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The Approximate MLE in a Skew-Symmetric Laplace Distribution

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Abstract

We define a skew-symmetric Laplace distribution by a symmetric Laplace distribution and evaluate its coefficient of skewness. And we derive an approximate maximum likelihood estimator(AME) and a moment estimator(MME) of a skewed parameter in a skew-symmetric Laplace distribution, and hence compare simulated mean squared errors of those estimators. We compare asymptotic mean squared errors of two defined estimators of reliability in two independent skew-symmetric distributions.

Keywords: Approximate MLE, Reliability, Right-Tail Probability, Skew-Symmetric Laplace Distribution

1. Introduction

Many authors had studied estimation and characterization in a symmetric Laplace distribution in Johnson et al(1995). Ali and Woo(2006) and Woo(2006) studied moments of several skew-symmetric reflected distributions which were derived from symmetric distributions about origin, and Woo(2007) studied reliability in a half-triangle distribution and a skew-symmetric distribution,

Balakrishnan(1989) proposed the approximate MLE of the scale parameter in the Rayleigh distribution. Han & Kang(2006) studied the approximate maximum likelihood estimator(AMLE) of parameters in several distributions with censored samples.

It's not easy for us to estimate a skewed parameter α in a skew-symmetric distribution, so we'd like to consider in estimating the skewed parameter in a

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skew-symmetric Laplace distribution based on a method of finding the approximate MLE.

In this paper we define a skew-symmetric Laplace distribution by a symmetric Laplace distribution and evaluate its coefficient of skewness. And we derive an AMLE and a moment estimator(MME) of a skewed parameter in a skew-symmetric Laplace distribution, and hence compare simulated mean squared errors of those estimators. We compare asymptotic mean squared errors of two defined estimators of the reliability in two independent skew-symmetric distributions.

2. A skew-symmetric Laplace distribution

Let X and Y be two independent identical distributed continuous random variables with the probability density function(pdf) $f(x)=F'(x)$ which is symmetric about $\theta \in R^1$. Then

$$For\ \forall \alpha \in R^1, \quad \frac{1}{2} = P\{(X-\theta)-\alpha(Y-\theta) \leq 0\}$$
$$= \int_{-\infty}^{\infty} f(t)F(\theta+\alpha(t-\theta))dt$$

$$\text{Therefore, } f(z;\alpha) \equiv 2f(z)F(\alpha(z-\theta)+\theta), \tag{2.1}$$

where α is called a skewed parameter of the distribution.

The density $f(z;\alpha)$ becomes a skewed density of a random variable Z derived from a symmetric distribution which is symmetric about $\theta \in R^1$(see Ali and Woo(2006)). Especially if $\alpha=0$, $then\, f(z;0)$ becomes the original symmetric Laplace density.

From the skewed density (2.1), the cdf of the skewed random variable Z with the density (2.1) is given by:

$$F(z;\alpha) = 2\int_{-\infty}^{z} f(t)\int_{-\infty}^{\alpha(t-\theta)+\theta} f(s)\,ds\,dt. \tag{2.2}$$

2-1. Distribution of a skew-symmetric Laplace random variable

From the density (2.1) and the cdf of Laplace in Johnson et al (1995), we obtain a skew-symmetric Laplace density as below:

$$f(z;\alpha)=\frac{1}{2\beta}\cdot e^{-\frac{|z-\theta|}{\beta}}[1+sgn(\alpha(z-\theta))\cdot(1-e^{-\frac{|\alpha(z-\theta)|}{\beta}})] \quad z\in R^1,\ \alpha\in R^1 \quad (2.3)$$

$$where \quad sgn(x)=\begin{cases}1, & \text{if } x>0\\ -1, & \text{if } x<0\end{cases}.$$

From the cdf (2.2) and the density of Laplace in Johnson et al (1995), the cdf of a skew-symmetric Laplace random variable Z is given by:

$$F(z;\alpha)=\frac{1}{2}[1+sgn(z-\theta)\cdot(1-e^{-\frac{|(z-\theta)|}{\beta}})]+\frac{sgn(\alpha)}{2}\cdot[\frac{1}{1+\alpha}e^{-\frac{1+\alpha}{\beta}|z-\theta|}-e^{-\frac{|z-\theta|}{\beta}}] \quad (2.4)$$

From the density (2.3) and the formula 3.381(4) in Gradshteyn & Ryzhik(1965, p.317), we obtain the moment generating function of the skew-symmetric Laplace random variable Z:

$$m_Z(t;\alpha)=\frac{1}{2}e^{\theta t}\left[\frac{2}{1-sgn(\alpha)\cdot\beta t}+\frac{sgn(\alpha)}{2}\cdot\left(\frac{1}{1+|\alpha|+\beta t}-\frac{1}{1+|\alpha|-\beta t}\right)\right], \text{ if } |t|<\frac{1}{\beta} \quad (2.5)$$

From the density (2.3) and the formulas 3.2 & 3.3 in Oberhettinger(1974, p.25), the k-th moment of the skew-symmetric Laplace random variable Z can be guaranteed as the following:

$$\begin{aligned}E(Z^k;\alpha)=\beta^k\cdot[2\cdot e^{sgn(\alpha)\cdot\theta/\beta}\cdot(\psi(\alpha)\cdot\Gamma(k+1;\theta/\beta)+\psi(-\alpha)\cdot\gamma(k+1;\theta/\beta))+\\ sgn(\alpha)\cdot((1+|\alpha|)^{-k-1}\cdot e^{-(1+|\alpha|)\theta/\beta}\cdot\gamma(k+1;(1+|\alpha|)\theta/\beta)-\\ (1+|c|)^{-k-1}\cdot e^{(1+|c|)\theta/\beta}\cdot\Gamma(k+1;(1+|c|)\theta/\beta))], \quad (2.6)\end{aligned}$$

$$where\ \psi(\alpha)=\begin{cases}1, & \text{if } \alpha>0\\ 0, & \text{if } \alpha<0\end{cases},$$

$$\gamma(a,x)\equiv\int_0^x e^{-t}\cdot t^{a-1}dt,\ \Gamma(a,x)\equiv\int_x^\infty e^{-t}\cdot t^{a-1}dt.$$

From now on we only consider the density (2.3) with $\theta=0$ and $\beta=1$ to have an emphasis primarily on estimating a skewed parameter α in the density (2.3), and $\theta=0$ and $\beta=1$ in the density (2.3) are assumed to get a simulation easily.

When $\theta=0$ and $\beta=1$ in the density (2.3), from k-th moment (2.6) of Z, the first, second, and third moments of Z for $\alpha>0$ are given by:

$$E[Z]=1-\frac{1}{(\alpha+1)^2},\ E[Z^2]=2,\ \text{and } E[Z^3]=6\left(1-\frac{1}{(1+\alpha)^4}\right). \quad (2.7)$$

And from the moments (2.7) and for $\alpha < 0$, $E(Z^k;\alpha) = (-1)^k E(Z^k;-\alpha)$ in Ali and Woo(2006), we obtain the following means, variances, and skewness as below:

If $\alpha = \pm 1$, then mean=± 0.75, variance=1.4375 and skewness=± 1.14230.

If $\alpha = \pm 2$, then mean=± 0.88889, variance=1.20988 and skewness=± 1.50080.

If $\alpha = \pm 5$, then mean=± 0.97222, variance=1.05478 and skewness=± 1.84619.

If $\alpha = \pm 10$, then mean=± 0.99174, variance=1.01646 and skewness=± 1.95161.

From these coefficients of skewness, we observe the following Fact 1:

Fact 1. Let $\theta = 0$ and $\beta = 1$ in the skew-symmetric Laplace density (2.3). Then

(a) the skew-symmetric Laplace density (2.3) is skewed to the right when $\alpha > 0$, and the density is skewed to the left when $\alpha < 0$.

(b) the density (2.3) is more skewed as the absolute values of α are larger.

2-2. Estimating a skewed parameter α

We consider estimation of α in a skew-symmetric Laplace distribution when the skewed parameter α is positive. because of the results in Ali and Woo(2006).

Assume $Z_1, Z_2, \cdots, Z_n$ be an iid random variables with the density (2.3) with $\theta = 0$ and $\beta = 1$ of Z. Then, by method of finding AMLE of a parameter in a distribution in Balakrishnan and Cohen (1991), from log-likelihood function of α and Taylor series, we can obtain the AMLE $\hat{\alpha}$ of α as the following:

For $Z_1 = z_1, Z_2 = z_2, \ldots, Z_n = z_n$, the likelihood function is

$$f(\alpha; z_1, \ldots, z_n) = \left(\frac{1}{2}\right)^n e^{-\sum |z_i|} \cdot \prod_{i=1}^{n} \left[1 + \alpha \cdot sgn(z_i)\left(1 - e^{-\alpha|z_i|}\right)\right], \text{ and}$$

$$\frac{d \ln f(\alpha; z_1, \ldots, z_n)}{d\alpha} = \sum_{i=1}^{n} \frac{sgn(z_i)\left(1 - e^{-\alpha|z_i|} + \alpha|z_i| \cdot e^{-\alpha|z_i|}\right)}{1 + \alpha \cdot sgn(z_i)\left(1 - e^{-\alpha|z_i|}\right)} = 0,$$

and hence, as taking first two terms in an expansion of Taylor series of $g(\alpha \mid z_i)$ about $\alpha = c$, we obtain the following:

$$g(\alpha \mid z_i) \approx g(c \mid z_i) + g'(c \mid z_i)(\alpha - c) = 0, \quad i = 1, 2, \ldots, n,$$

where c is any real number. $$g(c \mid z_i) \equiv \frac{sgn(z_i) \cdot \left\{1 - e^{-c|z_i|} + c|z_i| \cdot e^{-c|z_i|}\right\}}{1 + c \cdot sgn(z_i)\left(1 - e^{-c|z_i|}\right)}, \quad (2.8)$$

and $$g'(c \mid z_i) = \frac{A}{\left\{1 + c \cdot sgn(z_i) - c \cdot sgn(z_i) \cdot e^{-c|z_i|}\right\}^2},$$

$$A \equiv sgn(z_i)\left[\left\{2(sgn(z_i) + |z_i|) - c(1 - c \cdot sgn(z_i))z_i^2\right\} \cdot e^{-c|z_i|} - sgn(z_i) \cdot e^{-2c|z_i|} - sgn(z_i)\right]$$

Therefore, we obtain the AMLE $\hat{\alpha}$ of α as below:

$$\hat{\alpha} = \frac{\sum_{i=1}^{n}\left\{c g'(c \mid Z_i) - g(c \mid Z_i)\right\}}{\sum_{i=1}^{n} g'(c \mid Z_i)} \quad (2.9)$$

And also, from the density (2.3) with $\theta=0, \beta=1$, we can obtain the moment estimator(MME) $\tilde{\alpha}$ of α:

$$\tilde{\alpha}=\frac{1}{\sqrt{1-\frac{1}{n}\sum_{i=1}^{n} Z_i}}-1, \quad \text{if } \frac{1}{n}\sum_{i=1}^{n} Z_i<1. \tag{2.10}$$

To simulate MSEs of two estimators (2.9) & (2.10), by the cdf (2.4) with $\theta=0$ and $\beta=1$ it can be obtained that a transformed random variable U follows a uniform distribution over (0.1). By the computer 1000-simulations based on the uniform dis- tribution, in the Appendix, Table 1 shows 1000-averages and mean squared errors of AMLE $\hat{\alpha}$ when n=10(10)30, $\alpha=1.0$, and c= 0,8, 0.9. 1.0, 1.1, and 1.2, and MME $\tilde{\alpha}$ in skew-symmetric Laplace distribution with $\theta=0$ and $\beta=1$ when n=10(10)30 and $\alpha=1.0$. Table 2 shows 1000-averages and mean squared errors of AMLE $\hat{\alpha}$ when n=10(10)30, $\alpha=1.0$, and c= 0,8, 0.9. 1.0, 1.1, and 1.2, and MME $\tilde{\alpha}$ in skew- symmetric Laplace distribution with $\theta=0$ and $\beta=1$ when n=10(10)30 and $\alpha=2.0$. From Tables 1 & 2 in the Appendix, we observe the following Fact 2:

Fact 2. For the density (2.3) with $\theta=0$, $\beta=1$, when the true values α=1, 2 in (2.3),
(a) mean squared errors of AMLE are smaller as values of c approach near to the true value of α.
(b) the AMLE performs better than the MME in a sense of simulated mean squared error.
(c) the average values of the MME are underestimated,

Remark 1. In practical data problem, the AMLE $\hat{\alpha}$ in (2.9) would be recommended more favorable estimator of α than the MME $\bar{\alpha}$. as the value c in (2.8) is replaced by the moment estimate $\bar{\alpha}$ in (2.10).

2-3. Reliability

Here we consider estimation of the right-tail probability of the skew-symmetric Laplace variate with density (2.3), and consider estimation of reliability of two independent skew-symmetric Laplace random variables with different skewed parameters α_1 and α_2.

First we consider estimation of the right-tail probability of the skew-symmetric Laplace variate. From the cdf (2.4) of the skew-symmetric Laplace variable Z, the right-tail probability $R(t;\alpha)=P(Z>t)$ of Z is given by:

$$R(t;\alpha)=\frac{1}{2}[1-sgn(t-\theta)\cdot(1-e^{-\frac{|(t-\theta)|}{\beta}})]-\frac{sgn(\alpha)}{2}\cdot[\frac{1}{1+\alpha}e^{-\frac{1+\alpha}{\beta}|t-\theta|}-e^{-\frac{|t-\theta|}{\beta}}] \tag{2.11}$$

Let $\theta=0$ and $\beta=1$. Then $dR(t;\alpha)/d\alpha$ is positive for $\alpha>0$, and so if $\alpha>0$, then the right-tail probability $R(t;\alpha)$ is a monotone increasing function of $\alpha>0$, and hence, because inference on $R(t;\alpha)$ is equivalent to inference on $\alpha>0$(see McCool(1991)), from the AMLE (2.9) and MME (2.10) of α and Fact 2(b), we obtain the following:

Fact 3. If $\alpha>0$ in the density (2.3), then the estimator $\hat{R}(t;\alpha)\equiv R(t;\hat{\alpha})$ performs better than $\tilde{R}(t;\alpha)\equiv R(t;\tilde{\alpha})$ in a sense of approximate MSE.

Remark 2. If $\alpha<0$, then, from $R(t;\alpha)=1-R(-t;-\alpha)$ in Ali and Woo(2006), $R(t;\alpha)$ is a monotone decreasing function of α, and therefore, as its estimation can be applied to the result of McCool(1991) by the same method, we obtain the same result of Fact 3.

Next we consider estimation of reliability of two independent skew-symmetric Laplace random variables. Let Z and W be independent skew-symmetric Laplace random variables each having the density (2.3) with $\theta=0$, $\beta=1$ and two different skewed parameters α_1 and α_2, respectively. Then, from the density $f(x;\alpha)$ in (2.3), the cdf $F(x;\alpha)$ in (2.4), and the formula 3.381(4) in Gradshteyn & Ryzhik(1965, p.317) we obtain the reliability P(Z<W) : For $\alpha_i>0$, i=1 and 2,

$$\begin{aligned}R(\alpha_1,\alpha_2)\equiv P(Z<W)&=\int_0^\infty (f(x;\alpha_2)F(x;\alpha_1)+f(-x;\alpha_2)F(-x;\alpha_1))dx\\&=\frac{1}{2}+\frac{1}{2}\rho,\end{aligned} \tag{2.12}$$

where $\rho\equiv\frac{1}{1+\alpha_1}-\frac{1}{1+\alpha_2}+\frac{1}{2+\alpha_2}-\frac{1}{2+\alpha_1}$.

Especially if Z and W are identical random variables with $\alpha_1=\alpha_2$, then it is no wonder that the reliability is 1/2.

From (2.12), since the reliability $R(\alpha_1,\alpha_2)$ is a monotone function of ρ, and hence, because inference on $R(\alpha_1,\alpha_2)$ is equivalent to inference on ρ (see McCool(1991)).

it's sufficient for us to consider estimation of ρ instead of estimating reliability $R(\alpha_1,\alpha_2)$.

Assume $Z_1,Z_2,\ldots,Z_n$ and $W_1,W_2,\ldots,W_m$ be two independent samples each having the density $f(z;\alpha_1)$ and $f(z;\alpha_2)$ in the density (2.3) with $\theta=0$ and $\beta=1$, respectively. Then, by using the AMLE $\hat{\alpha}$ and MME $\tilde{\alpha}$ of α in (2.9) and (2.10). the following two proposed estimators of ρ in the reliability $R\equiv R(\alpha_1,\alpha_2)$ are defined by:

$$\hat{\rho}\equiv \frac{1}{1+\widehat{\alpha_1}}-\frac{1}{1+\widehat{\alpha_2}}+\frac{1}{2+\widehat{\alpha_2}}-\frac{1}{2+\widehat{\alpha_1}} \text{ and } \tilde{\rho}\equiv \frac{1}{1+\widetilde{\alpha_1}}-\frac{1}{1+\widetilde{\alpha_2}}+\frac{1}{2+\widetilde{\alpha_2}}-\frac{1}{2+\widetilde{\alpha_1}}, \tag{2.13}$$

where $\widehat{\alpha_1}=\dfrac{\sum_{i=1}^{n}\{c_1 g'(c_1\mid Z_i)-g(c_1\mid Z_i)\}}{\sum_{i=1}^{n} g'(c_1\mid Z_i)}$, $\widehat{\alpha_2}=\dfrac{\sum_{i=1}^{m}\{c_2 g'(c_2\mid W_i)-g(c_2\mid W_i)\}}{\sum_{i=1}^{m} g'(c_2\mid W_i)}$

$\widetilde{\alpha_1}=\dfrac{1}{\sqrt{1-\frac{1}{n}\sum_{i=1}^{n}Z_i}}-1$, and $\widetilde{\alpha_2}=\dfrac{1}{\sqrt{1-\frac{1}{m}\sum_{i=1}^{m}W_i}}-1$, if $\frac{1}{n}\sum_{i=1}^{n}Z_i<1$ and $\frac{1}{m}\sum_{i=1}^{m}W_i<1$.

To simulate MSEs of two estimators (2.13), by the cdf (2.4) with $\theta=0$ and $\beta=1$ it can be obtained that a transformed random variable U follows a uniform distribution over (0.1). By the computer 1000-simulations based on the uniform distribution, in the Appendix, Table 3 shows asymptotic mean squared errors of two estimators $\hat{\rho}$ and $\tilde{\rho}$ in the reliability $R\equiv R(\alpha_1,\alpha_2)$ in the skew-symmetric Laplace distribution with $\theta=0$ and $\beta=1$ when n and m are 10(10)30, $\alpha_i=1.0$, 2.0, and $c_i=\alpha_i$ for each i=1 and 2 on account of the result in Fact 2(a). From Table 3 in the Appendix and equivalence between inferences on $R(\alpha_1,\alpha_2)$and ρ in McCool(1991), we observe the following Fact 4:

Fact 4. For the density (2.3) with $\theta=0$, $\beta=1$, when (α_1,α_2)=(1,2) and (2,1) in (2.3), then (a) the $\hat{R}=R(\widehat{\alpha_1},\widehat{\alpha_2})$ of the reliability $R\equiv R(\alpha_1,\alpha_2)$ performs better than another estimator $\tilde{R}=R(\widetilde{\alpha_1},\widetilde{\alpha_2})$ in a sense of simulated mean squared error.

(b) absolute average values of $\hat{R}=R(\widehat{\alpha_1},\widehat{\alpha_2})$ are overestimated when n=10,20,30, but absolute average values of $\tilde{R}=R(\widetilde{\alpha_1},\widetilde{\alpha_2})$ are underestimated when n=10 and 20.

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Appendix

<Table 1> Simulated MSEs of AMLE and MME of skewed parameter α in the skew-symmetric Laplace distribution with $\theta=0$ and $\beta=1$ when $\alpha=1.0$.

(a) $c=0.8$

sample size	Average value		MSE	
	AMLE	MME	AMLE	MME
10	0.819945	0.726052	0.062317	0.133528
20	0.868509	0.810714	0.017290	0.094128
30	0.904596	0.865238	0.009102	0.054750

(b) $c=0.9$

sample size	Average value		MSE	
	AMLE	MME	AMLE	MME
10	0.915713	0.726052	0.034308	0.133528
20	0.932200	0.810714	0.015502	0.094128
30	0.970051	0.865238	0.006238	0.054750

(c) $c=1.0$

sample size	Average value		MSE	
	AMLE	MME	AMLE	MME
10	1.048566	0.726052	0.015827	0.133528
20	1.046368	0.810714	0.004493	0.094128
30	1.002073	0.865238	0.002645	0.054750

(d) $c = 1.1$

sample size	Average value		MSE	
	AMLE	MME	AMLE	MME
10	1.105936	0.726052	0.023296	0.133528
20	1.066475	0.810714	0.010803	0.094128
30	1.043508	0.865238	0.003654	0.054750

(e) $c = 1.2$

sample size	Average value		MSE	
	AMLE	MME	AMLE	MME
10	1.170049	0.726052	0.058615	0.133528
20	1.113337	0.810714	0.018471	0.094128
30	1.087120	0.865238	0.007590	0.054750

<Table 2> Simulated MSEs of AMLE and MME of skewed parameter α in the skew-symmetric Laplace distribution with $\theta = 0$ and $\beta = 1$ when $\alpha = 2.0$.

(a) $c = 1.8$

sample size	Average value		MSE	
	AMLE	MME	AMLE	MME
10	1.599224	1.725991	0.139233	0.147498
20	1.602906	1.784652	0.086463	0.091317
30	1.612423	1.843387	0.027183	0.037844

(b) $c = 1.9$

sample size	Average value		MSE	
	AMLE	MME	AMLE	MME
10	1.730581	1.725991	0.089952	0.147498
20	1.752528	1.786850	0.064151	0.091317
30	1.772059	1.843387	0.033822	0.037844

(c) $c = 2.0$

sample size	Average value		MSE	
	AMLE	MME	AMLE	MME
10	1.895278	1.725991	0.036619	0.147498
20	1.935906	1.784652	0.008659	0.091317
30	1.980910	1.843387	0.001415	0.037844

(d) $c = 2.1$

sample size	Average value		MSE	
	AMLE	MME	AMLE	MME
10	2.099374	1.725991	0.038116	0.147498
20	2.074622	1.784652	0.009850	0.091317
30	2.055158	1.843387	0.006954	0.037844

(e) $c = 2.2$

sample size	Average value		MSE	
	AMLE	MME	AMLE	MME
10	2.199453	1.725991	0.040497	0.147498
20	2.156158	1.784652	0.024385	0.091317
30	2.105147	1.843387	0.011056	0.037844

<Table 3> Simulated MSEs of $\hat{\rho}$ and $\tilde{\rho}$ in the skew-symmetric Laplace distribution.

(a) $\alpha_1 = 1.0 = c_1$, $\alpha_2 = 2.0 = c_2$, $\rho = \frac{1}{12} = 0.083333$

sample size		Average value		MSE	
n	m	$\hat{\rho}$	$\tilde{\rho}$	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.110188	0.152661	0.000198	0.004726
	20	0.103902	0.136256	0.000145	0.002790
	30	0.093949	0.104644	0.000103	0.000954
20	10	0.094002	0.106991	0.000176	0.000560
	20	0.090052	0.100861	0.000114	0.000307
	30	0.087673	0.090179	0.000019	0.000147
30	10	0.099017	0.105663	0.000299	0.000546
	20	0.087898	0.098854	0.000030	0.000241
	30	0.085544	0.091496	0.000005	0.000067

(b) $\alpha_1 = 2.0 = c_1$, $\alpha_2 = 1.0 = c_2$, $\rho = -\frac{1}{12} = -0.083333$

sample size		Average value		MSE	
n	m	$\hat{\rho}$	$\tilde{\rho}$	$\hat{\rho}$	$\tilde{\rho}$
10	10	-0.103546	-0.030803	0.000409	0.002759
	20	-0.095714	-0.054029	0.000310	0.000859
	30	-0.093561	-0.061664	0.000105	0.000570
20	10	-0.100162	-0.050467	0.000215	0.001080
	20	-0.094667	-0.056784	0.000162	0.000705
	30	-0.092206	-0.065673	0.000079	0.000312
30	10	-0.096351	-0.101860	0.000169	0.000643
	20	-0.090103	-0.096875	0.000046	0.000377
	30	-0.084525	-0.093595	0.000001	0.000105