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ON STRICT STATIONARITY OF NONLINEAR ARMA PROCESSES WITH NONLINEAR GARCH INNOVATIONS[†]

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ABSTRACT

We consider a nonlinear autoregressive moving average model with nonlinear GARCH errors, and find sufficient conditions for the existence of a strictly stationary solution of three related time series equations. We also consider a geometric ergodicity and functional central limit theorem for a nonlinear autoregressive model with nonlinear ARCH errors. The given model includes broad classes of nonlinear models. New results are obtained, and known results are shown to emerge as special cases.

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1. INTRODUCTION

In the last three decades, there were many papers in the literature discussing the stationarity of various types of nonlinear time series models such as Priestley (1980), Nummelin (1984), Tong (1990), Tjøstheim (1990), Meyn and Tweedie (1993), *etc.* The typical nonlinear ARMA(p,q) model is given by

$$y_t = \phi(y_{t-1}, y_{t-2}, \ldots, y_{t-p}, e_{t-1}, \ldots, e_{t-q}) + e_t,$$

where $\{e_t\}$ are independent and identically distributed *(i.i.d.).* While this model has constant variance, conditional variances of many types of economic and financial data depend on past information. The most well known example of stochastic volatility model is ARCH (autoregressive conditional heteroscedasticity) process, which was introduced by Engle (1982) to explain the time series with conditional

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heteroscedastic variances. The ARCH model was extended to generalized ARCH (GARCH) time series by Bollerslev (1986). As another extension of the ARCH process, a class of autoregressive models with ARCH errors proposed first by Weiss (1984) and Tong (1990) suggested a threshold model with an ARCH error, which is entitled the SETAR-ARCH model. The family of ARCH models have proven useful in financial applications and have attracted great attention in economics and statistical literature (see, for instance, Bollerslev *et al.*, 1992; Bougerol and Picard, 1992; Guégan and Diebolt, 1994; Lu, 1996; Li and Li, 1996; Wong and Li, 1997; Ling, 1999; He and Teräsvirta, 1999; Francq and Zakoïan, 2000; Ling and McAleer, 2002; van Dijk *et al.*, 2002; Hwang and Kim, 2004).

In the GARCH model, conditional variance is a linear function of the squared past disturbances and/or past observations, but some data show that this linearity is not adequate and the conditional variances are asymmetric conditional on previous returns (see, *e.g.*, Rabemananjara and Zakoian, 1993; Liu, *et al.*, 1997). In order to accommodate the asymmetric conditionality of conditional variance, the double threshold AR-ARCH model and the double threshold ARMA-GARCH model were introduced (see, Li and Li, 1996; Ling, 1999).

In this paper, we consider a nonlinear ARMA models with nonlinear GARCH innovations, which is a natural extension of double threshold ARMA-GARCH model. This model combines the advantages of the nonlinear ARMA model which targets on the conditional means given the past and the nonlinear GARCH model which concentrates on the conditional variances given the past. Therefore, this model is capable of modeling time series by changing the conditional mean and the conditional variance via nonlinear methods.

Let $\{y_t\}$ be the nonlinear autoregressive moving average time series with nonlinear GARCH errors given by

$$y_t = \phi(y_{t-1}, \ldots, y_{t-p}, \varepsilon_{t-1}, \ldots, \varepsilon_{t-q}) + \varepsilon_t, \tag{1.1}$$

$$\varepsilon_t = h_t^{\frac{1}{2}} e_t, \tag{1.2}$$

$$h_t = \alpha_0 + \psi(\varepsilon_{t-1}, \ldots, \varepsilon_{t-r}, h_{t-1}, \ldots, h_{t-s}), \tag{1.3}$$

where ϕ and ψ are real-valued measurable functions defined on R^{p+q} and R^{r+s}, respectively, $p \geq 0$, $q \geq 0, r \geq 0, s \geq 0, \alpha_0 > 0$, and $\{e_t\}$ is a sequence of *i.i.d.* random variables with mean zero and unit variance. The process obtained by (1.1)–(1.3) includes various well known nonlinear models such as nonlinear ARMA models with constant variance, TARMA, (β)-ARCH, SETAR-ARCH, double threshold ARMA-GARCH, asymmetric power GARCH model, augmented GARCH model, *etc.*

Our aim is to derive sufficient conditions for stationarity, geometric ergodicity and finiteness of moments of the model given above. We study the stationarity of y_t by applying the Tweedie's result (Tweedie, 1988; Tong, 1990) to the associated Markov chain and then derive desired results from that for the Markov chain.

Obtained results improve and extend those given earlier by, for example, Brockwell *et al.* (1992), An and Huang (1996), Ling (1999) and Lee (2000).

For terminologies and relevant results in Markov chain theory, we refer to Meyn and Tweedie (1993).

Section 2 presents the main results and their proofs are in Section 3. Several examples are given in Section 4.

2. Main Results

Consider the following nonlinear autoregressive moving average model with nonlinear GARCH innovations:

$$y_t = \phi(y_{t-1}, \ldots, y_{t-p}, \varepsilon_{t-1}, \ldots, \varepsilon_{t-q}) + \varepsilon_t, \tag{2.1}$$

$$\varepsilon_t = \sqrt{h_t} \cdot e_t, \tag{2.2}$$

$$h_t = \alpha_0 + \psi(\varepsilon_{t-1}, \ldots, \varepsilon_{t-r}, h_{t-1}, \ldots, h_{t-s}), \tag{2.3}$$

where ϕ and ψ are real-valued measurable functions on R^{p+q} and R^{r+s} respectively, $p \geq 0,\ q \geq 0,\ r \geq 0,\ s \geq 0,\ \psi \geq 0,\ \alpha_0 > 0$, and $\{e_t\}$ is a sequence of *i.i.d.* random variables with zero mean and unit variance.

To avoid unnecessary technicalities, we assume without loss of generality that $q \geq r$. Let $\{y_0, y_{-1}, \ldots, y_{-p+1}, \varepsilon_0, \ldots, \varepsilon_{-q+1}, h_0, \ldots, h_{-s+1}\}$ be arbitrarily specified real-valued random variables independent of $\{e_t; t \geq 1\}$.

Denote

$$\mathbf{X}_t = (y_t, \ldots, y_{t-p+1}, \varepsilon_t, \ldots, \varepsilon_{t-q+1}, \varepsilon_t^2, \ldots, \varepsilon_{t-r+1}^2, h_t, \ldots, h_{t-s+1})^t, \tag{2.4}$$

then $\{\mathbf{X}_t; t \geq 0\}$ is a Markov chain with state space $\mathbf{S}$ which is given by

$$\mathbf{S} = \{(u_1, \ldots, u_p, z_1, \ldots, z_q, z_1^2, \ldots, z_r^2, w_1, \ldots, w_s)^t \mid u_i, z_j, w_k \in R,\ 1 \leq i \leq p,\ 1 \leq j \leq q,\ 1 \leq k \leq s\}.$$

We make the following assumptions:

(A.1) There exist constants $\lambda, 0 < \lambda < 1$, $\theta > 0$, and c_1 such that for $(x_1, \ldots, x_{p+q})^t$ in R^{p+q},

$$|\phi(x_1, \ldots, x_{p+q})| \leq \lambda \max_{1 \leq i \leq p} \{|x_i|\} + \theta \max_{1 \leq i \leq q} \{|x_{p+i}|\} + c_1. \tag{2.5}$$

(A.2) There exist constants $\alpha_i \geq 0$, $\beta_j \geq 0$, $i = 1, \ldots, r$, $j = 1, \ldots, s$ with $\sum_{i=1}^r \alpha_i + \sum_{j=1}^s \beta_j < 1$ and c_2 such that for $(x_1, \ldots, x_{r+s})^t$ in R^{r+s},

$$\psi(x_1, \ldots, x_{r+s}) \leq \sum_{i=1}^r \alpha_i x_i^2 + \sum_{j=1}^s \beta_j |x_{r+j}| + c_2. \tag{2.6}$$

(A.3) $\{\mathbf{X}_t\}$ is a Feller chain, *i.e.*, for each bounded continuous function g, $E[g(\mathbf{X}_t)|\mathbf{X}_{t-1} = \mathbf{x}]$ is continuous in $\mathbf{x}$.

REMARK 2.1. (Feller continuity). $\mathbf{X}_t$ in (2.4) can be rewritten as an iterative model:

$$\mathbf{X}_t = F(\mathbf{X}_{t-1}, e_t),$$

where F is the proper measurable function on $\mathbf{S} \times R$ to $\mathbf{S}$. If ϕ and ψ in (2.1) and (2.3) are continuous, the function $x \to F(x, e)$ is continuous for any e, and hence, by dominated convergence theorem, $\{\mathbf{X}_t\}$ is a Feller chain.

Followings are our main results.

THEOREM 2.1. *Under the assumptions (A.1)–(A.3), there exists a stationary solution (y_t, ε_t) satisfying (2.1)–(2.3), and $E_{\pi_1}(|y_t|)$ and $E_{\pi_2}(\varepsilon_t^2)$ are finite, where π_1 and π_2 are the stationary distributions of $\{y_t\}$ and $\{\varepsilon_t\}$, respectively.*

COROLLARY 2.1. *In addition to the assumptions (A.1)–(A.3), suppose that the Markov chain $\{\mathbf{X}_t\}$ is aperiodic φ-irreducible. Then $\{\mathbf{X}_t\}$ is geometrically ergodic.*

Unfortunately, it is not an easy task to prove the irreducibility of $\{\mathbf{X}_t\}$ when $p > 1$ and $q \geq 1$.

To obtain the geometric ergodicity, we restrict ourselves to the nonlinear autoregressive model with nonlinear ARCH errors, that is, the model given by (2.1)–(2.3) with $q = s = 0$, and define Markov chain in a different way as follows:

$$\mathbf{Z}_t = (y_t, y_{t-1}, \ldots, y_{t-p-r+1})^t. \tag{2.7}$$

Here if y_t, $t = -p-r+1, \dots, -1, 0$ are arbitrarily defined random variables independent of $\{e_t : t \geq 1\}$, then $\{\mathbf{Z}_t\}$ is a Markov chain.

For the remaining part of this section, we assume that $q = s = 0$.

Now we need additional assumptions on e_t and the function ψ:

(A.4) The distribution of e_t is absolutely continuous with a probability density function $q(\cdot)$ which is positive almost everywhere (with respect to the Lebesgue measure μ_1) and $E|e_t|^m < \infty$ for some $m > 0$.

(A.5) $h(\mathbf{z})/\|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to \infty$ where $h(\mathbf{z})$ is given by $h(z_1, \dots, z_{p+r}) = \sqrt{\psi(w_1, \dots, w_r)}$, $w_i = z_i - \phi(z_{i+1}, \dots, z_{i+p})$, $1 \leq i \leq r$ and $\|\cdot\|$ is any norm on R^{p+r} .

THEOREM 2.2. *Suppose (A.1), (A.3), (A.4) and (A.5) hold. Then* $\mathbf{Z}_t$ *is a geometrically ergodic Markov chain and* $E_{\pi_1}|y_t|^m < \infty$ *with a unique invariant probability* π_1 *of* $\{y_t\}$.

The assumption (A.3) in Theorem 2.2 is used to ensure that every compact set is small. We can obtain that property by adding some mild conditions on $q(\cdot)$, ϕ and ψ.

THEOREM 2.3. *In addition to (A.1), (A.4) and (A.5), we assume that* $q(\cdot)$ *is lower semi-continuous and* ϕ *and* ψ *are bounded on compacts. Then the conclusion of Theorem 2.2 holds.*

When a Markov process $\{\mathbf{Z}_t\}$ given in (2.7) is geometrically ergodic, we can obtain a class of functions under which the functional central limit theorem holds. Let π denote the invariant initial probability of $\{\mathbf{Z}_t\}$ and let $\|\cdot\|_2$ be L^2-norm.

THEOREM 2.4. *Suppose the assumptions in Theorem 2.2 (or Theorem 2.3) hold. Let* V *be the test function given in (3.21) in the proof of Theorem 2.2. If* $f^2 \leq V + K$ *for some constant* $K > 0$*, the functional central limit theorem holds for* f*, that is,* $Y_n(t) = (1/\sqrt{n}) \sum_{k=0}^{[nt]} (f(\mathbf{Z}_k) - \pi(f))$, $t \geq 0$ *converges in distribution to a Brownian motion with mean zero and variance parameter* $\|g\|_2^2 - \|Pg\|_2^2$*, where* $g - Pg = f - \pi(f)$, $Pg(x) = \int g(y)P(x, dy)$ *and* $\pi(f) = \int f d\pi$. *In particular, the functional central limit theorem holds for every bounded measurable function* f.

3. Proofs

For the convenience of readers, we state the next theorem.

THEOREM 3.1. (Tweedie, 1988). *Suppose $\{X_t\}$ is a Feller chain with transition probability function $P(x, dy)$.*

(a) If there exists, for some compact set A, a nonnegative function g and an $\epsilon > 0$ such that

$$\int_{A^c} P(x, dy) g(y) \leq g(x) - \epsilon, \quad x \in A^c, \tag{3.1}$$

then there exists a σ-finite invariant measure μ for P with $0 < \mu(A) < \infty$.

(b) Further, if

$$\int_A \mu(dx) \left\{ \int_{A^c} P(x, dy) g(y) \right\} < \infty, \tag{3.2}$$

then μ is finite.

(c) Further, if

$$\int_{A^c} P(x, dy) g(y) \leq g(x) - f(x), \quad x \in A^c, \tag{3.3}$$

then μ admits a finite f-moment.

The main parts of the proofs for Theorem 2.1 to Theorem 2.3 in Section 2 are to construct a proper test function $g(\cdot)$ (or $V(\cdot)$) under which (3.1)–(3.3) hold.

PROOF OF THEOREM 2.1. Define a test function $g : R^{p+q+r+s} \to R$ by for any $(x_1, x_2, \ldots, x_{p+q+r+s})^t$ in $R^{p+q+r+s}$,

$$g(x_1, \ldots, x_{p+q+r+s}) = 1 + \max_{1 \leq i \leq p} \{\gamma_i |x_i|\} + \sum_{i=1}^{q+r+s} \gamma_{p+i} |x_{p+i}|, \tag{3.4}$$

where γ_i, $i = 1, 2, \ldots, p+q+r+s$ are to be defined later.

For any $\mathbf{x} = (u_1, \ldots, u_p, z_1, \ldots, z_q, z_1^2, \ldots, z_r^2, w_1, \ldots, w_s)^t \in \mathbf{S}$, we have that

$$\begin{aligned}
&E\left[g(\mathbf{X}_t) | \mathbf{X}_{t-1} = \mathbf{x}\right] \\
&= E[g\{\phi(u_1, \ldots, u_p, z_1, \ldots, z_q) + \varepsilon_t, u_1, \ldots, u_{p-1}, \\
&\qquad \varepsilon_t, z_1, \ldots, z_{q-1}, \varepsilon_t^2, z_1^2, \ldots, z_{r-1}^2, h_t, w_1, \ldots, w_{s-1}\} | \mathbf{X}_{t-1} = \mathbf{x}]
\end{aligned}$$

$$\begin{aligned}
&\leq 1+\max\{\gamma_1\lambda\max\{|u_1|,\dots,|u_p|\},\gamma_2|u_1|,\dots,\gamma_p|u_{p-1}|\}\\
&\quad+\gamma_1\theta\max\{|z_1|,\dots,|z_q|\}+\gamma_{p+2}|z_1|+\cdots+\gamma_{p+q}|z_{q-1}|\\
&\quad+\gamma_{p+q+1}E[\varepsilon_t^2|\mathbf{X}_{t-1}=\mathbf{x}]+\sum_{i=1}^{r-1}\gamma_{p+q+1+i}\;z_i^2+\gamma_{p+q+r+1}h_t\\
&\quad+\sum_{i=1}^{s-1}\gamma_{p+q+r+1+i}\;w_i+(\gamma_1+\gamma_{p+1})E[|\varepsilon_t||\mathbf{X}_{t-1}=\mathbf{x}]+c_1\gamma_1\\
&\leq I+II+III+K,
\end{aligned}\tag{3.5}$$

where

$$I=\max\{\gamma_1\lambda\max\{|u_1|,\dots,|u_p|\},\gamma_2|u_1|,\dots,\gamma_p|u_{p-1}|\},\tag{3.6}$$

$$II=\gamma_1\theta\max\{|z_1|,\dots,|z_q|\}+\gamma_{p+2}|z_1|+\cdots+\gamma_{p+q}|z_{q-1}|,\tag{3.7}$$

$$\begin{aligned}
III&=(\gamma_{p+q+1}+\gamma_{p+q+r+1})\left(\sum_{i=1}^{r}\alpha_i z_i^2+\sum_{i=1}^{s}\beta_i w_i\right)+\sum_{i=1}^{r-1}\gamma_{p+q+1+i}\;z_i^2\\
&\quad+\sum_{i=1}^{s-1}\gamma_{p+q+r+1+i}\;w_i+(\gamma_1+\gamma_{p+1})\sqrt{h_t},
\end{aligned}\tag{3.8}$$

$$K=1+c_1\gamma_1+(\alpha_0+c_2)(\gamma_{p+q+1}+\gamma_{p+q+r+1}).\tag{3.9}$$

III and K are obtained from assumptions (A.2), $E(e_t^2)=1$ and $E|e_t|\leq 1$.

We now choose $\gamma_1>0$ arbitrarily and define

$$\gamma_k=\lambda^{\frac{1}{p}}\gamma_{k-1},\quad k=2,3,\dots,p.\tag{3.10}$$

Then

$$\begin{aligned}
I&=\max\{\gamma_1\lambda\max\{|u_1|,\dots,|u_p|\},\gamma_2|u_1|,\dots,\gamma_p|u_{p-1}|\}\\
&=\lambda^{\frac{1}{p}}\max\{\gamma_p\max\{|u_1|,\dots,|u_p|\},\gamma_1|u_1|,\dots,\gamma_{p-1}|u_{p-1}|\}\\
&=\lambda^{\frac{1}{p}}\max\{\gamma_p|u_p|,\gamma_1|u_1|,\dots,\gamma_{p-1}|u_{p-1}|\}.
\end{aligned}\tag{3.11}$$

The last equality in (3.11) follows from the fact that $\gamma_p\leq\gamma_k$ for $1\leq k\leq p$.

Next choose η in $(0,1)$ and fix, and define $\gamma_{p+1},\dots,\gamma_{p+q}$ by

$$\gamma_{p+q-i}=\left(\frac{1}{\eta}\right)^{i+1}(1+\eta+\cdots+\eta^i)\gamma_1\theta,\quad i=0,1,\dots,q-1.\tag{3.12}$$

It is obvious, from (3.7) and (3.12), that

$$\begin{aligned}
II&\leq(\gamma_1\theta+\gamma_{p+2})|z_1|+(\gamma_1\theta+\gamma_{p+3})|z_2|+\cdots+(\gamma_1\theta+\gamma_{p+q})|z_{q-1}|+\gamma_1\theta|z_q|\\
&=\eta\sum_{i=1}^{q}\gamma_{p+i}|z_i|.
\end{aligned}\tag{3.13}$$

To consider part III, we define an $(r+s)\times(r+s)$ matrix A by

$$A=\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_r & \beta_1 & \beta_2 & \cdots & \beta_s \\ & I_{(r-1)\times(r-1)} & & 0_{(r-1)\times 1} & & \mathbf{O}_{(r-1)\times s} & & \\ \alpha_1 & \alpha_2 & \cdots & \alpha_r & \beta_1 & \beta_2 & \cdots & \beta_s \\ & \mathbf{O}_{(s-1)\times r} & & & & I_{(s-1)\times(s-1)} & & 0_{(s-1)\times 1} \end{pmatrix}$$

and modify the method adopted by Ling (1999).

Since $\det(xI-A)=x^{r+s}-x^s\sum_{i=1}^r\alpha_i x^{r-i}-x^r\sum_{i=1}^s\beta_i x^{s-i}$, from the assumption $\sum_{i=1}^r\alpha_i+\sum_{i=1}^s\beta_i<1$, we obtain that all roots of the characteristic polynomial of A lie inside the unit circle, that is $\rho(A)<1$, where $\rho(A)$ is the spectral radius of A. It is known that if $\rho(A)<1$, then $(I-A)$ is invertible and $(I-A)^{-1}=\sum_{k=0}^\infty A^k$ (see, *e.g.*, Horn and Johnson, 1990). Moreover, since each component of A is nonnegative, we can choose a vector $M_1>0$ such that $M=(I-A^t)^{-1}M_1=M_1+\sum_{k=1}^\infty(A^t)^kM_1>0$, and hence we have $(I-A^t)M=M_1>0$. Here a vector $M>0$ means that every component of M is positive. Take

$$(\gamma_{p+q+1},\gamma_{p+q+2},\dots,\gamma_{p+q+r+s})^t=M.$$

Then

$$\begin{aligned} III &= \mathbf{x}_2^tA^tM+(\gamma_1+\gamma_{p+1})\sqrt{h_t} \\ &= \mathbf{x}_2^tM-\mathbf{x}_2^t(I-A^t)M+(\gamma_1+\gamma_{p+1})\sqrt{h_t}, \end{aligned} \tag{3.14}$$

where $\mathbf{x}_2=(z_1^2,\dots,z_r^2,w_1,\dots,w_s)^t$.

For simplicity of notation, let $g_1(\mathbf{x}_1)=\max_{1\le i\le p}\{\gamma_i|u_i|\}+\sum_{i=1}^q\gamma_{p+i}|z_i|$ for $\mathbf{x}_1=(u_1,\dots,u_p,z_1,\dots,z_q)^t$ and let $\eta'=\max\{\lambda^{1/p},\eta\}<1$.

Combining (3.5), (3.11), (3.13) and (3.14), we have that

$$\begin{aligned} &E[g(\mathbf{X}_t)|\mathbf{X}_{t-1}=\mathbf{x}] \\ &\le \eta'g_1(\mathbf{x}_1)+\mathbf{x}_2^tM-\mathbf{x}_2^t(I-A^t)M+(\gamma_1+\gamma_{p+1})\sqrt{h_t}+K \\ &= g(\mathbf{x})-(1-\eta')g_1(\mathbf{x}_1)-\mathbf{x}_2^t(I-A^t)M+(\gamma_1+\gamma_{p+1})\sqrt{h_t}+K-1 \\ &= g(\mathbf{x})\left(1-\frac{(1-\eta')g_1(\mathbf{x}_1)+\mathbf{x}_2^t(I-A^t)M}{g(\mathbf{x})}+\frac{(\gamma_1+\gamma_{p+1})\sqrt{h_t}+K-1}{g(\mathbf{x})}\right). \end{aligned} \tag{3.15}$$

Let

$$\begin{aligned} m_1 &= \min\{\text{all components of M}\}>0, \\ m_2 &= \max\{\text{all components of M}\}>0, \\ m_3 &= \min\{\text{all components of M}_1\}>0 \end{aligned}$$

and for $k > 0$, define $\mathbf{B}_k = \{\mathbf{x} \in \mathbf{S} \mid g(\mathbf{x}) \le k\}$ and $\mathbf{B}_k^c = \mathbf{S} - \mathbf{B}_k$.

For $\mathbf{x} = (\mathbf{x}_1^t, \mathbf{x}_2^t)^t \in \mathbf{B}_k^c$, we have following two inequalities:

$$\begin{aligned} \frac{(1-\eta')g_1(\mathbf{x}_1) + \mathbf{x}_2^t(I - A^t)M}{g(\mathbf{x})} &\ge \frac{\min\{(1-\eta'), m_3\}(g_1(\mathbf{x}_1) + \sum z_i^2 + \sum w_i)}{1 + \max\{1, m_2\}(g_1(\mathbf{x}_1) + \sum z_i^2 + \sum w_i)} \\ &\ge \frac{\min\{(1-\eta'), m_3\}}{2\max\{1, m_2\}} \\ &> 0 \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \frac{(\gamma_1 + \gamma_{p+1})\sqrt{h_t} + K - 1}{g(\mathbf{x})} &\le \frac{(\gamma_1 + \gamma_{p+1})\sqrt{\sum z_i^2 + \sum w_i} + K'}{g(\mathbf{x})} \\ &\le \frac{(\gamma_1 + \gamma_{p+1})\sqrt{\sum z_i^2 + \sum w_i} + K'}{\min\{1, m_1\}(g_1(\mathbf{x}_1) + \sum z_i^2 + \sum w_i)}, \end{aligned} \tag{3.17}$$

where $K' = (\gamma_1 + \gamma_{p+1})(\sqrt{\alpha_0} + \sqrt{c_2}) + K - 1$.

Therefore, from (3.15)–(3.17) and the fact that we may choose k so large that for $x \in \mathbf{B}_k^c$, the last term of (3.17) is arbitrary small, there exist $\epsilon < 1$ and k such that

$$E[g(\mathbf{X}_t)|\mathbf{X}_{t-1} = \mathbf{x}] \le (1-\epsilon)g(\mathbf{x}), \quad \mathbf{x} \in \mathbf{B}_k^c. \tag{3.18}$$

Clearly,

$$\sup_{\mathbf{x} \in \mathbf{B}_k} E[g(\mathbf{X}_t)|\mathbf{X}_{t-1} = \mathbf{x}] < \infty. \tag{3.19}$$

Therefore, (3.18) and (3.19) imply that (3.1)–(3.3) in Theorem 3.1 hold with a compact set $\mathbf{B}_k$, and the existence of stationary $\mathbf{X}_t$ is shown.

Moreover, by (3.18) and the part (c) of Theorem 3.1, we have that

$$\int g(\mathbf{x})\pi(d\mathbf{x}) < \infty, \tag{3.20}$$

where π is a stationary distribution of $\{\mathbf{X}_t\}$. Thus, from (3.20), $\int |y_t| d\pi_1 < \infty$ and $\int \varepsilon_t^2 d\pi_2 < \infty$ where π_1 and π_2 are the first and $(p+q+1)^{th}$ projection measures of π respectively. This completes the proof. □

PROOF OF COROLLARY 2.1. It is obvious that $\{\mathbf{X}_t\}$ is aperiodic. The conclusion follows from Theorem 2.1 and Theorem A1.5 (p. 457) in Tong (1990). □

Let $\mathcal{B}^k$ denote the Borel sigma field on R^k, and let μ_k be the Lebesgue measure on $(R^k, \mathcal{B}^k)$, $k = 1, 2, \ldots$.

PROOF OF THEOREM 2.2. By definition of $h(\cdot)$, $y_t = \phi(y_{t-1}, \ldots, y_{t-p}) + (\alpha_0 + h^2(\mathbf{Z}_{t-1}))^{1/2} e_t$, $\mathbf{Z}_{t-1} = (y_{t-1}, \ldots, y_{t-p-r})^t$. Define

$$u(\mathbf{z}, \mathbf{y}) = \prod_{j=1}^{p+r} q_j(\mathbf{y}), \quad (\mathbf{z}, \mathbf{y} \in R^{p+r}),$$

where for $a_j (1 \le j \le p + r)$ such that

$$\begin{aligned} a_1 &= \{\alpha_0 + h^2(z_1, \ldots, z_{p+r})\}^{-\frac{1}{2}}, \\ a_j &= \{\alpha_0 + h^2(y_{j-1}, \ldots, y_1, z_1, \ldots, z_{p+r-j+1})\}^{-\frac{1}{2}}, \quad 2 \le j \le p + r, \end{aligned}$$

$q_j(\mathbf{y})$, $1 \le j \le p + r$ are given by

$$\begin{aligned} q_1(\mathbf{y}) &= a_1 q[a_1\{y_1 - \phi(z_1, \ldots, z_p)\}], \\ q_j(\mathbf{y}) &= a_j q[a_j\{y_j - \phi(y_{j-1}, \ldots, y_1, z_1, \ldots, z_{p-j+1})\}], \ 2 \le j \le p, \\ q_j(\mathbf{y}) &= a_j q[a_j\{y_j - \phi(y_{j-1}, \ldots, y_{j-p})\}], \ p + 1 \le j \le p + r. \end{aligned}$$

Then $u(\mathbf{z}, \mathbf{y})$ is a conditional density of $\mathbf{Z}_{p+r}$ at $\mathbf{y}$ given $\mathbf{Z}_0 = \mathbf{z}$, and hence, by assumption (A.4), $P(\mathbf{Z}_{p+r} \in A \mid \mathbf{Z}_0 = \mathbf{z}) = \int_A u(\mathbf{z}, \mathbf{y}) \mu_{p+r}(d\mathbf{y}) > 0$ for all $\mathbf{z} \in R^{p+r}$ and $A \in \mathcal{B}^{p+r}$ with $\mu_{p+r}(A) > 0$, which implies that $\{\mathbf{Z}_t\}$ is μ_{p+r}-irreducible.

Now define a test function $V : R^{p+r} \to R$ by for $\mathbf{z} = (z_1, \ldots, z_{p+r})^t$,

$$V(\mathbf{z}) = 1 + \max_{1 \le i \le p+r} \{\gamma_i |z_i|^m\}, \tag{3.21}$$

where $m > 0$ is given in assumption (A.4), $\gamma_1 > 0$ is chosen arbitrarily and γ_k, $k = 2, \ldots, p + r$ are taken by $\gamma_k = (\lambda^m)^{1/p} \gamma_{k-1}$ recursively.

We first consider the case that $0 < m \le 1$. In the same manner used to obtain (3.11), we have

$$\begin{aligned} &E[V(\mathbf{Z}_t) | \mathbf{Z}_{t-1} = \mathbf{z}] \\ &\le \max\{\gamma_1 \lambda^m \max_{1 \le i \le p} \{|z_i|^m\}, \gamma_2 |z_1|^m, \ldots, \gamma_{p+r} |z_{p+r-1}|^m\} \\ &\quad + \gamma_1 (\alpha_0 + h^2(\mathbf{z}))^{\frac{m}{2}} E|e_t|^m + \gamma_1 c_1^m + 1 \\ &\le \delta \max_{1 \le i \le p+r} \{\gamma_i |z_i|^m\} + \gamma_1 (\alpha_0 + h^2(\mathbf{z}))^{\frac{m}{2}} E|e_t|^m + \gamma_1 c_1^m + 1 \\ &\le \left(1 + \max_{1 \le i \le p+r} \{\gamma_i |z_i|^m\}\right) (\delta + \Delta(\mathbf{z})), \end{aligned} \tag{3.22}$$

where $\delta = (\lambda^m)^{1/p}$ and

$$\Delta(\mathbf{z}) = \frac{\gamma_1 E|e_t|^m(\alpha_0^{\frac{m}{2}} + h(\mathbf{z})^m) + \gamma_1 c_1^m + 1 - \delta}{\max\{\gamma_i |z_i|^m\}}.$$

Since $(\max\{\gamma_i|z_i|^m\})^{1/m}$, $m > 0$ is a norm on R^{p+r} and norms on R^{p+r} are equivalent, $(\max\{\gamma_i|z_i|^m\})^{1/m} \geq c\|\mathbf{z}\|$ for some $c > 0$, and hence by assumption (A.5), we have that $\Delta(\mathbf{z}) \to 0$ as $\|\mathbf{z}\| \to \infty$. Thus there exist $\epsilon > 0$ with $\delta + \epsilon < 1$ and $M > 0$ such that

$$\Delta(\mathbf{z}) < \epsilon \quad \text{if} \quad \|\mathbf{z}\| > M \tag{3.23}$$

and therefore,

$$E[V(\mathbf{Z}_t)|\mathbf{Z}_{t-1} = \mathbf{z}] \leq (\delta + \epsilon)V(\mathbf{z}), \quad \|\mathbf{z}\| > M. \tag{3.24}$$

Now suppose $m > 1$. For simplicity of notation we assume that $c_1 = 0$. The case $c_1 > 0$ is entirely analogous. Note that

$$\begin{aligned} &|\phi(z_1, \ldots, z_p) + (\alpha_0 + h^2(z))^{\frac{1}{2}} e_t|^m \\ &\leq |\phi(z_1, \ldots, z_p)|^m + R(\mathbf{z}, e_t), \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} R(\mathbf{z}, e_t) = &\sum_{i=1}^{[m]} \binom{[m]}{i} |\phi|^{m-i} \; |(\alpha_0 + h^2(\mathbf{z}))^{\frac{1}{2}} e_t|^i \\ &+ |\phi|^{[m]} \; |(\alpha_0 + h^2(\mathbf{z}))^{\frac{1}{2}} e_t|^s \\ &+ \sum_{i=1}^{[m]} \binom{[m]}{i} |\phi|^{[m]-i} \; |(\alpha_0 + h^2(\mathbf{z}))^{\frac{1}{2}} e_t|^{s+i} \end{aligned} \tag{3.26}$$

and $[m]$ is the Gauss number of m and $s = m - [m] \geq 0$.

From (3.21) and (3.25) we have

$$\begin{aligned} &E[V(Z_t)|Z_{t-1} = \mathbf{z}] \\ &\leq \max\{\gamma_1|\phi|^m, \gamma_2|z_1|^m, \ldots, \gamma_{p+r}|z_{p+r-1}|^m\} + \gamma_1 E[R(\mathbf{z}, e_t)] + 1 \\ &\leq \delta \max\{\gamma_i|z_i|^m\} + \gamma_1 E[R(\mathbf{z}, e_t)] + 1. \end{aligned} \tag{3.27}$$

On the other hand, for $1 \leq i \leq [m]$,

$$\frac{|\phi|^{m-i}E|(\alpha_0+h^2(\mathbf{z}))^{\frac{1}{2}}e_t|^i}{\max_i\{\gamma_i|z_i|^m\}} \leq \frac{2^{i-1}\lambda^{m-i}E|e_t|^i(\alpha_0^{\frac{i}{2}}+|h(\mathbf{z})|^i)\max\{|z_i|^{m-i}\}}{c^m\|\mathbf{z}\|^m}$$
$$= \frac{2^{i-1}E|e_t|^i\lambda^{m-i}}{c^m}\cdot\frac{\alpha_0^{\frac{i}{2}}+|h(\mathbf{z})|^i}{\|\mathbf{z}\|^i}\cdot\frac{\max\{|z_i|^{m-i}\}}{\|\mathbf{z}\|^{m-i}}$$
$$\to 0 \quad \text{as} \quad \|\mathbf{z}\|\to\infty. \tag{3.28}$$

By the same argument, the fraction of the expectation of the remaining terms of (3.26) and $\|\mathbf{z}\|^m$ goes to zero as $\|\mathbf{z}\|\to\infty$ and hence

$$\frac{E[R(\mathbf{z},e_t)]}{\max\{\gamma_i|z_i|^m\}}\to 0 \quad \text{as} \quad \|\mathbf{z}\|\to\infty. \tag{3.29}$$

Combining (3.27)–(3.29), we have that, for $\epsilon>0$ given in (3.23)

$$E[V(Z_t)|Z_{t-1}=\mathbf{z}]\leq(\delta+\epsilon)V(\mathbf{z}) \quad \text{if} \quad \|\mathbf{z}\|>M' \tag{3.30}$$

for sufficiently large $M'>0$. Clearly,

$$\sup_{\|\mathbf{z}\|\leq M'} E[V(Z_t)|Z_{t-1}=\mathbf{z}]<\infty. \tag{3.31}$$

Since $\{\mathbf{Z}_t\}$ is a μ_{p+r}-irreducible Feller chain, every compact set is small. Therefore, by (3.24), (3.30), (3.31), and Theorem A1.5 in Tong (1990), geometric ergodicity of the process is obtained. $E_{\pi_1}|y_t|^m<\infty$ follows from (3.30) and part (c) of Theorem 3.1. □

PROOF OF THEOREM 2.3. Suppose that ϕ and ψ are bounded on compacts and $q(\cdot)$ is lower semi-continuous. Then for every $A\in\mathcal{B}^{p+r}$ such that $\mu_{p+r}(A)>0$ and every compact set $B\subset R^{p+r}$, we have

$$\inf_{\mathbf{z}\in B}\int_A u(\mathbf{z},\mathbf{y})\mu_{p+r}(d\mathbf{y})>0.$$

Hence B is small, and the results in Theorem 2.3 follow. □

PROOF OF THEOREM 2.4. From (3.24), (3.30) and (3.31), we have, for some $\lambda<1$, $b<\infty$ and compact set C,

$$PV(\mathbf{z})\leq\lambda V(\mathbf{z})+bI_C \tag{3.32}$$

and $\pi(V) < \infty$. Choose a large K such that $(1 - \lambda^{1/2})K^{1/2} \geq 1$ and take $V_0 = V + K$. Then by (3.32) and the conditional Jensen's inequality, we obtain, for some $b_0 < \infty$,

$$PV_0^{\frac{1}{2}} \leq V_0^{\frac{1}{2}} - (1 - \lambda^{\frac{1}{2}})V_0^{\frac{1}{2}} + b_0^{\frac{1}{2}} I_C.$$

Suppose that $|f| \leq V_0^{1/2}$. Then $f \in L^2(\pi)$ and there exists a function $g \in L^2(\pi)$ such that $f - \pi(f) = g - Pg$. Since $g(\mathbf{Z}_t) - Pg(\mathbf{Z}_{t-1})$ is a sequence of martingale differences, the functional central limit theorem holds for f (for details, see, *e.g.*, Billingsley, 1968, Theorem 23.1; Glynn and Meyn, 1996, Theorem 2.3). Suppose f is bounded and measurable with $f \leq K_0$ for some $K_0 < \infty$. Then by taking $K \geq K_0^2$, we have $f^2 \leq V + K$ and hence the conclusion follows. □

4. Examples

For the following examples, we assume that $\{e_t\}$ is a sequence of *i.i.d.* random variables with mean zero. If there is no specification, let $Ee_t^2 = 1$. In each case, the corresponding Markov chain is assumed to be a Feller chain, if necessary.

Example 4.1. (NARMA). The classical nonlinear autoregressive moving average model with order p and q is given by

$$y_t = \phi(y_{t-1}, \ldots, y_{t-p}, e_{t-1}, \ldots, e_{t-q}) + e_t \tag{4.1}$$

with nonlinear measurable function $\phi : R^{p+q} \to R$. The above model has been studied by many authors such as Tong (1990), Tjøstheim (1990), An *et al.* (1996) and Lee (2000). Taking $r = s = 0$ in (2.3) yields (4.1). Theorem 2.1 ensures the existence of a strictly stationary solution of (4.1) under the assumption that for some $\lambda < 1$ and some constant θ,

$$|\phi(u_1, \ldots, u_p, z_1, \ldots, z_q)| \leq \lambda \max_{1 \leq i \leq p}\{|u_i|\} + \theta \max_{1 \leq i \leq q}\{|z_i|\}. \tag{4.2}$$

Note that if $|\phi(u_1, \ldots, u_p, z_1, \ldots, z_q)| \leq \sum_{i=1}^{p} \lambda_i |u_i| + \sum_{i=1}^{q} \theta_i |z_i|$ with $\sum_{i=1}^{p} \lambda_i < 1$, then (4.2) holds. Threshold ARMA(p,q) model is a special case of (4.1). In this case ϕ is given by

$$\phi(y_{t-1}, \ldots, y_{t-p}, e_{t-1}, \ldots, e_{t-q}) = \sum_{j=1}^{l} \left(\alpha_0^{(j)} + \sum_{i=1}^{p} \alpha_i^{(j)} y_{t-i} + \sum_{i=1}^{q} \beta_i^{(j)} e_{t-i} \right) I_{(a_{j-1} \leq y_{t-d} < a_j)}, \tag{4.3}$$

where $\alpha_i^{(j)}$ and $\beta_i^{(j)}$ are constants, $d \in \{1, \dots, p\}$, and $-\infty = a_0 < a_1 < \cdots < a_l = \infty$.

Applying Theorem 2.1 yields that $\max_j \sum_{i=1}^{p} |\alpha_i^{(j)}| < 1$ is sufficient for the existence of a strictly stationary solution of (4.3). Note that the stationarity condition does not depend on the coefficients of moving average part $\beta_i^{(j)}$, $1 \leq i \leq p$ and $1 \leq j \leq l$.

EXAMPLE 4.2. (NAR-threshold β ARCH errors). Suppose that $\{y_t\}$ is generated by

$$y_t = \phi(y_{t-1}, \dots, y_{t-p}) + \varepsilon_t, \tag{4.4}$$

$$\varepsilon_t = \sqrt{h_t} \cdot e_t, \tag{4.5}$$

$$h_t = \sum_{j=1}^{l} \left(\alpha_0^{(j)} + \sum_{i=1}^{r} \alpha_i^{(j)} \varepsilon_{t-i}^{2\beta} \right) I_{(\varepsilon_{t-d} \in [b_{j-1}, b_j))}, \tag{4.6}$$

where $d \in \{1, \dots, r\}$, $\alpha_0^{(j)} > 0$, $\alpha_i^{(j)} \geq 0$, $-\infty = b_0 < b_1 < \cdots < b_l = \infty$ and $0 < \beta < 1$. Suppose the function ϕ satisfies the assumption (A.1). Take $\alpha_i = \max_j \alpha_i^{(j)}$, $i = 0, 1, \dots, r$. Then

$$h_t \leq \alpha_0 + \sum_{i=1}^{r} \alpha_i \, |y_{t-i} - \phi(y_{t-i-1}, \dots, y_{t-i-p})|^{2\beta}$$

and

$$\begin{aligned} h(z_1, \dots, z_{p+r}) &\leq \sum_{i=1}^{r} \sqrt{\alpha_i} \, |z_i - \phi(z_{i+1}, \dots, z_{i+p})|^{\beta} \\ &\leq \sum_{i=1}^{r} \sqrt{\alpha_i} \, [|z_i|^{\beta} + \lambda^{\beta} \max_{1 \leq j \leq p} \{|z_{i+j}|^{\beta}\}]. \end{aligned}$$

Since $0 < \beta < 1$, $\{h(z_1, \dots, z_{p+r})\}/\|(z_1, \dots, z_{p+r})\| \to 0$ as $\|(z_1, \dots, z_{p+r})\| \to \infty$. Therefore if e_t satisfies the condition (A.4) with a lower semi-continuous density $q(\cdot)$, then $\{y_t\}$ is geometrically ergodic and $E_{\pi_1}|y_t|^m < \infty$.

EXAMPLE 4.3. (Double threshold ARMA-GARCH). The process $\{y_t\}$ is said to be a double threshold ARMA-GARCH model (DTARMACH) if it is defined by

$$y_t = \phi_0^{(j)} + \sum_{i=1}^{p} \phi_i^{(j)} y_{t-i} + \sum_{i=1}^{q} \theta_i^{(j)} \varepsilon_{t-i} + \varepsilon_t, \quad a_{j-1} \leq y_{t-b} < a_j, \tag{4.7}$$

$$\varepsilon_t = \sqrt{h_t} \cdot e_t, \tag{4.8}$$

$$h_t = \alpha_0^{(k)} + \sum_{i=1}^{r} \alpha_i^{(k)} \varepsilon_{t-i}^2 + \sum_{i=1}^{s} \beta_i^{(k)} h_{t-i}, \quad b_{k-1} \le \varepsilon_{t-d} < b_k, \tag{4.9}$$

where $j = 1, \ldots, l_1$, $k = 1, \ldots, l_2$, $-\infty = a_0 < \cdots < a_{l_1} = \infty$, $-\infty = b_0 < \cdots < b_{l_2} = \infty$, $\phi_i^{(j)}$, $\theta_i^{(j)}$, $\alpha_i^{(k)}$, $\beta_i^{(k)}$ are constants with $\alpha_0^{(k)} > 0$, $\alpha_i^{(k)} \ge 0$, $1 \le i \le r$, $\beta_j^{(k)} \ge 0$, $1 \le j \le s$. This model is studied in Li and Li (1996), Liu *et al.* (1997) and Ling (1999).

(4.7) can be rewritten as

$$y_t = \sum_{j=1}^{l_1} \left(\phi_0^{(j)} + \sum_{i=1}^{p} \phi_i^{(j)} y_{t-i} + \sum_{i=1}^{q} \theta_i^{(j)} \varepsilon_{t-i} \right) I_{(y_{t-b} \in [a_{j-1}, a_j))} + \varepsilon_t,$$

and hence

$$\left| \sum_{j=1}^{l_1} (\phi_0^{(j)} + \sum_{i=1}^{p} \phi_i^{(j)} y_{t-i}) I_{(y_{t-b} \in [a_{j-1}, a_j))} \right|$$
$$\le \max_{1 \le j \le l_1} \{|\phi_0^{(j)}|\} + \left(\max_{1 \le j \le l_1} \sum_{i=1}^{p} |\phi_i^{(j)}| \right) \cdot (\max\{|y_{t-1}|, \ldots, |y_{t-p}|\}).$$

On the other hand,

$$h_t \le \max_k \{\alpha_0^{(k)}\} + \sum_{i=1}^{r} \max_k \{\alpha_i^{(k)}\} \varepsilon_{t-i}^2 + \sum_{i=1}^{s} \max_k \{\beta_i^{(k)}\} h_{t-i} + c.$$

Therefore if $\max_j \sum_{i=1}^{p} |\phi_i^{(j)}| < 1$ and $\sum_{i=1}^{r} \max_k \{\alpha_i^{(k)}\} + \sum_{i=1}^{s} \max_k \{\beta_i^{(k)}\} < 1$, then the assumptions (A.1) and (A.2) hold and hence, by Theorem 2.1, a strictly stationary solution satisfying (4.7)–(4.9) exists. Compare this result with that of Ling (1999), where $\sum_{i=1}^{p} \max_j |\phi_i^{(j)}| < 1$ is required.

EXAMPLE 4.4. (MTAR with GARCH errors). The TAR model is one of the most widely used models to explain the asymmetric behaviors of economic and financial variables. But some authors pointed out that many economic variables are asymmetric in that they respond more sharply to negative shocks than to positive shocks. Enders and Granger (1998) proposed a modified version of the TAR model, the momentum TAR (MTAR) model given by

$$y_t = \begin{cases} \rho_1 y_{t-1} + \varepsilon_t, \ y_{t-1} - y_{t-2} \ge 0 \\ \rho_2 y_{t-1} + \varepsilon_t, \ y_{t-1} - y_{t-2} < 0, \end{cases} \tag{4.10}$$

where $\{\varepsilon_t\}$ is *i.i.d.* MTAR model is studied in Caner and Hansen (2001), Lee and Shin (2000), Shin and Lee (2001), *etc.* Consider the MTAR model with GARCH errors, that is $\{y_t\}$ is given by (4.9) where $\{\varepsilon_t\}$ is generated by (2.2) and (2.3). Here

$$\begin{aligned}|\phi(u_1, u_2)| &= |\rho_1 u_1 I_{\{u_1 - u_2 \geq 0\}} + \rho_2 u_1 I_{\{u_1 - u_2 < 0\}}| \\ &\leq \max\{|\rho_1|, |\rho_2|\} \cdot |u_1|.\end{aligned}$$

Therefore if $\max\{|\rho_1|, |\rho_2|\} < 1$ and (A.2) hold, then by Theorem 2.1, a strictly stationary solution of (4.10) exists.

REMARK 4.1. We can derive a sufficient condition for Feller continuity of each model considered in Example 4.1–4.4. For instance, if ϕ in (4.1) is continuous, $\{\mathbf{X}_t\} = (y_t, \ldots, y_{t-p+1}, e_t, \ldots, e_{t-q+1})$ has the Feller property. In particular, ϕ in (4.3) is continuous if for given $\mathbf{X}_{t-1}$,

$$\alpha_0^{(j)} + \sum_{i=1}^{p} \alpha_i^{(j)} y_{t-i} + \sum_{i=1}^{q} \beta_i^{(j)} e_{t-i} = \alpha_0^{(j+1)} + \sum_{i=1}^{p} \alpha_i^{(j+1)} y_{t-i} + \sum_{i=1}^{q} \beta_i^{(j+1)} e_{t-i}$$

holds whenever $y_{t-b} = a_j$, $j = 1, \ldots, l-1$. We can give analogous relations to ϕ and ψ in Example 4.3. Compare these conditions with the assumptions for Feller continuity given in Ling (1999) and Liu *et al.* (1997). Cline and Pu (1998, 2002) studied a process generated by a continuous nonlinear function and threshold ARMA(p,q) model, and found sufficient conditions for chains to be an aperiodic ψ-irreducible T-chain. Note that if the process is aperiodic ψ -irreducible T-chain, then every compact set is small.

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