Weight Functions for Notched Structures with Anti-plane Deformation

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Weight functions in fracture mechanics represent the stress intensity factors as weighted averages of the externally impressed boundary tractions and body forces. We extended the weight function theory for cracked linear elastic materials to calculate the notch stress intensity factor of a notched structure with anti-plane deformation. The well-known method of deriving weight functions by differentiation cannot be used for notched structures. By combining an appropriate singular field with a regular field, we derived weight functions for the notch stress intensity factor. Closed expressions of weight functions for notched cylindrical bodies are given as examples.

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NOMENCLATURE

c = notch stress intensity factor
$\vec{F}$ = body force density vector
K_I, K_{II}, K_{III} = stress intensity factors for modes I, II, and III
$\vec{n}$ = unit vector normal to S
$\vec{T}$ = surface traction vector
$\vec{W}_f$ = displacement of a fundamental field (weight function)
z , η = complex variables
ϕ_1, ϕ_2 = harmonic functions, *i.e.*, $\nabla^2\phi_1 = \nabla^2\phi_2 = 0$
μ = shear modulus

1. Introduction

Bueckner's weight function is useful in fracture mechanics because it gives a universal form for calculating stress intensity factors once it has been established. A weight function, which is the displacement of a fundamental field[1], is generally different for each body shape containing a crack. However, for a given geometry, the weight function can be used to compute the stress intensity factor for an arbitrary distribution of loads. We considered an elastic body V with a crack c (Fig. 1). The three stress intensity factors at a generic point Q, $K_I(Q)$, $K_{II}(Q)$, and $K_{III}(Q)$, are linear functions of the fields $\vec{F}$ and $\vec{T}$,

$$K_J(Q) = \int_V \vec{F}\cdot\vec{W}_f dV + \int_S \vec{T}\cdot\vec{W}_f dS \qquad (J = I, II, III) \tag{1}$$

The displacement field $\vec{W}_f$, or weight function, depends on the field location, Q, and the fracture mode, J, but is independent of $\vec{F}$ and $\vec{T}$. This independence of the loading condition makes weight function theory of considerable value. Several methods have been developed to calculate the weight function for a given problem.

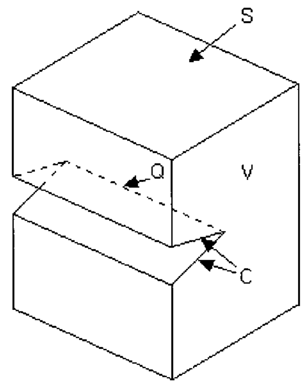

Fig. 1 Diagram of a cracked body

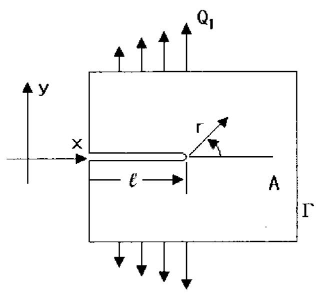

Fig. 2 Symmetric loading

For the case with a planar crack in which both the body and all load systems under consideration are symmetrical about the plane of the crack (Fig. 2), Rice derived a weight function by differentiating the displacement solutions with respect to the crack length.[2] Parks and Kamenetzky[3] and Vanderglas[4] used similar methods to virtually extend the crack and produce weight functions. If the loading condition is not symmetric with respect to the crack, or if the structure has a notch instead of a crack, the method of differentiation cannot be used. Because weight function theory is based on the reciprocal theorem, a weight function can be calculated wherever the reciprocal theorem can be applied. Bueckner[5] and An[6] showed that the weight function for a given geometry can be obtained by combining appropriate singular and rectangular solutions. Weight function theory is not restricted to cracked bodies; it can be extended to structures with notches in which the notch angle is not zero. For an anti-plane deformation, the governing equation becomes Laplace's or Poisson's equation. In this case, the reciprocal theorem can be derived from Maxwell and Betti's reciprocity theorem[7], or from the second form of Green's theorem[8],

$$\int_V (\phi_1\nabla^2\phi_2 - \phi_2\nabla^2\phi_1)dV = \int_S \phi_1\frac{\partial\phi_2}{\partial n}ds - \int_S \phi_2\frac{\partial\phi_1}{\partial n}ds \tag{2}$$

Choosing a suitable field ϕ_2, the weight functions for a given structure can be obtained. We demonstrate this procedure for an anti-plane case with longitudinal shear deformations.

2. Fundamental Results of Mode Ⅲ Deformation

Figure 3 shows a cylindrical body with generators parallel to the x_3 axis of a rectangular Cartesian coordinate system (x_1, x_2, x_3). The cross-section of the body in the (x_1, x_2) plane is bounded by a polygon O, A, B, C, D, where the sides OA and OD form the notch. We assume mirror symmetry with respect to the x_1 axis, as well as mode Ⅲ loading. The notch angle is $2(\pi - \gamma)$. In polar coordinates (r, θ), the two flanks OD and OA have polar angles of γ and $-\gamma$, respectively.

For mode Ⅲ loading, the only nonzero component of the displacement is w along axis x_3, where w is a function of the x_1 and x_2 coordinates only. As a consequence, all strain components vanish identically, except for the longitudinal shears. Within the context of the usual assumptions for an isotropic material and small deformations, all stresses vanish except the longitudinal shears. From Hooke's law, the components of the stress tensor become

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$$

$$\tau_{23} = \mu\frac{\partial w}{\partial x_2},\ \tau_{31} = \mu\frac{\partial w}{\partial x_1} \tag{3}$$

In the absence of body forces, the equilibrium condition requires

$$\frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} = 0 \tag{4}$$

From Eqs. (3) and (4),

$$\frac{\partial^2 w}{\partial x_1^{\,2}} + \frac{\partial^2 w}{\partial x_2^{\,2}} = 0 \tag{5}$$

i.e., w is harmonic. It is well known that the harmonic function can be described with the aid of $w(z)$, which is holomorphic in $z = x_1 + ix_2$ inside the polygon. The real and imaginary parts of $w(z)$ are harmonic. The imaginary part of $w(z)$ yields

$$\mu\ w = \mathrm{Im}\,\omega \tag{6}$$

for the displacement w in the x_3 direction. Using the Cauchy-Riemann equation

$$\frac{\partial\,\mathrm{Re}\,\omega}{\partial x_1} = \frac{\partial\,\mathrm{Im}\,\omega}{\partial x_2} \tag{7}$$

we obtain

$$\omega'(z) = \tau_{23} + i\tau_{31} \tag{8}$$

for the two shearing stresses τ_{23} and τ_{31}. From Eq. (8), it follows that the material to the right of an oriented line element ds in Fig. 3 applies a force $T\,ds$ to the material to the left of the element per unit length

$$Tds = \tau_{31}dx_2 - \tau_{23}dx_1 = -\mathrm{Re}\,d\omega \tag{9}$$

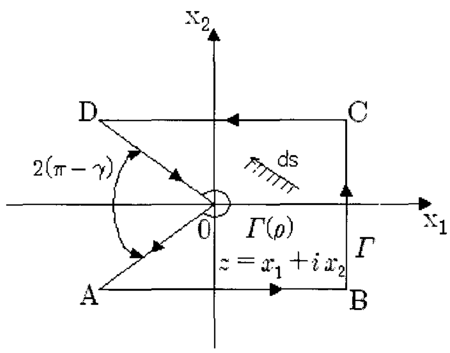

Fig. 3 Notched cylindrical body

3. Asymptotic Field Quantities near the Notch Tip and the Fundamental Field

For the notch, we assume for simplicity that the flanks bear no traction in the neighborhood of O. In this case, Eq. (9) implies that $\omega(z)$ has a constant real part on either flank for sufficiently small r. To verify this condition, we consider the special case

$$\omega(z) = \frac{2\sqrt{2}\,c}{\sqrt{\pi}}z^{\lambda} \tag{10}$$

with a real coefficient c and a real exponent λ. With $\mathrm{Re}\,\omega$ = a constant on either flank,

$$\cos\lambda\gamma = 0 \tag{11}$$

so that

$$\lambda = \pi / 2\gamma \tag{12}$$

gives the smallest possible positive λ. From this, we infer that the asymptotic behavior of a generic field of deformation is of the form

$$\omega(z) \sim \frac{2\sqrt{2}\,c}{\sqrt{\pi}}z^{\lambda},\ \lambda = \pi / 2\gamma \tag{13}$$

It follows that the stresses are asymptotically

$$\tau_{23} + i\tau_{31} \sim \lambda\frac{2\sqrt{2}\,c}{\sqrt{\pi}}z^{\lambda-1} \tag{14}$$

Thus, $\lambda < 1$ for $\gamma > \pi / 2$; in this case, the stresses are unbounded near $z = 0$ and we can designate the coefficient c in Eq. (13) as the notch stress intensity factor. If $\gamma = \pi$, the notch becomes a crack and the notch stress intensity factor c becomes a stress intensity factor K_{III}.[9] From Eq. (14), the stresses near the front of the crack tip become

$$\tau_{23} + i\tau_{31} \sim K_{III} \sqrt{2/\pi z}^{-1/2} \tag{15}$$

We use fundamental fields and weight functions to determine c. The λ and $-\lambda$ are simultaneous solutions to Eq. (11), and we define the *fundamental field* as any field derived from a holomorphic function $\Omega(z)$ within the polygon such that

$$\Omega(z) \sim z^{-\lambda} \text{ near } O \tag{16}$$

If λ is given by Eq. (12), the field will be *regular*.

4. Weight Function Formula

We apply the reciprocity theorem to fundamental and regular fields by excluding the interior of the cylinder $r = \rho$ shown in Fig. 3. The cross-section of the remaining body is bounded by a circular arc $\Gamma(\rho)$ of radius ρ, and by Γ of the polygon for which $r \geq \rho$. We distinguish the quantities of regular and fundamental fields by the subscripts r and f, respectively. The Maxwell and Betti reciprocity theorem can now be written in the form

$$\int_{\Gamma(\rho)} \left(w_r T_f - w_f T_r\right) ds = \int_{\Gamma} w_f T_r ds \tag{17}$$

The above equation can also be derived from Eq. (2). By setting

$$\phi_1 = \operatorname{Im}\omega, \ \phi_2 = \operatorname{Im}\Omega \tag{18}$$

the displacement and traction of regular and fundamental fields come from ϕ_1 and ϕ_2, respectively. Here, we have used $T_f = 0$ on Γ. Using Eqs. (9), (10), and (16), we treat the integral over $\Gamma(\rho)$ as follows:

$$\int_{\Gamma(\rho)} \left(w_r T_f - w_f T_r\right) ds$$

$$\approx \frac{2\sqrt{2}c}{\mu\sqrt{\pi}} \int_{\Gamma(\rho)} \left[\operatorname{Im} z^{-\lambda} d \operatorname{Re} z^{\lambda} - \operatorname{Im} z^{\lambda} d \operatorname{Re} z^{-\lambda}\right] \tag{19}$$

Integrating by parts with respect to $d \operatorname{Re} z^{-\lambda}$, we obtain

$$\frac{\mu\sqrt{\pi}}{2\sqrt{2}c} \int_{\Gamma(\rho)} \left(w_r T_f - w_f T_r\right) ds = \operatorname{Im} \int_{\Gamma(\rho)} z^{-\lambda} dz^{\lambda} = -\pi$$

In the limit as $\rho \to 0$, we may write

$$-\frac{2\sqrt{2\pi}}{\mu} c = \int_{\Gamma} w_f T_r ds \tag{20}$$

which represents the weight function formula for c. The above formula can be extended to regular fields with body forces.[5] Apart from arbitrary rigid body motion, the fundamental field is unique.

5. Examples

The fundamental field can be given in closed form for two special cases. For the infinite structure in Fig. 4, the fundamental field is

$$\Omega(z) = z^{-\lambda}, \ \lambda = \pi / 2\gamma \tag{21}$$

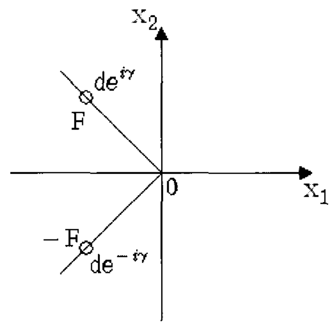

Fig. 4 Infinite structure with a notch

For the structure in Fig. 5, which consists of the portion of the structure in Fig. 4 where $r \leq a$,

$$\Omega(z) = z^{-\lambda} - \left(z / a^2\right)^{\lambda} \tag{22}$$

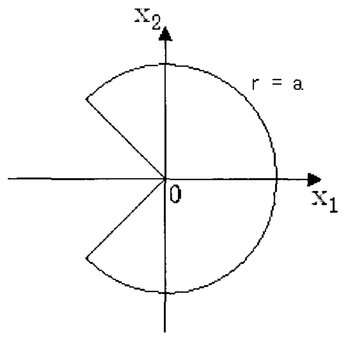

Fig. 5 Circular cross-section with a notch

For the case of Fig. 4, we consider a regular field responding to the concentrated forces F, $-F$ at the boundary points $z = de^{ir}$, $z = de^{-ir}$, respectively. Here, $F > 0$ indicates a force in the direction of the positive x_3 axis. From Eqs. (6) and (21),

$$\mu w_f = d^{-\lambda} \sin(-\lambda\gamma) \text{ at } z = de^{ir} \tag{23}$$

This and Eq. (20) lead to

$$c = \frac{1}{\sqrt{2\pi}} d^{-\lambda} F \tag{24}$$

for the notch stress intensity factor of the regular field. The same result can also be obtained by a direct analysis of the regular field. In our particular case, we consider the following transformation:

$$\eta = \left(\frac{z}{d}\right)^{\lambda} \tag{25}$$

This maps the shape of Fig. 4 in the z plane onto the right half-plane of Re $\eta > 0$. Considering the traction discontinuity along the flanks of the cylindrical body given by Eq. (9), the holomorphic function

$\omega(\eta)$ can be found in closed form,

$$\omega(\eta) = \frac{iF}{\pi} \log \frac{\eta + i}{\eta - i} \tag{26}$$

in the complex η plane. Expanding Eq. (26) near $z = 0$ gives

$$\omega(z) = \frac{iF}{\pi} \left[\log(-1) - i\, 2\left(\frac{z}{d}\right)^{\lambda} + \cdots \right] \tag{27}$$

and we can confirm the value of c given by Eq. (24).

6. Conclusion

The well-established weight function theory for cracked linear elastic materials was extended for notched structures with anti-plane deformation. We constructed the fundamental fields directly by combining an appropriate singular field with a regular field. Closed forms of weight functions were given for cylindrical-shaped structures as examples.

ACKNOWLEDGEMENTS

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