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A MARTINGALE APPROACH TO A RUIN MODEL WITH SURPLUS FOLLOWING A COMPOUND POISSON PROCESS†

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Abstract

We consider a ruin model whose surplus process is formed by a compound Poisson process. If the level of surplus reaches $V > 0$, it is assumed that a certain amount of surplus is invested. In this paper, we apply the optional sampling theorem to the surplus process and obtain the expectation of period T, time from origin to the point where the level of surplus reaches either 0 or V. We also derive the total and average amount of surplus during T by establishing a backward differential equation.

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Keywords. Backward differential equation, compound Poisson process, optional sampling theorem, surplus process.

1. Introduction

In this paper, we consider a surplus process whose premium rate is constant $c > 0$. The claims are aggregated according to a compound Poisson process with arrival rate $\lambda > 0$. The amounts of claims are assumed to be independent and exponentially distributed with mean μ. Let $U(t)$ be the surplus at time t, then

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i,$$

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where $N(t)$ is a Poisson process with rate $\lambda > 0$, X_i is the amount of the i^{th} claim and $u = U(0)$ is the initial surplus. A sample path of $U(t)$ is shown in Figure 1.1.

Many authors have studied the surplus process with compound Poisson claims and the core results, specially the ruin probabilities, are well summarized in Klugman *et al.* (2004). Meanwhile, the first passage time to a certain level in the surplus process with compound Poisson claims was introduced by Gerber (1990). Thereafter, Gerber and Shiu (1997) obtained the joint distribution of the time to ruin, the surplus before ruin and the deficit at ruin, and Dickson and Willmot (2005) calculated the density of the time to ruin by an inversion of its Laplace transform.

Lee and Kinateder (2000) introduced a finite dam of capacity V. They applied the optional sampling theorem (Karlin and Taylor, 1975, pp. 257–262) to the level of water in the reservoir and obtained the expected first passage time to either 0 or V.

We, in this paper, extend the analysis of Lee and Kinateder (2000) to the surplus process with compound Poisson claims and obtain the total and average surplus during a random period T, time from the origin to the point where the surplus either reaches V or goes below 0. Here, V is a kind of target amount of surplus so that we can invest a certain amount of those surplus to other place.

In Section 2, we define two martingales by transforming the surplus process

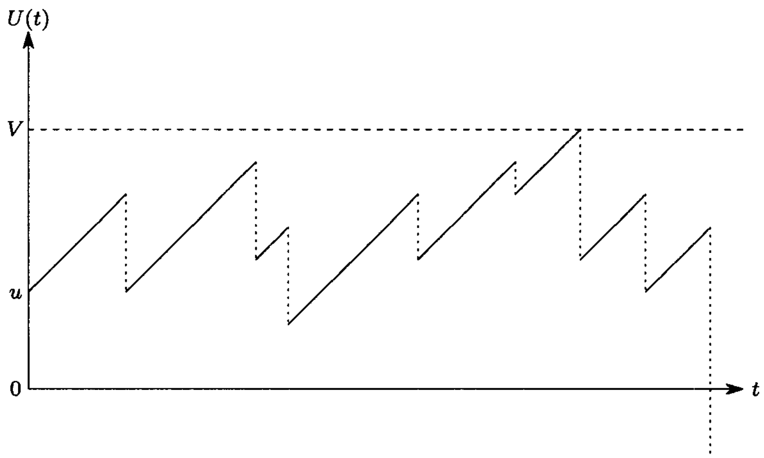

FIGURE 1.1 *A sample path of surplus process $U(t)$.*

and derive $E(T)$ by applying the optional sampling theorem to the martingales. In Section 3, we establish a backward differential equation and solve the equation to obtain the expected total surplus during T. The average surplus during T is obtained by dividing the expected total surplus by $E(T)$. The total and average surplus during T are useful when we predict or estimate the interest of the accumulated surplus.

2. THE FIRST EXIT TIME

Define $T = \inf\{t > 0 | U(t) \notin (0, V)\}$ as the first exit time for $U(t)$ either to go below 0 or to reach $V > 0$. To derive $E(T)$, consider two martingales which are transformed from the surplus process $U(t)$ as follows:

$$W(t) = \frac{e^{\theta U(t)+\theta u}}{E[e^{\theta U(t)}]}$$

and

$$P(t) = U(t) - E[U(t)] + u.$$

$W(t)$ is formed by considering the moment generating function of $U(t)$ and $P(t)$ by subtracting the expectation from $U(t)$. Both $W(t)$ and $P(t)$ are the martingale with respect to filtration $\mathcal{F}_t = \sigma\{U(s), s \leq t\}$. We can show that $W(t)$ and $P(t)$ are martingales by an argument similar to that in Lee and Kinateder (2000). Specially, when $\theta = (\mu\lambda - c)/c\mu$, $W(t) = e^{\theta U(t)}$.

We, first, obtain the probability that the surplus reaches V before going below 0 and the probability that the surplus goes below 0 before reaching V.

LEMMA 2.1. *Let* $P_0^u = P\{U(T) \leq 0 | U(0) = u\}$ *and* $P_V^u = P\{U(T) = V | U(0) = u\}$, *then*

$$P_V^u = \frac{\mu\lambda e^{\theta u} - c}{\mu\lambda e^{\theta V} - c} \quad \textit{and} \quad P_0^u = \frac{\mu\lambda e^{\theta V} - \mu\lambda e^{\theta u}}{\mu\lambda e^{\theta V} - c}.$$

PROOF. Note that $\{U(t), t \geq 0\}$ is a continuous time Markov process and $U(t)$ eventually either reaches V or goes below 0 with probability 1. Hence, $T = \inf\{t > 0 | U(t) \notin (0, V)\}$ is finite with probability 1. Applying the optional sampling theorem to $W(t) = e^{\theta U(t)}$ with respect to T gives that

$$\begin{aligned} e^{\theta u} &= E[W(0)|U(0)=u] = E[W(T)|U(0)=u] \\ &= P[U(T)\le 0|U(0)=u]\int_{-\infty}^{0} e^{\theta y}\frac{1}{\mu}e^{\frac{y}{\mu}}dy + P[U(T)=V|U(0)=u]e^{\theta V} \\ &= P_0^u\frac{c}{\mu\lambda} + P_V^u e^{\frac{\mu\lambda - c}{c\mu}V}. \end{aligned}$$

The 3^{rd} equality follows from the memoryless property of the exponential distribution. Since $P_0^u + P_V^u = 1$, the result follows. □

We are, now, ready to obtain $E(T)$.

LEMMA 2.2. *Let* $T = \inf\{t > 0|U(t) \notin (0, V)\}$, *then*

$$E(T) = \frac{(\mu\lambda e^{\theta u} - c)(\mu + V) - (\mu\lambda e^{\theta V} - c)(\mu + u)}{(\mu\lambda e^{\theta V} - c)(c - \mu\lambda)}, \quad 0 < u < V.$$

PROOF. Applying the optional sampling theorem to $P(t)$ gives that

$$\begin{aligned} u &= E[P(0)|U(0)=u] = E[P(T)|U(0)=u] \\ &= P[U(T)\le 0|U(0)=u]\int_{-\infty}^{0} y\frac{1}{\mu}e^{\frac{y}{\mu}}dy \\ &\quad + P[U(T)=V|U(0)=u]V - (c-\mu\lambda)E(T). \end{aligned}$$

Since $P_0^u = P\{U(T) \le 0|U(0) = u\}$ and $P_V^u = P\{U(T) = V|U(0) = u\}$ are obtained in Lemma 2.1, we have the result, after some algebras. □

3. THE TOTAL AND AVERAGE SURPLUS

The expected total surplus during T is defined by

$$M(u) = E\left[\int_0^T U(t)dt|U(0) = u\right].$$

We obtain $M(u)$ by establishing and solving a backward differential equation.

Conditioning on whether a claim occurs in a small interval $(0, h)$ and on the amount of the claim, we have the following three mutually exclusive events:

(i) no claim occurs, then

$$M(u) = \frac{(2u + ch)h}{2} + M(u + ch),$$

(ii) a claim occurs with $X \geq u + ch', h' \leq h$, then

$$M(u) = \frac{(2u + ch')h'}{2},$$

(iii) a claim occurs with $Y \leq u + ch', h' \leq h$, then

$$M(u) = \frac{(2u + ch')h'}{2} + M(u + ch' - Y),$$

where Y is the amount of a claim and the probability of the event that two claims occur during the interval $(0, h)$ is $o(h)$. Hence, we have, for $0 < u < V$,

$$\begin{aligned} M(u) &= \{1 - \lambda h + o(h)\}\left\{\frac{(2u + ch)h}{2} + M(u + ch)\right\} \\ &\quad + \{\lambda h + o(h)\}\left\{\frac{(2u + ch')h'}{2}\right\} Pr(X \geq u + ch') \\ &\quad + \{\lambda h + o(h)\}\int_0^{u+ch'}\left\{\frac{(2u + ch')h'}{2} + M(u + ch' - y)\right\}\frac{1}{\mu}e^{-\frac{y}{\mu}}dy + o(h) \\ &= uh + M(u + ch) - \lambda h M(u + ch) \\ &\quad + \lambda h\int_0^{u+ch'}\left\{\frac{(2u + ch')h'}{2} + M(u + ch' - y)\right\}\frac{1}{\mu}e^{-\frac{y}{\mu}}dy + o(h). \end{aligned}$$

Subtracting $M(u + ch)$ from each side of the above equation, dividing by h and letting $h \to 0$, we have

$$M'(u) = -\frac{u}{c} + \frac{\lambda}{c}M(u) - \frac{\lambda}{c\mu}\int_0^u M(y)e^{-\frac{u-y}{\mu}}dy. \tag{3.1}$$

The unique solution of equation (3.1) is given in the following Lemma 3.1.

LEMMA 3.1.

$$M(u) = M(0) + \frac{\mu^2\lambda}{k^2}u - \frac{1}{2k}u^2 - \left\{\frac{\mu\lambda}{k}M(0) - \frac{\mu^3 c\lambda}{k^3}\right\}\left(e^{-\frac{k}{\mu c}u} - 1\right),$$

where $M(0) = \{(\mu^2\lambda V/k) - (V^2/2) + (\mu^3\lambda c/k^2)(e^{-kV/\mu c} - 1)\}/(\mu\lambda e^{-kV/\mu c} - c)$ *and* $k = c - \mu\lambda$.

PROOF. Differentiating both sides of equation (3.1) with respect to u, we have

$$M''(u) = -\frac{1}{c} + \frac{\lambda}{c}M'(u) - \frac{\lambda}{c\mu}M(u) + \frac{\lambda}{c\mu^2}\int_0^u M(y)e^{-\frac{u-y}{\mu}}dy. \tag{3.2}$$

Multiplying $1/\mu$ on both sides of equation (3.1) and adding the resulting equation to equation (3.2), we have

$$M''(u) + \frac{c-\mu\lambda}{\mu c}M'(u) = -\frac{\mu+u}{c\mu}. \tag{3.3}$$

Multiplying $e^{(c-\mu\lambda/\mu c)u}$ on both sides of equation (3.3) and integrating from 0 to V give

$$M'(u) = \frac{\mu c}{k^2} - \frac{\mu+u}{k} + \left\{M'(0) - \frac{\mu c}{k^2} + \frac{\mu}{k}\right\}e^{-\frac{k}{\mu c}u}, \tag{3.4}$$

where $k = c - \mu\lambda$ which is the increasing rate of the surplus process. Integrating equation (3.4) again up to u, we have

$$M(u) = M(0) + \frac{\mu^2\lambda}{k^2}u - \frac{1}{2k}u^2 - \left\{\frac{\mu c}{k}M'(0) - \frac{\mu^3 c\lambda}{k^3}\right\}\left(e^{-\frac{k}{\mu c}u} - 1\right). \tag{3.5}$$

By putting $u = 0$ in equation (3.1), we have $M'(0) = (\lambda/c)M(0)$. To get $M(0)$, we use a boundary condition $M(V) = 0$, then

$$M(0) = \frac{\frac{\mu^2\lambda V}{k} - \frac{V^2}{2} + \left(\frac{\mu^3\lambda c}{k^2}\right)\left(e^{-\frac{kV}{\mu c}} - 1\right)}{\mu\lambda e^{-\frac{kV}{\mu c}} - c}.$$

□

Finally, the average surplus during T is given by

$$A(u) = \frac{M(u)}{E(T)}, \quad 0 < u < V,$$

where $M(u)$ is given in Lemma 3.1 and $E(T)$ in Lemma 2.2.

As a numerical example, we illustrate $E(T)$, $M(u)$ and $A(u)$ in the following figures when

(i) the target amount of surplus, V is 50;

(ii) the premium rate, c is 1;

(iii) the arrival rate of claims, λ is 0.6 and

(iv) the expected amount of a claim, μ is 2.

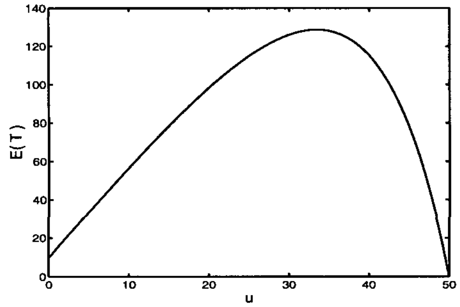

FIGURE 3.1 *The expected first exit time $E(T)$.*

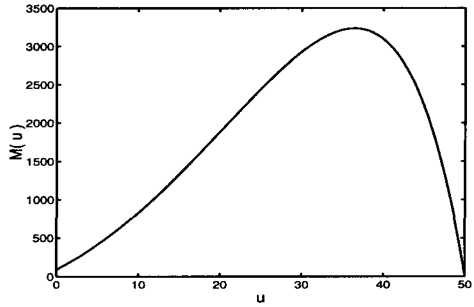

FIGURE 3.2 *The expected total surplus $M(u)$.*

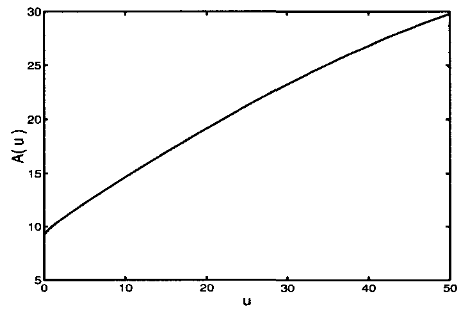

FIGURE 3.3 *The average surplus $A(u)$.*

REFERENCES

DICKSON, D. C. M. AND WILLMOT, G. E. (2005). "The density of the time to ruin in the classical Poisson risk model", *Astin Bulletin*, **35**, 45–60.

GERBER, H. U. (1990). "When does the surplus reach a given target?", *Insurance: Mathematics & Economics*, **9**, 115–119.

GERBER, H. U. AND SHIU, E. S. W. (1997). "The joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin", *Insurance: Mathematics & Economics*, **21**, 129–137.

KARLIN, S. AND TAYLOR, H. M. (1975). *A First Course in Stochastic Processes*, 2nd ed., Academic Press, New York-London.

KLUGMAN, S. A., PANJER, H. H. AND WILLMOT, G. E. (2004). *Loss Models: From Data to Decisions*, 2nd ed., John Wiley & Sons, New Jersey.

LEE, E. Y. AND KINATEDER, K. K. J. (2000). "The expected wet period of finite dam with exponential inputs", *Stochastic Processes and their Applications*, **90**, 175–180.