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ON $g.\gamma$-CLOSED SETS AND γ-T_* SPACES

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Abstract. In this paper, we introduce the notion of $g.\gamma$-closed sets and study its basic properties. Also we introduce the notion of γ-T_* spaces and investigate relationships among these spaces and γ-T_i spaces $(i = 0, 1/2, 1)$ due to Ogata [5].

1. Introduction

In 1979, Kasahara [3] defined the notion of an operation on topological space and introduced the notion of an operation-closed graph of a function. By its theory the several topological results are unified. Janković [2] continued the investigation of the topological properties with help of a certain operation. Ogata and Fukutake [5, 6, 7] introduced the notion of γ-open sets of a topological space and investigated the related topological properties of the associated operation-topology and the original topology. Furthermore, Ogata [5] introduced the notion of γ-T_i spaces which generalize one of T_i spaces $(i = 0, 1/2, 1, 2)$ (see, [4] and [1]) and studied some topological properties on them.

In section 2, we introduce the notion of $g.\gamma$-closed sets and study the properties of $g.\gamma$-closed sets relative to union, intersection and subspaces and improve the results of Ogata [5] for γ-g.closed sets. The notion of $g.\gamma$-open sets are also introduced in section 3. By using this notion, in section 4, we introduce the notion of γ-T_* spaces and investigate

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relationships among these spaces and γ-T_i spaces ($i = 0, 1/2, 1$). Finally, we show that every γ-symmetric γ-T_0 space is γ-T_1.

Let (X, τ) be a topological space (simply, space) and A be a subset of (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An operation γ on the topology τ is a mapping from τ into the power set $P(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denoted the value of γ at V. Throughout the present paper, the operation will be denoted by $\gamma : \tau \to P(X)$ (or, simply γ).

Definition 1.1. A subset A of (X, τ) is called *γ-open* [5] if for each $x \in A$, there exists an open set U containing x such that $U^\gamma \subset A$. τ_γ denotes the set of all γ-open sets in (X, τ).

Definition 1.2. Let (X, τ) be a space. An operation γ is said to be

(a) *regular* [6] if for each open neighborhood U and V of each $x \in X$, there exists an open neighborhood W of x such that $W^\gamma \subset U^\gamma \cap V^\gamma$;

(b) *open* [5] if for each open neighborhood U of each $x \in X$, there exists a γ-open set V such that $x \in V$ and $V \subset U^\gamma$.

Proposition 1.1. [5] *Let $\gamma : \tau \to P(X)$ be a regular operation on τ.*

(a) *If A and B are γ-open, then $A \cap B$ is γ-open.*

(b) *τ_γ is a topology on X such that $\tau_\gamma \subset \tau$.*

Definition 1.3. A point x of X is in the *γ-closure* [5] of $A \subset X$, denoted by $\mathrm{Cl}_\gamma(A)$, if $U^\gamma \cap A \neq \emptyset$ for any open neighborhood U of x. A subset A of X is said to be *γ-closed* (in the sense of Janković [2]) if $\mathrm{Cl}_\gamma(A) = A$. For the family τ_γ, γ-closure in the sense of Ogata [5] of A defined as follows:

$$\tau_\gamma\text{-}\mathrm{Cl}(A) = \cap\{F : F \supset A,\ X \setminus F \in \tau_\gamma\}.$$

A point x of X is in the *γ-interior* [7] of A, denoted by $\mathrm{Int}_\gamma(A)$, if $U^\gamma \subset A$ for some open neighborhood U of x.

Proposition 1.2. *Let $\gamma : \tau \to P(X)$ be an operation on τ and A, B be subset of X.*

(a) $A \subset \mathrm{Cl}(A) \subset \mathrm{Cl}_\gamma(A) \subset \tau_\gamma\text{-}\mathrm{Cl}(A)$.

(b) *If* $A \subset B$, *then* $\mathrm{Cl}_\gamma(A) \subset \mathrm{Cl}_\gamma(B)$.
(c) $\mathrm{Cl}_\gamma(A \cup B) = \mathrm{Cl}_\gamma(A) \cup \mathrm{Cl}_\gamma(B)$.
(d) *If* γ *is open operation,* $\mathrm{Cl}_\gamma(\mathrm{Cl}_\gamma(A)) = \mathrm{Cl}_\gamma(A)$.
(e) $X \setminus \mathrm{Int}_\gamma(A) = \mathrm{Cl}_\gamma(X \setminus A)$.

2. $g.\gamma$-closed sets

To introduce the notion of $g.\gamma$-closed sets and investigate the relationship between $g.\gamma$-closed sets and γ-g.closed sets, we first recall that a subset A of (X, τ) is called g-closed [4] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is open in X.

Definition 2.1. A subset A of (X, τ) is said to be
(a) *γ-g.closed* [5] if $\mathrm{Cl}_\gamma(A) \subset U$ whenever $A \subset U$ and U is γ-open in (X, τ);
(b) *$g.\gamma$-closed* if $\mathrm{Cl}_\gamma(A) \subset U$ whenever $A \subset U$ and U is open in (X, τ).

Remark 2.1. From above definition and Definition 2.1 of [4], we obtain the following diagram:

γ-closed	$\rightarrow$	$g.\gamma$-closed	$\rightarrow$	γ-g.closed
$\downarrow$		$\downarrow$		
closed	$\rightarrow$	g-closed		

Example 2.1. (a) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Let $\gamma : \tau \to P(X)$ be an operation defined by $\{a, b\}^\gamma = \{a, b\}$ and $A^\gamma = \mathrm{Cl}(A)$ if $A (\neq \{a, b\}) \in \tau$. Then $\{a\}$ is γ-g.closed in (X, τ) but not g-closed. Also $\{b, c\}$ is $g.\gamma$-closed and closed in (X, τ) but not γ-closed.

(b) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$. Let $\gamma : \tau \to P(X)$ be an operation defined by $\{a\}^\gamma = \{a, c\}$ and $A^\gamma = \mathrm{Cl}(A)$ if $A (\neq \{a\}) \in \tau$. Then $\{b, c\}$ is closed in (X, τ) but not γ-g.closed. Also, $\{a, b\}$ is $g.\gamma$-closed in (X, τ) but not closed.

Ogata [5] proved that if for subset A of (X, τ), τ_γ-$\mathrm{Cl}(\{x\}) \cap A \neq \emptyset$ for each $x \in \mathrm{Cl}_\gamma(A)$, then A is γ-g.closed. To improve this result, we need the following lemma.

Lemma 2.1. (a) *If A is any subset and B is γ-open set in (X, τ) with $A \cap B = \emptyset$, then τ_γ-$\mathrm{Cl}(A) \cap B = \emptyset$.*

(b) *For any subset A of (X, τ), τ_γ-$\mathrm{Cl}(A)$ is γ-closed.*

Proof. (a): It follows from Proposition 3.3 of [5].

(b): Let $x \notin \tau_\gamma$-$\mathrm{Cl}(A)$. By Proposition 3.3 of [5], there exists a γ-open set U containing x such that $U \cap A = \emptyset$ and thus by (a), $U \cap \tau_\gamma$-$\mathrm{Cl}(A) = \emptyset$. Hence $X \setminus \tau_\gamma$-$\mathrm{Cl}(A)$ is γ-open. By Theorem 3.7 of [5], τ_γ-$\mathrm{Cl}(A)$ is γ-closed. □

Proposition 2.1. *Let A be a subset of (X, τ) and $\gamma : \tau \to P(X)$ be any operation. Then A is γ-g.closed in (X, τ) if and only if τ_γ-$\mathrm{Cl}(\{x\}) \cap A \neq \emptyset$ for every $x \in \mathrm{Cl}_\gamma(A)$.*

Proof. By Lemma 2.1 (b), τ_γ-$\mathrm{Cl}(\{x\})$ is γ-closed, the proof is same manner as Proposition 4.6 of [5]. □

Theorem 2.1. *Let A be a subset of (X, τ) and $\gamma : \tau \to P(X)$ be any operation. Then*

(a) *If A is γ-g.closed in (X, τ), then $\mathrm{Cl}_\gamma(A) \setminus A$ contains no nonempty γ-closed set. The converse is true if γ is an open operation.*

(b) *A subset A is $g.\gamma$-closed if and only if $\mathrm{Cl}_\gamma(A) \setminus A$ contains no nonempty closed set.*

Proof. (a) Let F be a γ-closed subset of $\mathrm{Cl}_\gamma(A) \setminus A$. Now, $A \subset X \setminus F$ and since A is γ-g.closed, we have $\mathrm{Cl}_\gamma(A) \subset X \setminus F$ or $F \subset X \setminus \mathrm{Cl}_\gamma(A)$. Thus $F \subset \mathrm{Cl}_\gamma(A) \cap X \setminus \mathrm{Cl}_\gamma(A) = \emptyset$ and F is empty. Suppose that $A \subset U$ and U is γ-open. If $\mathrm{Cl}_\gamma(A) \not\subset U$, then by Theorem 3.6 (iii) of [5], $\mathrm{Cl}_\gamma(A) \cap X \setminus U$ is a nonempty γ-closed subset of $\mathrm{Cl}_\gamma(A) \setminus A$, a contradiction. Hence A is γ-g.closed.

(b) The proof is similar to (a) by using Theorem 3.6 (i) of [5]. □

Recall that (X, τ) is called *γ-regular space* [6] if for each $x \in X$ and each open neighborhood U of x there exists an open neighborhood V of x such that $V^\gamma \subset U$.

Corollary 2.1. (a) *Let $\gamma : \tau \to P(X)$ be an open operation. Then γ-g.closed set A is γ-closed if and only if $\mathrm{Cl}_\gamma(A) \setminus A$ is γ-closed.*

(b) *Let* (X, τ) *be a* γ*-regular space. Then* $g.\gamma$*-closed set* A *is closed if and only if* $\mathrm{Cl}_\gamma(A) \setminus A$ *is closed.*

Proof. (a) If γ-g.closed set A is γ-closed, $\mathrm{Cl}_\gamma(A) \setminus A = \emptyset$. Conversely, suppose that $\mathrm{Cl}_\gamma(A) \setminus A$ is γ-closed. But A is a γ-closed set and $\mathrm{Cl}_\gamma(A) \setminus A$ is a γ-closed subset of itself. By Theorem 2.1 (a), $\mathrm{Cl}_\gamma(A) \setminus A = \emptyset$ and hence $\mathrm{Cl}_\gamma(A) = A$.

(b) If $g.\gamma$-closed set A is closed, by Theorem 3.6 (ii) of [5], $\mathrm{Cl}(A) \setminus A = \mathrm{Cl}_\gamma(A) \setminus A = \emptyset$. Conversely, suppose that $\mathrm{Cl}_\gamma(A) \setminus A$ is closed. But A is a $g.\gamma$-closed set and $\mathrm{Cl}_\gamma(A) \setminus A$ is a closed subset of itself. By Theorem 2.1 (b), $\mathrm{Cl}_\gamma(A) \setminus A = \emptyset$ and hence by Theorem 3.6 (ii) of [5], $\mathrm{Cl}(A) = A$. □

Theorem 2.2. (a) *If* A *and* B *are* γ-g*.closed, then* $A \cup B$ *is* γ-g*.closed.*
(b) *If* A *and* B *are* $g.\gamma$*-closed, then* $A \cup B$ *is* $g.\gamma$*-closed.*

Proof. (a) If $A \cup B \subset U$ and if U is γ-open, then $\mathrm{Cl}_\gamma(A \cup B) = \mathrm{Cl}_\gamma(A) \cup \mathrm{Cl}_\gamma(B) \subset U$.

(b) The proof is similar to (a). □

The intersection of two $g.\gamma$-closed (resp. γ-g.closed) sets is generally not $g.\gamma$-closed (resp. γ-g.closed).

Example 2.2. (a) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}$. Let $\tau \to P(X)$ be an operation defined by $A^\gamma = A$ if $A = \emptyset$ or $\{a\}$, $A^\gamma = X$ if $A \neq \{a\}$ or $\emptyset$. Then $\tau_\gamma = \{X, \emptyset, \{a\}\}$. If $A = \{a, b\}$ and $B = \{a, c\}$, then A and B are γ-g.closed but not $A \cap B$ is not γ-g.closed.

(b) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Let $\gamma : \tau \to P(X)$ be an operation defined by $X^\gamma = X$, $\emptyset^\gamma = \emptyset$ and $\{a\}^\gamma = \{a, b\}$. If $A = \{a, b\}$ and $B = \{a, c\}$, then A and B are $g.\gamma$-closed but $A \cap B$ is not $g.\gamma$-closed.

Now we define an operation on subspace topology as follows:

Definition 2.2. Let (A, τ_A) be a subspace of a space (X, τ) and $\gamma : \tau \to P(X)$ be an operation on X. We define the restriction of γ to τ_A, denoted by γ_A, to be the mapping τ_A into $P(A)$ such that for each $U \in \tau_A$, $U^{\gamma_A} = V^\gamma \cap A$ for some $V \in \tau$ with $U = V \cap A$.

Proposition 2.2. *Let (A, τ_A) be a subspace of (X, τ) and γ_A be the restriction of γ to τ_A.*

(a) *If B is γ-open in X, then $B \cap A$ is γ_A-open in A.*

(b) *If γ is open and B is γ_A-open in A, then there exists a γ-open set C such that $B = C \cap A$.*

Proof. (a) Let $x \in B \cap A$. Since B is γ-open in X, there exists a $U \in \tau$ such that $x \in U$ and $U^\gamma \subset B$. Then $U \cap A$ is τ_A-open set such that $x \in U \cap A$ and $(U \cap A)^{\gamma_A} = U^\gamma \cap A \subset B \cap A$. Hence $B \cap A$ is γ_A-open in A.

(b) Since B is γ_A-open in A, for each $x \in B$, there exists a $U_x \in \tau$ such that $x \in U_x$ and $(U_x \cap A)^{\gamma_A} = U_x^\gamma \cap A \subset B$. Since γ is open, there exists a γ-open set V_x such that $x \in V_x$ and $V_x \subset U_x^\gamma$. Put $C = \cup_{x \in B} V_x$. Then C is γ-open in X and $B \subset C \cap A = (\cup_{x\in B} V_x) \cap A \subset (\cup_{x \in B} U_x^\gamma) \cap A = \cup_{x \in B}(U_x \cap A)^{\gamma_A} \subset B$. □

Proposition 2.3. *Let (A, τ_A) be a subspace of (X, τ) and $B \subset A \subset X$. If γ_A is the restriction of γ to τ_A, then $\mathrm{Cl}_{\gamma_A}(B) = \mathrm{Cl}_\gamma(B) \cap A$.*

Proof. Let $x \in \mathrm{Cl}_{\gamma_A}(B)$ and U be a τ-open neighborhood of x. Then $(U \cap A)^{\gamma_A} \cap B = U^\gamma \cap B \neq \emptyset$ and hence $x \in \mathrm{Cl}_\gamma(B) \cap A$. On the other hand, let $x \in \mathrm{Cl}_\gamma(B) \cap A$ and V be τ_A-open neighborhood of x. Then $V = U \cap A$ for some an τ-open neighborhood U of x. Since $x \in \mathrm{Cl}_\gamma(B)$, $(V)^{\gamma_A} \cap B = (U^\gamma \cap A) \cap B = U^\gamma \cap B \neq \emptyset$. Hence $x \in \mathrm{Cl}_{\gamma_A}(B)$. □

Theorem 2.3. *Let (A, τ_A) be a subspace of (X, τ) and γ_A be the restriction of γ to τ_A.*

(a) *If B is γ_A-g.closed in A, A is γ-g.closed in X and γ is open operation, then B is γ-g.closed in X.*

(b) *If B is g.γ_A-closed in A and A is g.γ-closed in X, then B is g.γ-closed in X.*

Proof. (a) Let U be γ-open in X and $B \subset U$. Then by Proposition 2.2 (a), $U \cap A$ is γ_A-open in A and $B \subset U \cap A$. By hypothesis, $\mathrm{Cl}_{\gamma_A}(B) = \mathrm{Cl}_\gamma(B) \cap A \subset U \cap A$ and then $A \subset U \cup (X \setminus \mathrm{Cl}_\gamma(B))$. Since A is γ-g.closed and γ is open, $\mathrm{Cl}_\gamma(A) \subset U \cup (X \setminus \mathrm{Cl}_\gamma(B))$. Hence $\mathrm{Cl}_\gamma(B) \subset \mathrm{Cl}_\gamma(A) \subset U \cup (X \setminus \mathrm{Cl}_\gamma(B))$ and thus $\mathrm{Cl}_\gamma(B) \subset U$.

(b) The proof is similar to (a). □

Corollary 2.2. (a) *If γ is open, A is γ-g.closed in X and F is γ-closed in X, then $A \cap F$ is γ-g.closed in X.*

(b) *If A is $g.\gamma$-closed in X and F is γ-closed in X, then $A \cap F$ is $g.\gamma$-closed in X.*

Proof. (a) By Proposition 2.2 (a), $A \cap F$ is γ_A-closed in A and hence $A \cap F$ is γ_A-g.closed in A. By Theorem 2.3 (a), $A \cap F$ is γ-g.closed in X.

(b) Using Proposition 2.2 (a) and Theorem 2.3 (b), the proof is similar to (a). □

Theorem 2.4. (a) *If γ is open, A is γ-g.closed in X and $A \subset B \subset \mathrm{Cl}_\gamma(A)$, then B is γ-g.closed.*

(b) *If A is $g.\gamma$-closed in X and $A \subset B \subset \mathrm{Cl}_\gamma(A)$, then B is $g.\gamma$-closed.*

Proof. (a) $\mathrm{Cl}_\gamma(B) \setminus B \subset \mathrm{Cl}_\gamma(A) \setminus A$ and since $\mathrm{Cl}_\gamma(A) \setminus A$ has no non empty γ-closed sets, neither does $\mathrm{Cl}_\gamma(B) \setminus B$. By Theorem 2.1 (a), B is γ-g.closed.

(b) The proof is similar to (a) by using Theorem 2.1 (b). □

Theorem 2.5. *Let $B \subset A \subset X$, (A, τ_A) be a subspace of (X, τ) and γ_A be the restriction of γ to τ_A.*

(a) *If γ is open and B is γ-g.closed in X, then B is γ_A-g.closed in A.*

(b) *If B is $g.\gamma$-closed in X, then B is $g.\gamma_A$-closed in A.*

Proof. We prove only (a). Let U be γ_A-open in A and $B \subset U$. There exists a γ-open set V of X such that $U = A \cap V$ and thus $B \subset V$. Hence $\mathrm{Cl}_\gamma(B) \subset V$. By Proposition 2.3, $\mathrm{Cl}_{\gamma_A}(B) = \mathrm{Cl}_\gamma(B) \cap A \subset V \cap A = U$ and thus B is γ_A-g.closed in A. □

3. γ-g.open sets and $g.\gamma$-open sets

Definition 3.1. A subset A of (X, τ) is called *γ-g.open* (resp. *$g.\gamma$-open*) in X if the complement $X \setminus A$ is γ-g.closed (resp. $g.\gamma$-closed).

Theorem 3.1. (a) *A subset A is γ-g.open if and only if $F \subset \mathrm{Int}_\gamma(A)$ whenever F is γ-closed and $F \subset A$.*

(b) *A subset A is $g.\gamma$-open if and only if $F \subset \mathrm{Int}_\gamma(A)$ whenever F is closed and $F \subset A$.*

Proof. Straightforward. □

Theorem 3.2. (a) *If γ is open, A and B are γ-separated (i.e., $\mathrm{Cl}_\gamma(A) \cap B = \emptyset = A \cap \mathrm{Cl}_\gamma(B)$) γ-g.open sets, then $A \cup B$ is γ-g.open.*

(b) *If A and B are γ-separated $g.\gamma$-open sets, then $A \cup B$ is $g.\gamma$-open.*

Proof. We prove only (a). Let F be a γ-closed set and $F \subset A \cup B$. Since γ is open, $F \cap \mathrm{Cl}_\gamma(A)$ is γ-closed and $F \cap \mathrm{Cl}_\gamma(A) \subset A$, and hence by Theorem 3.1 (a), $F \cap \mathrm{Cl}_\gamma(A) \subset \mathrm{Int}_\gamma(A)$. Similarly, $F \cap \mathrm{Cl}_\gamma(B) \subset \mathrm{Int}_\gamma(B)$. Now we have $F = F \cap (A \cup B) \subset (F \cap \mathrm{Cl}_\gamma(A)) \cup (F \cap \mathrm{Cl}_\gamma(B)) \subset \mathrm{Int}_\gamma(A) \cup \mathrm{Int}_\gamma(B) \subset \mathrm{Int}_\gamma(A \cup B)$. Hence $F \subset \mathrm{Int}_\gamma(A \cup B)$ and thus $A \cup B$ is γ-g.open. □

Remark 3.1. The union of two γ-g.open (resp. $g.\gamma$-open) sets is generally not γ-g.open (resp. $g.\gamma$-open) (see Example 2.2).

Theorem 3.3. (a) *If a subset A is γ-g.open in X, then $U = X$ whenever U is γ-open and $\mathrm{Int}_\gamma(A) \cup (X \setminus A) \subset U$. The converse is true if γ is open.*

(b) *A subset A is $g.\gamma$-open in X if and only if $U = X$ whenever U is open and $\mathrm{Int}_\gamma(A) \cup (X \setminus A) \subset U$.*

Proof. (a) Let U be γ-open in X and $\mathrm{Int}_\gamma(A) \cup (X \setminus A) \subset U$. Then $X \setminus U \subset \mathrm{Cl}_\gamma(X \setminus A) \cap A = \mathrm{Cl}_\gamma(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus A$ is γ-g.closed and $X \setminus U$ is γ-closed, by Theorem 2.1 (a), $X \setminus U = \emptyset$ and hence $X = U$. Conversely, let F be γ-closed in X and $F \subset A$. By Theorem 3.1 (a), it suffices to show that $F \subset \mathrm{Int}_\gamma(A)$. Since $\mathrm{Int}_\gamma(A) \cup (X \setminus F)$ is γ-open and $\mathrm{Int}_\gamma(A) \cup (X \setminus A) \subset \mathrm{Int}_\gamma(A) \cup (X \setminus F)$, $\mathrm{Int}_\gamma(A) \cup (X \setminus F) = X$. Hence $F \subset \mathrm{Int}_\gamma(A)$.

(b) Using Theorems 2.1 (b) and 3.1 (b), the proof is similar to (a). □

Theorem 3.4. (a) *If γ is open, $\mathrm{Int}_\gamma(A) \subset B \subset A$ and A is γ-g.open, then B is γ-g.open.*

(b) *If* $\mathrm{Int}_\gamma(A) \subset B \subset A$ *and* A *is* $g.\gamma$*-open, then* B *is* $g.\gamma$*-open.*

Proof. (a) By Proposition 1.2 (e), $X \setminus A \subset X \setminus B \subset \mathrm{Cl}_\gamma(X \setminus A)$. Since $X \setminus A$ is γ-g.closed, by Theorem 2.4 (a), $X \setminus B$ is γ-g.closed and hence B is γ-g.open.

(2) Using Theorem 2.4 (b), the proof is similar to (a). □

Theorem 3.5. (a) *If* A *is* γ-g*.closed, then* $\mathrm{Cl}_\gamma(A) \setminus A$ *is* γ-g*.open. The converse is true if* γ *is open.*

(b) A *is* $g.\gamma$*-closed if and only if* $\mathrm{Cl}_\gamma(A) \setminus A$ *is* $g.\gamma$*-open.*

Proof. (a) Let A be γ-g.closed and $F \subset \mathrm{Cl}_\gamma(A) \setminus A$, where F is γ-closed. By Theorem 2.1 (a), $F = \emptyset$ and hence $F \subset \mathrm{Int}_\gamma(\mathrm{Cl}_\gamma(A) \setminus A)$. By Theorem 3.1 (a), $\mathrm{Cl}_\gamma(A) \setminus A$ is γ-g.open. Conversely, let U be γ-open and $A \subset U$. Now $\mathrm{Cl}_\gamma(A) \cap (X \setminus U) \subset \mathrm{Cl}_\gamma(A) \setminus A$ and since $\mathrm{Cl}_\gamma(A) \cap (X \setminus U)$ is γ-closed and $\mathrm{Cl}_\gamma(A) \setminus A$ is γ-g.open, it follows that $\mathrm{Cl}_\gamma(A) \cap (X \setminus U) \subset \mathrm{Int}_\gamma(\mathrm{Cl}_\gamma(A) \setminus A) = \emptyset$. Then $\mathrm{Cl}_\gamma(A) \subset U$ and hence A is γ-g.closed.

(b) Using Theorem 2.1 (b), the proof is similar to (a). □

4. γ-T_* spaces

In this section we define γ-T_* spaces and study investigate relationships among these spaces and γ-T_i spaces ($i = 0, 1/2, 1$).

Definition 4.1. [5] A space (X, τ) is called

(a) γ-T_0 if for each distinct points $x, y \in X$, there exists an open set U such that either $x \in U$ and $y \notin U^\gamma$, or $y \in U$ and $x \notin U^\gamma$;

(b) γ-T_1 if for each distinct points $x, y \in X$, there exist open sets U and V containing x and y, respectively, such that $y \notin U^\gamma$ and $x \notin V^\gamma$;

(c) γ-$T_{1/2}$ if every γ-g.closed set of (X, τ) is γ-closed.

Definition 4.2. A space (X, τ) is called γ-T_* if every $g.\gamma$-closed set of (X, τ) is γ-closed.

Theorem 4.1. *A space* (X, τ) *is* γ-T_* *if and only if for each* $x \in X$, $\{x\}$ *is closed or* γ*-open in* (X, τ).

Proof. Let $X \in X$. If $\{x\}$ is not closed, then $X \setminus \{x\}$ is not open and thus $g.\gamma$-closed. By hypothesis, $X \setminus \{x\}$ is γ-closed, i.e. $\{x\}$ is γ-open.

Conversely, let F be $g.\gamma$-closed in (X, τ). Let x be any point of $\mathrm{Cl}_\gamma(F)$. We consider the following two cases:

Case 1. Let $\{x\}$ be γ-open. Since $x \in \mathrm{Cl}_\gamma(F)$, $\{x\} \cap F = \{x\}^\gamma \cap F \neq \emptyset$ and so $x \in F$.

Case 2. Let $\{x\}$ be closed. If $x \notin F$, then $x \in \mathrm{Cl}_\gamma(F) \setminus F$. This is impossible from Theorem 2.1 (a). Hence $x \in F$.

So in both cases we have $\mathrm{Cl}_\gamma(F) \subset F$. Since the reverse inclusion is trivial, $\mathrm{Cl}_\gamma(F) = F$ or equivalently F is γ-closed. □

The following theorem is improvement of Proposition 4.10 of [5].

Theorem 4.2. *A space (X, τ) is γ-$T_{1/2}$ if and only if for each $x \in X$, $\{x\}$ is γ-closed or γ-open in (X, τ).*

Proof. The proof is similar to that of Theorem 4.1 by using Theorem 2.1 (a). □

Theorem 4.3. *Any subspace (A, τ_A) of γ-T_i space (X, τ) is also γ_A-T_i space, for $i = 0, 1/2, *, 1$.*

Proof. We prove only $i = 1/2$. Let $x \in A$. Then $\{x\}$ is either γ-open or γ-closed in (X, τ). If $\{x\}$ is γ-open, then, by Proposition 2.2 (a), $\{x\}$ is γ_A-open in (A, τ_A). If $\{x\}$ is γ-closed, then $\mathrm{Cl}_\gamma(\{x\}) = \{x\}$. By Proposition 2.3, $\mathrm{Cl}_{\gamma_A}(\{x\}) = \mathrm{Cl}_\gamma(\{x\}) \cap A = \{x\}$ and so $\{x\}$ is γ_A-closed. Hence (A, τ) is γ_A-$T_{1/2}$ space. □

From above definitions and results of [4], we obtain the following diagram:

$$
\begin{array}{ccccc}
\gamma\text{-}T_1 & \rightarrow & \gamma\text{-}T_{1/2} & \rightarrow & \gamma\text{-}T_0 \\
| & & \downarrow & & | \\
| & & \gamma\text{-}T_* & & | \\
\downarrow & \nearrow & \downarrow & & \downarrow \\
T_1 & \rightarrow & T_{1/2} & \rightarrow & T_0
\end{array}
$$

Example 4.1. (a) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Let $\gamma : \tau \to P(X)$ be an operation defined by $\emptyset^\gamma = \emptyset$ and $A^\gamma = X$ if $A \neq \emptyset (\in \tau)$. Then (X, τ) is $T_{1/2}$ space but not γ-T_*.

(b) Let (X, τ) be a space given in (a). Let $\gamma : \tau \to P(X)$ be an operation defined by $\{a\}^\gamma = \{a\}$, $\emptyset^\gamma = \emptyset$ and $A^\gamma = X$ if $(\emptyset \neq) A \neq \{a\} (\in \tau)$. Then (X, τ) is γ-T_* space. However the space (X, τ) is neither γ-$T_{1/2}$ nor T_1.

(c) Let (X, τ) be a space given in (a). Let $\gamma : \tau \to P(X)$ be an operation defined by $\{a\}^\gamma = \{a, b\}$ and $A^\gamma = \mathrm{Cl}(A)$ if $A \neq \{a\} (\in \tau)$. Then (X, τ) is γ-T_0 space but not γ-T_*.

(d) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$. Let $\gamma : \tau \to P(X)$ be an operation defined by $\{c\}^\gamma = \{c\}$, $\emptyset^\gamma = \emptyset$ and $A^\gamma = X$ if $(\emptyset \neq) A \neq \{c\} (\in \tau)$. Then (X, τ) is γ-T_* space but γ-T_0.

Throughout the rest of this section, let (X, τ) and (Y, σ) be spaces, and let $\gamma : \tau \to P(X)$ and $\beta : \sigma \to P(Y)$ be operations on τ and σ, respectively. Let id be an identity operation.

Definition 4.3. [5] A mapping $f : (X, \tau) \to (Y, \sigma)$ said to be

(a) (γ, β)-*continuous* if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U such that $x \in U$ and $f(U^\gamma) \subset V^\beta$;

(b) (γ, β)-*closed* if for any γ-closed F of (X, τ), $f(F)$ is β-closed in (Y, σ).

Proposition 4.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a (id, β)-closed mapping. Then*

(a) *If A is $g.\gamma$-closed in (X, τ) and f is continuous, then $f(A)$ is $g.\beta$-closed in (Y, σ).*

(b) *If A is γ-g.closed in (X, τ) and f is (γ, id)-continuous, then $f(A)$ is $g.\beta$-closed in (Y, σ).*

(c) *If A is $g.\gamma$-closed in (X, τ) and f is (id, β)-continuous, then $f(A)$ is β-g.closed in (Y, σ).*

Proof. (a) Let V be any open set of Y such $f(A) \subset V$. Since f is continuous and A is $g.\gamma$-closed, $\mathrm{Cl}_\gamma(A) \subset f^{-1}(V)$ and hence $f(\mathrm{Cl}_\gamma(A)) \subset V$. By Proposition 3.6 (i) of [5] and assumption, $f(\mathrm{Cl}_\gamma(A))$ is β-closed in (Y,σ). Thus we have $\mathrm{Cl}_\beta(f(A)) \subset \mathrm{Cl}_\beta(f(\mathrm{Cl}_\gamma(A))) = f(\mathrm{Cl}_\gamma(A)) \subset V$. This implies that $f(A)$ is g.β-closed in (Y,σ).

The proofs of (b) and (c) are similar to (a). □

Theorem 4.4. *Let $f:(X,\tau)\to(Y,\sigma)$ be a (id,β)-closed injective mapping. Then*

(a) *If (Y,σ) is β-T_* and f is continuous and (γ,β)-continuous, then (X,τ) is γ-T_*.*

(b) *If (Y,σ) is β-T_* and f is (γ,id)-continuous, then (X,τ) is γ-$T_{1/2}$.*

(c) *If (Y,σ) is β-$T_{1/2}$ and f is (γ,β)-continuous, then (X,τ) is γ-T_*.*

Proof. (a) Let A is be a $g.\gamma$-closed set of (X,τ). By Proposition 4.1 (a) and assumption, $f(A)$ is β-closed in (Y,σ). Since f is (γ,β)-continuous and injective, $A(=f^{-1}(f(A)))$ is γ-closed. Hence (X,τ) is γ-T_*.

(b), (c) Since every (γ,id)-continuous (resp. (γ,β)-continuous) function is (γ,β)-continuous (resp. (id,β)-continuous), the proofs are similar to (a). □

Proposition 4.2. *Let $f:(X,\tau)\to(Y,\sigma)$ be a (γ,β)-continuous mapping. Then*

(a) *If B is g.β-closed in (Y,σ) and f is closed, then $f^{-1}(B)$ is $g.\gamma$-closed in (X,τ).*

(b) *If B is β-g.closed in (Y,σ) and f is (id,β)-closed, then $f^{-1}(B)$ is $g.\gamma$-closed in (X,τ).*

Proof. (a) Let U be any open set of X such that $f^{-1}(B)\subset U$. Put $F=\mathrm{Cl}_\gamma(f^{-1}(B))\cap(X\setminus U)$. Then by Theorem 3.6 (i) of [5], F is closed in X. Since f is closed, $f(F)$ is closed in (Y,σ) and $f(F)\subset \mathrm{Cl}_\beta(B)\setminus B$ by using Proposition 4.13 of [5]. By Theorem 2.1 (b), $f(F)=\emptyset$ and so $F=\emptyset$, i.e. $\mathrm{Cl}_\gamma(f^{-1}(B))\subset U$. Hence $f^{-1}(B)$ is $g.\gamma$-closed in (X,τ).

(b) The proof is similar to (a) by using Theorem 2.1 (a). □

Theorem 4.5. *Let $f:(X,\tau)\to(Y,\sigma)$ be a (γ,β)-continuous surjective mapping. Then*

(a) *If (X,τ) is γ-T_* and f is (γ,β)-closed and closed, then (Y,σ) is β-T_*.*

(b) *If* (X,τ) *is* γ-T_* *and* f *is* (id,β)*-closed, then* (Y,σ) *is* β-$T_{1/2}$.

Proof. (a) Let B be a g.β-closed set of (Y,σ). By Proposition 4.2 (a), $f^{-1}(B)$ is $g.\gamma$-closed in (X,τ) and hence it is γ-closed because (X,τ) is γ-T_*. Since f is (γ,β)-closed, $B(=f(f^{-1}(B)))$ is β-closed. Hence (Y,σ) is β-T_*.

(b) Since every (id,β)-closed mapping is (γ,β)-closed, the proof is similar to (a). □

Definition 4.4. A space (X,τ) is called γ-*symmetric* if for x and y in X, $x\in \mathrm{Cl}_\gamma(\{y\})$ implies that $y\in \mathrm{Cl}_\gamma(\{x\})$.

Theorem 4.6. *If every singleton* $\{x\}$ *is* $g.\gamma$*-closed in* (X,τ)*, then* (X,τ) *is* γ*-symmetric. The converse is true if* γ *is open.*

Proof. Suppose that $x\in \mathrm{Cl}_\gamma(\{y\})$ but $y\notin \mathrm{Cl}_\gamma(\{x\})$. Then $\{y\}\subset X\backslash \mathrm{Cl}_\gamma(\{x\})$ and hence $\mathrm{Cl}_\gamma(\{y\})\subset X\backslash \mathrm{Cl}_\gamma(\{x\})$. Then $x\in X\backslash \mathrm{Cl}_\gamma(\{x\})$, a contradiction. Conversely, let $x\in X$ and U be open set containing x. If $\mathrm{Cl}_\gamma(\{x\})\not\subset U$, $\mathrm{Cl}_\gamma(\{x\})\cap X\setminus U\neq\emptyset$. Take $y\in \mathrm{Cl}_\gamma(\{x\})\cap X\setminus U$. Since (X,τ) is γ-symmetric and γ is open, $x\in \mathrm{Cl}_\gamma(\{y\})\subset X\setminus U$ and hence $x\notin U$, a contradiction. □

Corollary 4.1. *If a space* (X,τ) *is* γ-T_1*, then* (X,τ) *is* γ*-symmetric.*

Proof. Since singletons, in γ-T_1 space, are γ-closed and hence $g.\gamma$-closed. Thus by Theorem 4.6, the space is γ-symmetric. □

Corollary 4.2. *A space* (X,τ) *is* γ-T_1 *if and only if* (X,τ) *is* γ*-symmetric and* γ-T_0.

Proof. By Corollary 4.2, it suffices to prove only the necessity. Let x and y are distinct points in X. Since (X,τ) is γ-T_0, there exists an open set U such that $x\in U$ and $U^\gamma\subset X\setminus\{y\}$. Then $x\notin \mathrm{Cl}_\gamma(\{y\})$ and hence $y\notin \mathrm{Cl}_\gamma(\{x\})$. Then there exists an open set V such that $y\in V$ and $V^\gamma\subset X\setminus\{x\}$. Hence (X,τ) is γ-T_1. □

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