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Fuzzy ideals of subtraction algebras

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Abstract

The notion of ideals in subtraction algebras and characterizations of ideals introduced by Y.B.Jun et al. [?]. Using this idea, we consider the fuzzification of an ideal of a subtraction algebra. In this paper, we define the concept of a fuzzy ideal of a subtraction algebra and study characterizations of a fuzzy ideal. We give some conditions to show that a fuzzy set in a subtraction algebra is a fuzzy ideal of a subtraction algebra. We investigate the generalization of some properties of a fuzzy ideal of a subtraction algebra.

Key words : subtraction algebra, ideal, fuzzy ideal, level ideal.

1. Introduction

B. M. Schein [?] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\setminus$" (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [?]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [?] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [?] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [?], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. In this paper, we consider the fuzzification of an ideal, and define a fuzzy ideal of a subtraction algebra. We give characterizations of a fuzzy ideal.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;

(S2) $x - (x - y) = y - (y - x)$;

(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [?], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [?]):

(a1) $(x - y) - y = x - y$.

(a2) $x - 0 = x$ and $0 - x = 0$.

(a3) $(x - y) - x = 0$.

(a4) $x - (x - y) \leq y$.

(a5) $(x - y) - (y - x) = x - y$.

(a6) $x - (x - (x - y)) = x - y$.

(a7) $(x - y) - (z - y) \leq x - z$.

(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.

(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.

(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.

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(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

Definition 2.1. ([?]) A nonempty subset A of a subtraction algebra X is called an *ideal* of X, denoted by $A \lhd X$, if it satisfies:

(b1) $a - x \in A$ for all $a \in A$ and $x \in X$.

(b2) for all $a, b \in A$, whenever $a \vee b$ exists in X then $a \vee b \in A$.

Proposition 2.2. ([?]) *A nonempty subset A of a subtraction algebra X is an ideal of X if and only if it satisfies:*

(b3) $0 \in A$.

(b4) $(\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$.

Lemma 2.3. *An ideal A of a subtraction algebra X has the following property:*

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

3. Fuzzy ideals

In what follows let X be a subtraction algebra unless otherwise specified. Using Proposition **??**, we give the notion of fuzzy ideal as follows.

Definition 3.1. A fuzzy set $\mathscr{A}$ in X is called a *fuzzy ideal* of X if it satisfies:

(c1) $(\forall x \in X)\ (\mathscr{A}(0) \geq \mathscr{A}(x))$,

(c2) $(\forall x, y \in X)\ (\mathscr{A}(x) \geq \min\{\mathscr{A}(x - y),\ \mathscr{A}(y)\})$.

Example 3.2. Consider a subtraction algebra $X = \{0, a, b, c, d\}$ with the following Cayley table:

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

(1) Let $\mathscr{A}$ be a fuzzy set in X defined by

$$\mathscr{A}(x) = \begin{cases} m & \text{if } x \in \{0, a, d\}, \\ n & \text{otherwise} \end{cases} \tag{3.1}$$

for all $x \in X$, where $m, n \in [0, 1]$ with $m > n$. Then $\mathscr{A}$ is a fuzzy ideal of X.

(2) Let $\mathscr{B}$ be a fuzzy set in X defined by $0 \leq \mathscr{B}(d) < \mathscr{B}(b) = \mathscr{B}(c) < \mathscr{B}(a) < \mathscr{B}(0) \leq 1$. Then $\mathscr{B}$ is a fuzzy ideal of X.

Proposition 3.3. *Every fuzzy ideal $\mathscr{A}$ in X satisfies:*

$$(\forall x, y \in X)\ (x \leq y \Rightarrow \mathscr{A}(x) \geq \mathscr{A}(y)). \tag{3.2}$$

Proof. If $x \leq y$, then $x - y = 0$, and so

$$\begin{aligned} \mathscr{A}(x) &\geq \min\{\mathscr{A}(x - y), \mathscr{A}(y)\} \\ &= \min\{\mathscr{A}(0), \mathscr{A}(y)\} \\ &= \mathscr{A}(y). \end{aligned}$$

This completes the proof. □

Proposition 3.4. *Every fuzzy ideal $\mathscr{A}$ of X satisfies the following inequality:*

$$(\forall x, y, z \in X)(\mathscr{A}(x-z) \geq \min\{\mathscr{A}((x-y)-z), \mathscr{A}(y)\}). \tag{3.3}$$

Proof. Using (c2) and (S3), we have

$$\begin{aligned} \mathscr{A}(x - z) &\geq \min\{\mathscr{A}((x - z) - y), \mathscr{A}(y)\} \\ &= \min\{\mathscr{A}((x - y) - z), \mathscr{A}(y)\} \end{aligned}$$

for all $x, y, z \in X$. □

We give conditions for a fuzzy set to be a fuzzy ideal.

Theorem 3.5. *If a fuzzy set $\mathscr{A}$ in X satisfies conditions* (c1) *and* (**??**)*, then $\mathscr{A}$ is a fuzzy ideal of X.*

Proof. Taking $z = 0$ in (**??**) and using (a2), we obtain

$$\begin{aligned} \mathscr{A}(x) &= \mathscr{A}(x - 0) \\ &\geq \min\{\mathscr{A}((x - y) - 0), \mathscr{A}(y)\} \\ &= \min\{\mathscr{A}(x - y), \mathscr{A}(y)\} \end{aligned}$$

for all $x, y \in X$. Hence $\mathscr{A}$ is a fuzzy ideal of X. □

Corollary 3.6. *Let $\mathscr{A}$ be a fuzzy set in X. Then $\mathscr{A}$ is a fuzzy ideal of X if and only if it satisfies conditions* (c1) *and* (**??**).

The following is a characterization of a fuzzy ideal of X.

Theorem 3.7. *Let $\mathscr{A}$ be a fuzzy set in X. Then $\mathscr{A}$ is a fuzzy ideal of X if and only if it satisfies the following conditions:*

$$(\forall x, y \in X)(\mathscr{A}(x - y) \geq \mathscr{A}(x)), \tag{3.4}$$

$$(\forall x, a, b \in X)(\mathscr{A}(x-((x-a)-b)) \geq \min\{\mathscr{A}(a), \mathscr{A}(b)\}). \tag{3.5}$$

Proof. Assume that $\mathscr{A}$ is a fuzzy ideal of X. Using (a3), (c1) and (c2), we get

$$\begin{aligned} \mathscr{A}(x - y) &\geq \min\{\mathscr{A}((x - y) - x), \mathscr{A}(x)\} \\ &= \min\{\mathscr{A}(0), \mathscr{A}(x)\} = \mathscr{A}(x) \end{aligned}$$

for all $x, y \in X$. Since

$$(x - ((x - a) - b)) - a = (x - a) - ((x - a) - b) \leq b,$$

it follows from (**??**) that $\mathscr{A}((x-((x-a)-b))-a) \geq \mathscr{A}(b)$ so from (c2) that

$\mathscr{A}(x-((x-a)-b))$
$\geq \min\{\mathscr{A}((x-((x-a)-b))-a), \mathscr{A}(a)\}$
$\geq \min\{\mathscr{A}(a), \mathscr{A}(b)\}$.

Conversely let $\mathscr{A}$ be a fuzzy set in X satisfying conditions (**??**) and (**??**). Taking $y = x$ in (**??**). Then $\mathscr{A}(0) = \mathscr{A}(x-x) \geq \mathscr{A}(x)$ for all $x \in X$. Using (**??**), we obtain

$$\begin{aligned} \mathscr{A}(x) &= \mathscr{A}(x-0) \\ &= \mathscr{A}(x-((x-y)-(x-y))) \\ &= \mathscr{A}(x-((x-(x-y))-y)) \\ &\geq \min\{\mathscr{A}(x-y), \mathscr{A}(y)\} \end{aligned}$$

for all $x, y \in X$. Hence $\mathscr{A}$ is a fuzzy ideal of X. □

Proposition 3.8. *Every fuzzy ideal $\mathscr{A}$ of X satisfies the following assertion:*

$$(\forall x, y \in X)(\exists x \vee y \Rightarrow \mathscr{A}(x \vee y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y)\}). \quad (3.6)$$

Proof. Suppose there exists $x \vee y$ for $x, y \in X$. Let w be an upper bound of x and y. Then $x \vee y = w-((w-y)-x)$ is the least upper bound for x and y (see [**?**]), and so $\mathscr{A}(x \vee y) = \mathscr{A}(w-((w-y)-x)) \geq \min\{\mathscr{A}(x), \mathscr{A}(y)\}$ by (**??**). This completes the proof. □

Let $\mathscr{A}$ be a fuzzy set in X. For any $w \in X$, we consider the set

$$\uparrow\mathscr{A}(w) := \{x \in X \mid \mathscr{A}(x) \geq \mathscr{A}(w)\}.$$

Obviously, $w \in \uparrow\mathscr{A}(w)$. If $\mathscr{A}$ is a fuzzy ideal of X, then $0 \in \uparrow\mathscr{A}(w)$ by (c1). The following is our question: *For a fuzzy set $\mathscr{A}$ in X satisfying* (c1)*, is $\uparrow\mathscr{A}(w)$ an ideal of X?* But the following example provides a negative answer, that is, there exists an element $w \in X$ such that $\uparrow\mathscr{A}(w)$ is not an ideal of X.

Example 3.9. Consider a subtraction algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Let $\mathscr{A}$ be a fuzzy set in X defined by $\mathscr{A}(0) = 0.8$, $\mathscr{A}(a) = 0.5$, $\mathscr{A}(b) = 0.7$, and $\mathscr{A}(c) = 0.3$. Note that $\mathscr{A}$ is not a fuzzy ideal of X since $\mathscr{A}(c) < \min\{\mathscr{A}(c-b), \mathscr{A}(b)\}$. Then $\uparrow\mathscr{A}(a) = \{0, a, b\}$ is not an ideal of X since $c-b = a \in \uparrow\mathscr{A}(a)$ and $b \in \uparrow\mathscr{A}(a)$, but $c \notin \uparrow\mathscr{A}(a)$. Note that $\uparrow\mathscr{A}(b) = \{0, b\}$ is an ideal of X.

We give conditions for the set $\uparrow\mathscr{A}(w)$ to be an ideal.

Theorem 3.10. *Let $w \in X$. If $\mathscr{A}$ is a fuzzy ideal of X, then $\uparrow\mathscr{A}(w)$ is an ideal of X.*

Proof. Recall that $0 \in \uparrow\mathscr{A}(w)$. Let $x, y \in X$ be such that $x-y \in \uparrow\mathscr{A}(w)$ and $y \in \uparrow\mathscr{A}(w)$. Then $\mathscr{A}(w) \leq \mathscr{A}(x-y)$ and $\mathscr{A}(w) \leq \mathscr{A}(y)$. Since $\mathscr{A}$ is a fuzzy ideal of X, it follows from (c2) that

$$\mathscr{A}(x) \geq \min\{\mathscr{A}(x-y),\ \mathscr{A}(y)\} \geq \mathscr{A}(w)$$

so that $x \in \uparrow\mathscr{A}(w)$. Therefore $\uparrow\mathscr{A}(w)$ is an ideal of X. □

Theorem 3.11. *Let $\mathscr{A}$ be a fuzzy set in X and $w \in X$. Then*

1. *If $\uparrow\mathscr{A}(w)$ is an ideal of X, then $\mathscr{A}$ satisfies the following implication for all $x, y, z \in X$,*

$$(\mathscr{A}(x) \leq \min\{\mathscr{A}(y-z)\mathscr{A}(z)\} \Rightarrow \mathscr{A}(x) \leq \mathscr{A}(y)). \quad (3.7)$$

2. *If $\mathscr{A}$ satisfies* (c1) *and* (**??**)*, then $\uparrow\mathscr{A}(w)$ is an ideal of X.*

Proof. (1) Assume that $\uparrow\mathscr{A}(w)$ is an ideal of X for each $w \in X$. Let $x, y, z \in X$ be such that $\mathscr{A}(x) \leq \min\{\mathscr{A}(y-z), \mathscr{A}(z)\}$. Then $y-z \in \uparrow\mathscr{A}(x)$ and $z \in \uparrow\mathscr{A}(x)$. It follows from (b4) that $y \in \uparrow\mathscr{A}(x)$, that is, $\mathscr{A}(x) \leq \mathscr{A}(y)$.

(2) Suppose that $\mathscr{A}$ satisfies (c1) and (**??**). For each $w \in X$, let $x, y \in X$ be such that $x-y \in \uparrow\mathscr{A}(w)$ and $y \in \uparrow\mathscr{A}(w)$. Then $\mathscr{A}(x-y) \geq \mathscr{A}(w)$ and $\mathscr{A}(y) \geq \mathscr{A}(w)$, which imply that $\mathscr{A}(w) \leq \min\{\mathscr{A}(x-y),\ \mathscr{A}(y)\}$. Using (**??**), we have $\mathscr{A}(w) \leq \mathscr{A}(x)$ and so $x \in \uparrow\mathscr{A}(w)$. Since $\mathscr{A}$ satisfies (c1), it follows that $0 \in \uparrow\mathscr{A}(w)$. Therefore $\uparrow\mathscr{A}(w)$ is an ideal of X. □

For any $\alpha \in [0,1]$, we know $U(\mathscr{A};\alpha) = \{x \in X | \mathscr{A}(x) \geq \alpha\}$([**?**]).

Theorem 3.12. *Let $\mathscr{A}$ be a fuzzy set in X. Then $\mathscr{A}$ is a fuzzy ideal of X if and only if it satisfies:*

$$(\forall \alpha \in [0,1])\,(U(\mathscr{A};\alpha) \neq \emptyset \Rightarrow U(\mathscr{A};\alpha) \lhd X).$$

Proof. It follows from the Transfer Principle (see [**?**, Theorem 2.1]). □

The ideals $U(\mathscr{A};\alpha)$, $\alpha \in [0,1]$, in Theorem **??** are called *level ideals* of $\mathscr{A}$.

Theorem 3.13. *Any ideal of X can be realized as a level ideal of some fuzzy ideal of X.*

Proof. Let A be an ideal of X and let $\mathscr{A}$ be a fuzzy set in X defined by

$$\mathscr{A}(x) = \begin{cases} \alpha & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

where α is a fixed number in $(0, 1)$. Then

$$U(\mathscr{A}; \beta) = \begin{cases} X & \text{if } \beta = 0, \\ A & \text{if } 0 < \beta \leq \alpha, \\ \emptyset & \text{if } \alpha < \beta \leq 1, \end{cases} \tag{3.9}$$

and so $U(\mathscr{A}; \beta) \lhd X$ whenever $U(\mathscr{A}; \beta) \neq \emptyset$ for all $\beta \in [0, 1]$. It follows from Theorem **??** that $\mathscr{A}$ is a fuzzy ideal of X and clearly $U(\mathscr{A}; \alpha) = A$. □

Proposition 3.14. *Let $\mathscr{A}$ be a fuzzy set in X. Then $\mathscr{A}$ is a fuzzy ideal of X if and only if it satisfies:*

$$(\forall x, y, z \in X)\,(x - y \leq z \Rightarrow \mathscr{A}(x) \geq \min\{\mathscr{A}(y), \mathscr{A}(z)\}). \tag{3.10}$$

Proof. Assume that $\mathscr{A}$ is a fuzzy ideal of X and let $x, y, z \in X$ be such that $x - y \leq z$. Then $\mathscr{A}(z) \leq \mathscr{A}(x - y)$ by Proposition **??**. It follows from (c2) that $\mathscr{A}(x) \geq \min\{\mathscr{A}(x - y), \mathscr{A}(y)\} \geq \min\{\mathscr{A}(y), \mathscr{A}(z)\}$. Conversely, suppose that $\mathscr{A}$ satisfies (**??**). Since $0 - y \leq y$ for all $y \in X$, we have

$$\mathscr{A}(0) \geq \min\{\mathscr{A}(y), \mathscr{A}(y)\}$$

by (**??**). Thus (c1) is valid. Since $x - (x - y) \leq y$ for all $x, y \in X$ by (a4), it follows from (**??**) that $\mathscr{A}(x) \geq \min\{\mathscr{A}(x - y), \mathscr{A}(y)\}$. Hence $\mathscr{A}$ is a fuzzy ideal of X. □

As a generalization of Proposition **??**, we have the following results.

Theorem 3.15. *If a fuzzy set $\mathscr{A}$ in X is a fuzzy ideal of X, then*

$$\prod_{i=1}^{n} x - w_i = 0 \Rightarrow \mathscr{A}(x) \geq \min\{\mathscr{A}(w_i) \mid i = 1, 2, \cdots, n\} \tag{3.11}$$

for all $x, w_1, w_2, \cdots, w_n \in X$, where

$$\prod_{i=1}^{n} x - w_i = (\cdots((x - w_1) - w_2) - \cdots) - w_n.$$

Proof. The proof is by induction on n. Let $\mathscr{A}$ be a fuzzy ideal of X. Propositions **??** and **??** show that the condition (**??**) is valid for $n = 1, 2$. Assume that $\mathscr{A}$ satisfies the condition (**??**) for $n = k$, that is,

$$\prod_{i=1}^{k} x - w_i = 0 \Rightarrow \mathscr{A}(x) \geq \min\{\mathscr{A}(w_i) \mid i = 1, 2, \cdots, k\}$$

for all $x, w_1, w_2, \cdots, w_k \in X$.
Let $x, w_1, w_2, \cdots, w_k, w_{k+1} \in X$ be such that $\prod_{i=1}^{k+1} x - w_i = 0$. Then

$$\mathscr{A}(x - w_1) \geq \min\{\mathscr{A}(w_j) \mid j = 2, 3, \cdots, k + 1\}.$$

Since $\mathscr{A}$ is a fuzzy ideal of X, it follows from (c2) that

$$\begin{aligned} \mathscr{A}(x) &\geq \min\{\mathscr{A}(x - w_1), \mathscr{A}(w_1)\} \\ &\geq \min\{\mathscr{A}(w_1), \min\{\mathscr{A}(w_j) \mid j = 2, 3, \cdots, k + 1\}\} \\ &= \min\{\mathscr{A}(w_i) \mid i = 1, 2, \cdots, k + 1\}. \end{aligned}$$

This completes the proof. □

Now we consider the converse of Theorem **??**.

Theorem 3.16. *Let $\mathscr{A}$ be a fuzzy set in X satisfying the condition (**??**). Then $\mathscr{A}$ is a fuzzy ideal of X.*

Proof. Note that $(\cdots((0 - \underbrace{x) - x) - \cdots) - x}_{n \text{ times}} = 0$ for all $x \in X$. It follows from (**??**) that $\mathscr{A}(0) \geq \mathscr{A}(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $x - y \leq z$. Then

$$0 = (x - y) - z = (\cdots(((x - y) - z) - \underbrace{0) - \cdots) - 0}_{n-2 \text{ times}},$$

and so $\mathscr{A}(x) \geq \min\{\mathscr{A}(y), \mathscr{A}(z), \mathscr{A}(0)\} = \min\{\mathscr{A}(y), \mathscr{A}(z)\}$. Hence, by Proposition **??**, we conclude that $\mathscr{A}$ is a fuzzy ideal of X. □

References

[1] J. C. Abbott, *Sets, Lattices and Boolean Algebras*, Allyn and Bacon, Boston 1986.

[2] G. Birkhoff, *Lattice Theory,* Amer. Math. Soc. Colloq. Publ., Vol. 25, second edition 1984; third edition, 1967, Providence.

[3] Y. B. Jun and H. S. Kim, *On ideals in subtraction algebras,* Sci. Math. Jpn. **24** (2007), no. 1, 129–134, bf:e2006, 1081–1086.

[4] Y. B. Jun, H. S. Kim and E. H. Roh, *Ideal theory of subtraction algebras,* Sci. Math. Jpn. **61**(2005), no. 3, 459–464, **:e-2004**, 397–402.

[5] Y. B. Jun and M. Kondo, *On transfer principle of fuzzy BCK/BCI-algerbas,* Sci. Math. Jpn. **59** (2004), no. 1, 35-40, **:e9,** 95–100.

[6] B. M. Schein, *Difference Semigroups,* Comm. in Algebra **20** (1992), 2153–2169.

[7] B. Zelinka, *Subtraction Semigroups,* Math. Bohemica, **120** (1995), 445–447.

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