaces. If  $f : X \rightarrow Y$  is IVF semi-precontinuous and  $g : Y \rightarrow Z$  is IVF continuous, then  $g \circ f : X \rightarrow Z$  is IVF semi-precontinuous.*

*Proof.* Straightforward. □

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