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REMARKS ON DIGITAL HOMOTOPY EQUIVALENCE

SANG-EON HAN

Abstract. The notions of digital k-homotopy equivalence and digital (k_0, k_1)-homotopy equivalence were developed in [13, 16]. By the use of the digital k-homotopy equivalence, we can investigate digital k-homotopy equivalent properties of Cartesian products constructed by the minimal simple closed 4- and 8-curves in $\mathbf{Z}^2$.

1. Introduction

A digital image (X, k) in $\mathbf{Z}^n$ can be recognized as both a discrete topological space with k-adjacency on $\mathbf{Z}^n$ and a digital k-graph on $\mathbf{Z}^n$ [9, 10, 11, 12, 13, 14, 15, 16, 17]. The notion of *digital k-homotopy equivalence* was originally developed in [13] and it has been used to study some digital images in relation to both a classification of digital images (X, k) and a calculation of the digital fundamental group of some digital images. This has been studied in many papers including [4, 13, 16, 17, 18]. The paper [13] is important for its introduction of the digital k-homotopy equivalence for classifying Cartesian products of some digital images. Attempt to study the digital k-homotopy equivalence for classifying some Cartesian products also appeared in [13] (however there are insufficient presentations of some topics in [13]. Thus, we fix these errors in the current paper). Furthermore, since a digital image (X, k) can be recognized as a digital k-graph [16] (see

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also [9, 11, 17, 18]), the digital graph versions of the digital k-homotopy equivalence and the *digital* (k_0, k_1)*-homotopy equivalence* were originally introduced in [16]. Recently, the paper [4] investigated some properties of a digital k-homotopy equivalence (although, it should be noted those earlier papers such as [13, 16]).

The paper is organized as follows. In Section 2, we provide some basic notions as well as a (k_0, k_1)-isomorphism. In Section 3, we describe the notions of digital k-homotopy equivalence and (k_0, k_1)-homotopy equivalence and we investigate their properties. In Section 4, we investigate some properties of a digital k-surface in $\mathbf{Z}^n, n \geq 3$, and a digital (k_0, k_1)-covering. In Section 5, some homotopical properties of Cartesian products constructed by minimal simple closed k-curves in $\mathbf{Z}^2$ are investigated. Section 6 gives some remark on the origin of the notions of (k_0, k_1)*-isomorphism,* k*-homotopy equivalence, and* (k_0, k_1)*-homotopy equivalence.*

2. Digital (k_0, k_1)-isomorphism

Let $\mathbf{Z}$ represent the set of integers. A digital image in $\mathbf{Z}^n$ can be considered to be a discrete topological space in $\mathbf{Z}^n$ with k-adjacency and a digital k-graph in $\mathbf{Z}^n$, which has been studied by employing combinatorial topology, general topology, algebraic topology, graph theory, and so forth [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. A digital image with k-adjacency has been considered in $(\mathbf{Z}^n, k, \bar{k}, X)$, where $n \in \mathbf{N}$ is a natural number, $X \subset \mathbf{Z}^n$ is the set of finite points we regard as belonging to the image depicted, k represents an adjacency relation for X, and $\bar{k}$ represents an adjacency relation for $\mathbf{Z}^n - X$. We say that the pair (X, k) is a *digital image* or *a discrete topological space in* $\mathbf{Z}^n$ *with* k*-adjacency.*

As a generalization of the 4- and 8-adjacency of $\mathbf{Z}^2$, and the 6-, 18-, and 26-adjacency of $\mathbf{Z}^3$, the following adjacency relations of $\mathbf{Z}^n, n \geq 1$, were developed in [8, 9, 14, 15] and they have been used to study digital images in $\mathbf{Z}^n, n \geq 1$.

(2.1)

$$k \in \{2n, n \geq 1; 3^n - 1, n \geq 2; 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1, 2 \leq r \leq n-1, n \geq 3\},$$

where $C_t^n = \frac{n!}{(n-t)!t!}$.
For instance, we obtain the following from (2.1). [1, 3, 4, 6, 7, 8, 9, 22]

$$\begin{aligned}(n, l, k) \in \{&(1,1,2), (2,1,4), (2,2,8), (3,1,6), (3,2,18), (3,3,26),\\ &(4,1,8), (4,2,32), (4,3,64), (4,4,80)\}.\end{aligned}$$

From now on, a digital image (X, k) is considered in $(\mathbf{Z}^n, k, \bar{k}, X)$, $n \geq 1$, where $(k, \bar{k}) \in \{(k, 2n), (2n, \bar{k})\}$. But $k \neq \bar{k}$ except for $n = 1$ owing to the *digital k-connectivity paradox* [22]. For $a, b \in \mathbf{Z}$ with $a \lneq b$, the set $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} | a \leq n \leq b\}$ is called a *digital interval* [3].

A k-path from x to y in X is assumed to be a sequence $(x = x_0, x_1, x_2, \cdots, x_{m-1}, x_m = y)$ in X such that each point x_i is k-adjacent to x_{i+1} for $m \geq 1$ and $i \in [0, m-1]_{\mathbf{Z}}$. Then, the number m is called the *length* of this path [22]. If $x_0 = x_m$, then the k-path is said to be *closed.* For a digital image (X, k), two distinct points $x, y \in X$ are k-connected [22]. If there is a k-path from x to y in X, and if any two distinct points in X are k-connected, then X is called *k-connected.* For an adjacency relation k, a *simple k-path* with m elements in $\mathbf{Z}^n$ is assumed to be a sequence $(x_0, x_1, x_2, \cdots, x_{m-1}) \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $j = i + 1$ or $i = j + 1$ [3, 8, 9]. The following notion of digital neighborhood has been be used to study the digital k-surface and to establish another statement of digital continuity in Proposition 2.1.

Definition 1. [6, 7, 8, 9, 10, 11] *Let (X, k) be a digital image in $\mathbf{Z}^n$ and $\varepsilon \in \mathbf{N}$. We say that the k-neighborhood of $x_0 \in X$ with radius ε is the set*

$$N_k(x_0, \varepsilon) := \{x \in X | \quad l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\},$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x in X.

For a digital images (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, the following characterizes digital continuity in a fashion used later in this paper.

Proposition 2.1. [7, 8, 9] *Let (X, k_0) and (Y, k_1) be digital images. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x_0 \in X, \varepsilon \in \mathbf{N}$, and $N_{k_1}(f(x_0), \varepsilon) \subset Y$, there is $\delta \in \mathbf{N}$ such that the corresponding $N_{k_0}(x_0, \delta) \subset X$ satisfies $f(N_{k_0}(x_0, \delta)) \subset N_{k_1}(f(x_0), \varepsilon)$.*

For an adjacency relation k of $\mathbf{Z}^n$, a *simple closed k-curve* with l elements in $X \subset \mathbf{Z}^n$ is the image of a $(2, k)$-continuous function $f : [0, l-1]_{\mathbf{Z}} \to X$ such that $f(i)$ and $f(j)$ are k-adjacent if and only if either $i = j \pm 1 (mod\, l)$ [3, 8, 15] and it is denoted by $SC_k^{n,l}$ which is assumed to be a sequence $(c_i)_{i \in [0,l-1]_{\mathbf{Z}}}$, where $f(i) = c_i$ [15].

In [16] (see also [17, 18, 19, 20]), digital graph versions of the (k_0, k_1)-continuity, the (k_0, k_1)-homeomorphism, the digital (k_0, k_1)-covering, and the (k_0, k_1)-homotopy were developed. Let us introduce some necessary terminology for digital k-graph theory. A digital graph G on $\mathbf{Z}^n$ is considered in a quadruple $(\mathbf{Z}^n, k, \bar{k}, G)$, where $n \in \mathbf{N}$, G is a digital graph depicted on $\mathbf{Z}^n$, and k represents an adjacency relation for G [16]. Indeed, a digital image (X, k) can be recognized as a *digital graph with k-adjacency* called *a digital k-graph* [16] (see also [9, 11, 17, 18]). To be specific, we say that a *digital k-graph* is a graph on $\mathbf{Z}^n$ with k-adjacency and write it as $G_k = (V_k, E_k)$ consisting of both V_k and E_k which are the sets of vertices and *k-edges uv*, respectively. The k-edge uv is considered in such a way: $u \in N_k(v) = \{u | u$ is k-adjacent to $v\}$ and $N_k(v)$ is the k-neighbors of v in $\mathbf{Z}^n$ [22]. By the use of the above mentioned terminology, the concepts of (k_0, k_1)-homomorphism, *(k_0, k_1)-isomorphism*, and induced k-subgraph were originally developed in [16] (see also [11, 17, 18]).

Definition 2. [16] *For a digital k-graph $G_k = (V_k, E_k)$ and a subset $V' \subset V_k$, we say that the induced k-subgraph generated by V' is the subgraph of G_k consisting of vertices V' and only the k-edges which join vertices from V'.*

Definition 3. [16] *For a digital k-graph $G_k = (V_k, E_k)$, we say that the graph k-neighborhood of $v_0 \in V$ with radius $\varepsilon \in \mathbf{N}$ is the set $G_k(v_0, \varepsilon) = (V_k(v_0, \varepsilon), E_k(v_0, \varepsilon))$ which is an induced k-subgraph generated by $\{v \in V_k | l_k(v_0, v) \leq \varepsilon\} \cup \{v_0\}$, where $l_k(v_0, v)$ is the length of a shortest simple k-path from v_0 to v.*

Definition 4. [16] *Let $G_{k_i} := (V_{k_i}, E_{k_i})$ be a digital k_i-graph on $\mathbf{Z}^{n_i}$, $i \in \{0, 1\}$. A map $f : G_{k_0} \to G_{k_1}$ is called a (k_0, k_1)-homomorphism if for every vertex $v_0 \in V_{k_0}$ and $G_{k_1}(f(v_0), \varepsilon) \subset G_{k_1}$, there is $G_{k_0}(v_0, \delta) \subset G_{k_0}$ such that $f(G_{k_0}(v_0, \delta)) \subset G_{k_1}(f(v_0), \varepsilon)$, where $\varepsilon, \delta \in \mathbf{N}$.*

Definition 5. [16] *(see also* [9, 11, 17, 18]*) Let G_{k_i} be a k_i-graph on $\mathbf{Z}^{n_i}$, $i \in \{0, 1\}$. We say that $h : G_{k_0} \to G_{k_1}$ is a (k_0, k_1)-isomorphism if*
(1) the restriction map on V_{k_0}, briefly $h|_{V_{k_0}}$, is bijective and
(2) h and h^{-1} are, respectively, a (k_0, k_1)- and a (k_1, k_0)-homomorphism.

By Definitions 1, 2, 3, 4, and 5, we obtain the following.

Remark 2.2. [16] *(see also* [9, 11, 17, 18]*) A digital image (X, k) can be recognized as a digital k-graph $G_k = (V_k, E_k)$ and vice versa because $G_k(v_0, \varepsilon) \subset G_k$ in Definition 3 is equivalent to and $N_k(x_0, \varepsilon) \subset X$ in Definition 1 if $x_0 = v_0$, where $V_k = X$.*

By Remark 2.2, we can represent Definition 5 as follows.

Definition 6. [16]*(see also* [9, 11]*) For two digital images (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-isomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. Then, we use the notation $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$*

and $k_0 = k_1$, *then we call it a* k_0*-isomorphism and use the notation* $X \approx_{k_0} Y$.

By Remark 2.2, we obtain the following.

Remark 2.3. [9, 11, 16, 17, 18] *Since the digital graph theoretical approach is so convenient for studying a digital image, hereafter, we use a* (k_0, k_1)*-isomorphism instead of a* (k_0, k_1)*-homeomorphism.*

Recently, in [16] (see also [9, 11]), for a digital image (X, k) in $\mathbf{Z}^n$, the notion of *geometric realization* of (X, k) was also developed by the similar method as that of (X, k) in $\mathbf{Z}^3$ [5]. The geometric realization of a digital image (X, k) in $\mathbf{Z}^n$ is the simplicial complex $S(G_k)$ realized by the digital k-graph G_k derived from (X, k). Moreover, a (k_0, k_1)-continuous map $f : (X, k_0) \to (Y, k_1)$ induces the (k_0, k_1)-homomorphism $G(f) : G_{k_0} \to G_{k_1}$ characterizing a simplicial map $S(f) : S(G_{k_0}) \to S(G_{k_1})$ [9, 16].

3. Digital (k_0, k_1)-homotopy Equivalence

In [16], the notions of digital graph (k_0, k_1)-homotopy and *graph* (k_0, k_1)*-homotopy equivalence* were originally developed. By Remarks 2.2 and 2.3, we may consider the graph (k_0, k_1)-homotopy in [16] to be the (k_0, k_1)-homotopy in [3, 7, 10, 11]. For (X, k), consider a subset $(A, k) \subset (X, k)$. In relation to the strong k-deformation retract in [10, 17, 18], motivated by the pointed digital (k_0, k_1)-homotopy in [3], the following notion of (k_0, k_1)*-homotopy relative to* A was established in [7] (see also [9, 11, 17, 18]).

Definition 7. [7, 11, 17, 18] *Let* (X, k_0) *and* (Y, k_1) *be digital images in* $\mathbf{Z}^{n_0}$ *and* $\mathbf{Z}^{n_1}$, *respectively, and* $A \subset X$. *Let* $f, g : X \to Y$ *be* (k_0, k_1)*-continuous functions. Suppose that there exist* $m \in \mathbf{N}$ *and a function* $F : X \times [0, m]_{\mathbf{Z}} \to Y$ *such that*

- *for all $x \in X, F(x,0) = f(x)$ and $F(x,m) = g(x)$;*
- *for all $x \in X$, the induced function $F_x : [0,m]_{\mathbf{Z}} \to Y$ defined by $F_x(t) = F(x,t)$ is $(2,k_1)$-continuous for all $t \in [0,m]_{\mathbf{Z}}$;*
- *for all $t \in [0,m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ defined by $F_t(x) = F(x,t)$ is (k_0,k_1)-continuous for all $x \in X$.*

Then, F is called a (k_0,k_1)-homotopy between f and g, and f and g are (k_0,k_1)-homotopic in Y.

- *Furthermore, for all $t \in [0,m]_{\mathbf{Z}}$, then the induced map F_t on A is a constant which is the prescribed function from A to Y. In other words, $F_t(x) = f(x) = g(x)$ for all $x \in X$ and for all $t \in [0,m]_{\mathbf{Z}}$.*

Then, we call F a (k_0,k_1)-homotopy relative to A between f and g, and we say that f and g are (k_0,k_1)-homotopic relative to A in Y.

Remark 3.1. *In order to make the notion of (k_0,k_1)-homotopy relative to some digital image (A,k) clear, the fourth bullet item in Definition 2 of [7] and Definition 1 of [10] was changed into the current fourth bullet item in Definition 7.*

We say that a (k_0,k_1)-continuous function $f : X \to Y$ is k_1*-nullhomotopic* in Y if f is k_1-homotopic in Y to a constant function $c_{\{y_0\}}$ for some $y_0 \in Y$ [3]. In the following, we use the pointed (k_0,k_1)-homotopy. Due to the pointed digital homotopy for a pointed digital image (X,x_0), the k*-fundamental group* $\pi^k(X,x_0)$ was established with the Khalimsky operation in [21]. If x_0 and x_1 belong to the same k-connected component of X, then $\pi^k(X,x_0)$ and $\pi^k(X,x_1)$ are isomorphic [3]. Thus, for a k-connected digital image X, we need not fix a base point for the k-fundamental group. In particular, if $A = \{x_0\} \subset X$, then we say that F is a pointed (k_0,k_1)-homotopy at $\{x_0\}$ [3]. If the identity map 1_X is (k,k)-homotopic relative to $\{x_0\}$ in X to a constant map with image consisting of some $x_0 \in X$, then we say that (X,x_0) is *pointed k-contractible* [3]. The current k-contractibility is different from the contractibility in Euclidean topology [3, 7, 8, 9, 15].

Consider the following minimal simple closed k-curves in $\mathbf{Z}^2$, $k \in \{4,8\}$ [7, 8, 9, 10, 11, 12, 13, 14, 15]. These will be often used in the paper.

$$\begin{aligned} MSC_4: &\approx ((0,0),(1,0),(2,0),(2,1),(2,2),(1,2),(0,2),(0,1)) \approx SC_4^{2,8}, \\ MSC_8: &\approx ((0,0),(1,1),(1,2),(0,3),(-1,2),(-1,1)) \approx SC_8^{2,6}, \text{and} \\ MSC_8': &\approx ((0,0),(1,1),(0,2),(-1,1)) \approx SC_8^{2,4}. \end{aligned}$$

Then, we obtain the following [3, 6, 7, 8, 9].

- MSC_4 is not 4-contractible but 8-contractible.
- MSC_8 is not 8-contractible.
- MSC_8' is 8-contractible.

Motivated by the Khalimsky homotopy in [21], several digital homotopies have been used in digital topology [3, 7, 22, 24]. Let us now assume that $F^k(X,x_0) := \{f : [0,m]_{\mathbf{Z}} \to (X,x_0)$ is a pointed $(2,k)-$continuous map$\}$ and recall the *Khalimsky operation* in [21]. To be specific, for members $f : [0,m_1]_{\mathbf{Z}} \to X$, $g : [0,m_2]_{\mathbf{Z}} \to X$ of $F^k(X,x_0)$, we get a map $f * g : [0,m_1+m_2]_{\mathbf{Z}} \to X$ defined by

$$f * g(t) = \begin{cases} f(t), & \text{if } 0 \le t \le m_1; \\ g(t-m_1), & \text{if } m_1 \le t \le m_1+m_2. \end{cases}$$

Due to the *trivial extension* in [3], we compare the digital homotopy properties of loops whose domains may have different cardinality. Namely, if $m_f \le m_{f'}$, we can obtain a trivial extension of a loop $f : [0,m_f]_{\mathbf{Z}} \to X$ to a loop $f' : [0,m_{f'}]_{\mathbf{Z}} \to X$ via

$$f'(t) = \begin{cases} f(t), & \text{if } 0 \le t \le m_f; \\ f(m_f), & \text{if } m_f \le t \le m_{f'}. \end{cases}$$

Besides, if $f_1, f_2, g_1, g_2 \in F^k(X,x_0)$, $f_1 \in [f_2]$, and $g_1 \in [g_2]$, then $f_1 * g_1 \in [f_2 * g_2]$, *i.e.*, $[f_1 * g_1] = [f_2 * g_2]$ [3, 21]. Then, we use the notation $\pi^k(X,x_0) = \{[f] | f \in F^k(X,x_0)\}$ is a group [3] with the operation $[f] \cdot [g] = [f * g]$ and is called the *k-fundamental group* of (X,x_0) [3], where $*$ means the *Khalimsky operation* in [21]. Also, we have the following.

If $h : (X, k_0) \to (Y, k_1)$ is a pointed (k_0, k_1)-continuous map, then there is a homomorphism $h_* : \pi^{k_0}(X, x_0) \to \pi^{k_1}(Y, y_0)$ for which $h_*([f]) = [h \circ f]$, where $[f] \in \pi^{k_0}(X, x_0)$ [3]. Furthermore, if (X, k) is pointed k-contractible, then $\pi^k(X, x_0)$ is trivial [3]. Besides, in [15, 16], a (k_0, k_1)-isomorphism $h : (X, k_0) \to (Y, k_1)$ induces a group isomorphism defined by $h_* : \pi^{k_0}(X, x_0) \to \pi^{k_1}(Y, y_0)$ for which $h_*([f]) = [h \circ f]$, where $[f] \in \pi^{k_0}(X, x_0)$.

The notion of (k_0, k_1)-homotopy equivalence was originally introduced in [16] from the view point of digital k-graph theory. By Remark 2.2, the notion of *graph* (k_0, k_1)*-homotopy equivalence* developed in [16] can be represented as follows.

Definition 8. [16] *(see also* [17, 18]*) Consider two digital images* (X, k_0) *and* (Y, k_1) *in* $\mathbf{Z}^{n_0}$ *and* $\mathbf{Z}^{n_1}$*, respectively. If there are both a* (k_0, k_1)*-continuous map* $h : X \to Y$ *and a* (k_1, k_0)*-continuous map* $l : Y \to X$ *such that* $l \circ h$ *is* k_0*-homotopic to* 1_X *and* $h \circ l$ *is* k_1*-homotopic to* 1_Y*, then the map* $h : X \to Y$ *is called a* (k_0, k_1)*-homotopy equivalence and is denoted by* $X \simeq_{(k_0,k_1)\cdot h\cdot e} Y$.

In Definition 8, if $n_0 = n_1$ and $k_0 = k_1$, then we obtain the following.

Definition 9. [13] *Consider two digital images* (X, k) *and* (Y, k) *in* $\mathbf{Z}^n$*. If there are* (k, k)*-continuous maps* $h : X \to Y$ *and* $l : Y \to X$ *such that* $l \circ h$ *is* k*-homotopic to* 1_X *and* $h \circ l$ *is* k*-homotopic to* 1_Y*, then the map* $h : X \to Y$ *is called a* k*-homotopy equivalence and is denoted by* $X \simeq_{k\cdot h\cdot e} Y$.

By the use of the digital k-homotopy equivalence we can classify digital images. When X is k-homotopy equivalent to Y, there is no need to have the same cardinality of between X and Y [4].

4. Digital k-surface in $\mathbf{Z}^n$ and Digital (k_1, k_0)-covering

In order to investigate some properties of Cartesian products constructed by the minimal simple closed 4- and 8-curves in $\mathbf{Z}^2$, let us introduce some necessary terminology. A point $x \in X$ is called a *k-corner* if x is k-adjacent to two and only two points y, $z \in X$ such that y and z are k-adjacent to each other [1]. The k-corner x is called *simple* if y, z are not k-corners and if x is the only point k-adjacent to both y, z [2]. X is called a *generalized simple closed k-curve* if what is obtained by removing all simple k-corners of X is a simple closed k-curve [2, 24]. For a k-connected digital image (X, k) in $\mathbf{Z}^3$, we recall the following: $|X|^x := N_{26}^*(x) \cap X$, $N_{26}^*(x) = \{x' | x \text{ and } x' \text{ are 26-adjacent}\}$ [1, 2]. Thus, we can restate $|X|^x := N_{26}(x, 1) - \{x\}$ in $\mathbf{Z}^3$ by Definition 1. More generally, for a k-connected digital image (X, k) in $\mathbf{Z}^n, n \geq 3$, we can state $|X|^x = N_{3^n-1}^*(x) \cap X$, where $N_{3^n-1}^*(x) = \{x' | x \text{ and } x' \text{ are } (3^n-1)\text{-adjacent}\}$. In other words, $|X|^x = N_{3^n-1}(x, 1) - \{x\}$ in $X \subset \mathbf{Z}^n$ by Definition 1 [7, 9, 10].

In the earlier work in [7], a (simple) closed k-surface in $\mathbf{Z}^n$ was partially studied, where $(k, \bar{k}) = (3^n - 2^n - 1, 2n)$, which is the generalization of *Malgouyres' simple closed 18-surface* [24]. Now in order to define a closed $2n$-surface and a closed k-surface, where $k \neq 3^n - 2^n - 1$, the more generalized criterion of a closed k-surface in $\mathbf{Z}^n$ is required, which is motivated by the notions of both Malgouyres' 18-surface in $\mathbf{Z}^3$ in [24] and *Morgenthaler's and Rosenfeld's 6-, 26-surfaces* in $\mathbf{Z}^3$ [25], where the k-adjacency relations of $\mathbf{Z}^n$ is taken from (2-1). For a digital image (X, k), a *simple k-point* is one whose removal does not change the digital topological property of (X, k) [1, 16]. Indeed, some points in X including all simple k-points can be deleted by a strong k-deformation retract.

Definition 10. [9, 10, 18] *Let (X, k) be a digital image in $\mathbf{Z}^n, n \geq 3$, and $\bar{X} = \mathbf{Z}^n - X$. Then, X is called a closed k-surface if it satisfies the following.*

(1) In case that $(k, \bar{k}) \in \{(k, 2n), (2n, 3^n - 1)\}$, where the k-adjacency is taken from (2-1) and $k \neq 3^n - 2^n - 1$, then
(a) for each point $x \in X$, $|X|^x$ has exactly one k-component k-adjacent to x;
(b) $|\bar{X}|^x$ has exactly two $\bar{k}$-components $\bar{k}$-adjacent to x; we denote by C^{xx} and D^{xx} these two components; and
(c) for any point $y \in N_k(x) \cap X$, $N_{\bar{k}}(y) \cap C^{xx} \neq \phi$ and $N_{\bar{k}}(y) \cap D^{xx} \neq \phi$, where $N_k(x)$ and $N_{\bar{k}}(y)$ are the k- and $\bar{k}$-neighbors of the points x and y, respectively.

(2) In case that $(k, \bar{k}) = (3^n - 2^n - 1, 2n)$, then
(a) X is k-connected,
(b) for each point $x \in X$, $|X|^x$ is a generalized simple closed k-curve.
Furthermore, if a closed k-surface X does not have a simple k-point, then X is called simple.

Remark 4.1. *For a closed k-surface (X, k) in $\mathbf{Z}^n$ with $k \neq 2n$, if each point $x \in X$ satisfies that $|X|^x$ is a simple closed k-curve, then (X, k) is simple.*

By the use of various tools from digital (k_0, k_1)-covering theory in [8, 10, 11, 15, 16, 17, 18], both the calculation of the digital fundamental group of some digital images and the classification of digital images can proceed. For an efficient calculation of the digital fundamental group, the following digital (k_0, k_1)-covering has been often used.

Definition 11. [9, 10, 11, 17, 18] *Let (E, k_0) and (B, k_1) be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $p : E \to B$ be a (k_0, k_1)-continuous surjection. Suppose that for every $b \in B$ there exists $\varepsilon \in \mathbf{N}$ such that*

$(DC1)$ for some index set M,

$p^{-1}(N_{k_1}(b,\varepsilon)) = \cup_{i\in M} N_{k_0}(e_i,\varepsilon)$ with $e_i \in p^{-1}(b)$;

(DC2) if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i,\varepsilon) \cap N_{k_0}(e_j,\varepsilon) = \phi$; and

(DC3) the restriction map $p|_{N_{k_0}(e_i,\varepsilon)} : N_{k_0}(e_i,\varepsilon) \to N_{k_1}(b,\varepsilon)$ is a (k_0,k_1)-isomorphism for all $i \in M$.

Then, the map p is called a (k_0,k_1)-covering map and (E,p,B) is said to be a (k_0,k_1)-covering.

In Definition 11, $N_{k_1}(b,\varepsilon)$ is called an *elementary k_1-neighborhood* of b with radius ε. The collection $\{N_{k_0}(e_i,\varepsilon)|i \in M\}$ is a partition of $p^{-1}(N_{k_1}(b,\varepsilon))$ into slices.

The current (k_0,k_1)-covering is stronger than a graph covering in pure graph theory and it has different motivation from that of pure graph theory.

Remark 4.2. [10] *(see also* [11, 17, 18]*) In Definition 11, we may take $\varepsilon = 1$.*

Let $p : (E,e_0) \to (B,b_0)$ be a (k_0,k_1)-covering map which preserves the base point. Any k_1-path $f : [0,m]_{\mathbf{Z}} \to B$ beginning at b_0 has a *unique digital lifting* to a k_0-path $\tilde{f}$ in E beginning at e_0 [11, 17, 18]. Let us now restate this property in terms of the current (k_0,k_1)-covering as follows.

Lemma 4.3. [15] *(see also* [10, 11]*) For pointed digital images $((E,e_0),k_0)$ in $\mathbf{Z}^{n_0}$ and $((B,b_0),k_1)$ in $\mathbf{Z}^{n_1}$, let $p : (E,e_0) \to (B,b_0)$ be a pointed (k_0,k_1)-covering map. Any k_1-path $f : [0,m]_{\mathbf{Z}} \to B$ beginning at b_0 has a unique digital lifting to a k_0-path $\tilde{f}$ in E beginning at e_0.*

Definition 12. [8, 9, 10, 11, 16, 17, 18] *A (k_0,k_1)-covering (E,p,B) is called a radius n-(k_0,k_1)-covering if $\varepsilon \geq n$ in Definition 11.*

The *digital homotopy lifting theorem* was originally introduced in [8]. Let us now restate it in terms of the (k_0,k_1)-covering in Definition 11.

Lemma 4.4. [8] *(see also* [11, 17, 18]*) Let* $((E, e_0), k_0)$ *and* $((B, b_0), k_1)$ *be pointed digital images. Let* $p : (E, e_0) \to (B, b_0)$ *be a radius* 2-(k_0, k_1)*-covering map. For* k_0*-paths* $g_0 : [0, m_0]_{\mathbf{Z}} \to (E, e_0)$ *and* $g_1 : [0, m_1]_{\mathbf{Z}} \to (E, e_0)$ *that begin at* $e_0 = g_0(0) = g_1(0)$ *if* $p \circ g_0$ *and* $p \circ g_1$ *are* k_1*-homotopic relative to* $\{p(e_0), p(g_0(m_0)) = p(g_1(m_1))\}$ *in* B*, then* g_0 *and* g_1 *are* k_0*-homotopic relative to* $\{e_0, g_0(m_0) = g_1(m_1)\}$ *in* E*.*

The notion of *simply* k*-connected* is used in calculating the digital fundamental group of some digital image, classifying digital images, and studying a *discrete Deck's transformation group* [11] and a *universal* (k_0, k_1)*-covering* [11].

Definition 13. [15] *A pointed* k*-connected digital image* (X, x_0) *is called simply* k*-connected if* $\pi^k(X, x_0)$ *is a trivial group.*

By the use of Lemmas 4.3, 4.4 and the simply 2-connectedness of $(\mathbf{Z}, 2)$, we obtain the following.

Theorem 4.5. [10, 11, 17, 18] *For* $SC_k^{n,l}$ *not* k*-contractible,* $\pi^k(SC_k^{n,l})$ *is isomorphic to an infinite cyclic group, briefly* $\pi^k(SC_k^{n,l}) \simeq (l\mathbf{Z}, +)$.

5. Digital Surface Structure and Homotopical Properties of Cartesian Product Constructed by the Minimal Closed k-curves

In algebraic topology, it is well known that the Cartesian product $S \times S \subset \mathbf{R}^4$ is homeomorphic to a torus in $\mathbf{R}^3$, where S is a circle in $\mathbf{R}^2$. Motivated by this fact, the paper [13] studied the digital k-homotopy equivalent property of the Cartesian products of the following minimal simple closed k-curves in $\mathbf{Z}^2$, $k \in \{4, 8\}$; MSC_4, MSC_8, and MSC'_8. The author of [13] attempted to recognize the following Cartesian products $MSC_4 \times MSC_4$, $MSC'_8 \times MSC_8$, and $MSC_4 \times MSC'_8$ as the corresponding digital k-surfaces in $\mathbf{Z}^3$ (see pp.243 of [13]), which made the proofs

of Theorems 5.2, 5.3, and 5.4 misleading. Owing to non-existence of the corresponding above, this kind of approach cannot be not available in digital topology and discrete geometry (see the corrected one in [18]). The paper [13] has the following results in Section 5 of [13].
(1) MSC_4, MSC_8, and MSC'_8 are distinct up to k-homotopy equivalence, $k \in \{4, 8\}$, except that $MSC_4 \simeq_{8\cdot h\cdot e} MSC'_8$ (Theorem 5.2 of [13]).
(2) $\pi^k(MSC'_8 \times MSC'_8)$ is trivial, $k \in \{6, 18, 26\}$ (Theorem 5.3 of [13]).
(3) $MSC_4 \times MSC_4$, $MSC_4 \times MSC_8$, and $MSC_8 \times MSC'_8$ are different from each other up to digital k-homotopy equivalence, $k \in \{6, 18, 26\}$ (Theorem 5.4 of [13]).
In view of the results in Sections 3 and 4, and the notion of a digital (k_0, k_1)-homotopy equivalence, the above assertion should be represented:

Theorem 5.1. *(1) MSC_4 is not 4-homotopy equivalent to both MSC_8 and MSC'_8.*

(2) MSC_4 is 8-homotopy equivalent to MSC'_8.

(3) MSC_8 is not 8-homotopy equivalent to MSC'_8.

Proof: (1) While MSC_4 is 4-connected, both MSC_8 and MSC'_8 are not 4-connected but 8-connected. Thus, the proof is completed.

(2) The proof is obviously completed.

(3) While $\pi^8(MSC_8)$ is isomorphic to $(6\mathbf{Z}, +)$ [11, 13, 15], $\pi^8(MSC'_8)$ is trivial [3, 14, 15]. Thus, MSC_8 cannot be 8-homotopy equivalent to MSC'_8. To be specific, there is no $(8, 8)$-continuous map $l : MSC'_8 \to MSC_8$ establishing the 8-homotopy equivalence between MSC'_8 and MSC_8. □

Theorem 5.5 of [13] should be represented with $k \in \{8, 32, 64, 80\}$ instead of $k \in \{6, 18, 26\}$:

Theorem 5.2. $\pi^k(MSC'_8 \times MSC'_8)$ *is trivial,* $k \in \{8, 32, 64, 80\}$.

Proof: In the case of $k = 8$, since every point in $MSC'_8 \times MSC'_8$ is distinct up to 8-connectedness, each 8-loop in $MSC'_8 \times MSC'_8$ is trivial. Thus, the proof is completed.

In the case of $k \in \{32, 64, 80\}$, any k-loops in $MSC'_8 \times MSC'_8$ are k-nullhomotopic in the set. Thus, the proof is completed, as required. □

Theorem 5.3. [18] *The Cartesian product* $SC^{n_1,l_1}_{k_1} \times SC^{n_2,l_2}_{k_2} := (c_i)_{i\in[0,l_1-1]_{\mathbf{Z}}} \times (d_j)_{j\in[0,l_2-1]_{\mathbf{Z}}}$ *is a simple closed k-surface in* $\mathbf{Z}^{n_1+n_2}$ *if* $m_1 = m_2$, *where the k-adjacency of* $\mathbf{Z}^{n_1+n_2}$ *is determined by the number* m_2 *and* $(CON\star)$, *and* m_i *is also determined by the* k_i*-adjacency of* $\mathbf{Z}^{n_i}$ *via* $(CON\star)$, $i \in \{1, 2\}$.

Each of the Cartesian products established from MSC_4, MSC_8, and MSC'_8 has the following properties in relation to both a digital k-surface structure and a classification of the Cartesian products up to digital k-homotopy equivalence. By theorem 5.3, we obtain the following.

Theorem 5.4. *(1)*$MSC_4 \times MSC_4 \subset \mathbf{Z}^4$ *is a simple closed 8-surface* [18].

(2) $MSC'_8 \times MSC'_8$ *and* $MSC_8 \times MSC'_8$ *are simple closed k-surfaces in* $\mathbf{Z}^4$, $k \in \{32, 64, 80\}$.

(3) None of $MSC_4 \times MSC'_8$ *and* $MSC_4 \times MSC_8$ *can be a closed k-surface,* $k \in \{8, 32, 64, 80\}$

(4) $MSC_4 \times MSC_4$, $MSC_4 \times MSC_8$, *and* $MSC_8 \times MSC'_8$ *are different from each other up to digital k-homotopy equivalence,* $k \in \{8, 32, 64, 80\}$.

Proof: (1) For $SC^{n_i,l_i}_{k_i}$, $n_i \geq 2$, and $i \in \{1, 2\}$, $SC^{n_1,l_1}_{k_1} \times SC^{n_2,l_2}_{k_2}$ is a simple closed k-surface in $\mathbf{Z}^{n_1+n_2}$, where $(k_1, k_2, k) \in \{(2n_1, 2n_2, 2n_1 + 2n_2) | n_i \geq 2, i \in \{1, 2\}\}$ [18]. Thus, $MSC_4 \times MSC_4$ is a simple closed 8-surface.

(2) Let us verify that both $MSC'_8 \times MSC'_8$ and $MSC_8 \times MSC'_8$ are simple closed k-surfaces, $k \in \{32, 64, 80\}$.

(a) We now verify the closed 32-surface structure of $MSC'_8 \times MSC'_8 := T$. For each point $p \in T$, $|T|^p$ has exactly one 32-component 32-adjacent to p;
$|\overline{T}|^p$ has exactly two 8-components 8-adjacent to p; we denote by C^{pp} and D^{pp} these two components; and
for any point $y \in N_k(p) \cap T$, $N_8(y) \cap C^{pp} \neq \phi$ and $N_8(y) \cap D^{pp} \neq \phi$.
Furthermore, since T does not have a simple 32-point, $MSC'_8 \times MSC'_8 := T$ is a simple closed 32-surface.

(b) We now investigate the closed 64-surface structure of $MSC'_8 \times MSC'_8$.
First, $MSC_8 \times MSC'_8$ is 64-connected.
Second, for any point $p \in MSC'_8 \times MSC'_8$, $|MSC'_8 \times MSC'_8|^p$ is a simple closed 64-curve, which means that $MSC'_8 \times MSC'_8$ is a simple closed 64-surface by Remark 4.1.

(c) By the same method as the proof of the 32-surface structure of $MSC'_8 \times MSC'_8$, the proof of the 80-surface structure of $MSC'_8 \times MSC'_8$ is completed, as required.

(3) The following Cartesian products $MSC_4 \times MSC'_8$ and $MSC_4 \times MSC_8$ cannot be a closed k-surface in $\mathbf{Z}^4, k \in \{8, 32, 64, 80\}$ [18]. Precisely, we obtain the following.

First, $MSC_4 \times MSC'_8 \subset \mathbf{Z}^4$ is proved not to be a closed k-surface, $k \in \{8, 32, 64, 80\}$ by Theorem 5.3.

Second, by the same method as above, we can prove that $MSC_4 \times MSC_8$ cannot be a closed k-surface in $\mathbf{Z}^4, k \in \{8, 32, 64, 80\}$.

(4) By Theorem 5.1 and some Cartesian product property of a digital k-homotopy, the proof is completed, as required. □

6. Further Remarks

The papers [13, 16] have originally developed the notions of digital k-homotopy equivalence, digital graph (k_0, k_1)-homotopy equivalence and

(k_0, k_1)-isomorphism. Furthermore, the notion of (k_0, k_1)-isomorphism in [16] can be often used in studying a digital image because a digital image (X, k) in $\mathbf{Z}^n$ can be recognized as a digital k-graph on $\mathbf{Z}^n$ [9, 11, 12] and further, (X, k) can be considered to be a simplical complex in terms of a geometric realization in [9, 16]. Besides, we can study the Euler characteristic of a digital image by the use of the geometric realization in [9]. We can classify digital images up to digital k-homotopy equivalence, digital (k_0, k_1)-homotopy equivalence, and k-isomorphism. Recently, the paper [4] referred to the notion of a digital k-homotopy equivalence (although, it should be noted those earlier papers such as [13, 16] have studied the digital k-homotopy equivalence and the digital graph (k_0, k_1)-homotopy equivalence).

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Department of Computer and Applied Mathematics
Honam University,
Gwangju, 506-714, Korea
Tel: +82-62-940-5421, Fax:+82-62-940-5644
E-mail: sehan@honam.ac.kr