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On Reliability and UMVUE of Right-Tail Probability in a Half-Normal Variable

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Abstract

We consider parametric estimation in a half-normal variable and a UMVUE of its right-tail probability. Also we consider estimation of reliability in two independent half-normal variables, and derive k-th moment of ratio of two same variables.

Keywords : Reliability, Right-Tail Probability, UMVUE

1. Introduction

For two independent random variables X and Y, and any real number "c", the probability $P(Y<cX)$ induces the following facts: (i) the probability $P(Y<cX)$ is the reliability when the real number "c" equals one, (ii) the probability $P(Y<cX)$ is the distribution of the ratio $Y/(X+Y)$ when $c=t/(1-t)$ for $0<t<1$, and (iii) the probability $P(Y<cX)$ induces the density of a skewed-symmetric random variable if X and Y are symmetric random variables about origin.

Many authors considered inferences on reliability in several distributions. Especially Ali and Woo(2005a) had studied inferences on the reliability in a Levy distribution And Ali et al (2005b & 2006) had studied the ratio of two independent generalized uniform random variables and power function random variables. And in recent, Kim(2006a&b)), Lee(2006), and Lee & Won(2006) studied inferences on the reliability in an exponentiated uniform distribution and an exponential distribution.

A half-normal distribution has been introduced in Johnson et al(1994) as the following density:

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$$f(x)=\sqrt{\frac{2}{\pi\beta}}\,exp(-\frac{x^2}{2\beta}),\quad x>0\quad where\ \beta>0 \tag{1.1}$$

which the random variable has been applied to a truncated normal random variable only when it has a positive value in a normal random variable.

In this paper we consider parametric estimation in a half-normal variate X with the density (1.1) and a UMVUE of its right-tail probability. And we consider estimation of reliability P(Y<X) in two independent half-normal variables X and Y each having the density (1.1) with different parameters β_1 and β_2, respectively, and derive k-th moment of ratio R=Y/(X+Y) of two same variables.

2. Estimation of parameter

The k-th moment and moment generating function of a half-normal random variable with the density (1.1) have been known as the followings:
From the density (1.1) and the formula 5.42 in Oberhettinger and Badii(1973, p.43), the moment generating function of the half normal random variable can be obtained by:

$$m_X(t)=\exp(\beta t^2)\cdot Erfc(-t\sqrt{\beta/2}),\qquad -\infty<t<\infty \tag{2.1}$$

where $Erfc(x)=\frac{2}{\sqrt{\pi}}\int_0^{\infty}e^{-t^2}dt$ is an error function.

From the density (1.1) and the formula 3.381(4) in Gradshteyn & Ryzhik (1965, p.317), the k-th moment of X can be obtained :

$$m_k=E(X^k)=\frac{1}{\sqrt{\pi}}(2\beta)^{k/2}\cdot\Gamma(\frac{k+1}{2}),\quad k=1,2,3,.. \tag{2.2}$$

and hence it's well-known that mean and variance of the half normal random variable with the density (1.1) are $\sqrt{2\beta/\pi}$ and $(1-2/\pi)\beta$,respectively.

Assume $X_1,X_2,...,X_m$ be iid with the density (1.1). Then the MLE of β is given by: $\hat{\beta}=\sum_{i=1}^{m}X_i^2/m$, which its mean and variance are β and $2\beta^2/m$.

Since $\sum_{i=1}^{m}X_i^2$ is a complete sufficient statistic of β with a gamma density , it's a well-known fact that $\hat{\beta}=\sum_{i=1}^{m} X_i^2/m$ is a UMVUE of β, A proposed estimator is given by:

$$\tilde{\beta}=\sum_{i=1}^{m} X_i^2/(m+2) \ ,$$

which mean and variance are $m\beta/(m+2)$ and $2m\beta^2/(m+2)^2$, respectively.

From the preceding means and variances of $\hat{\beta}$ and $\tilde{\beta}$, we can obtain it:
Fact 1. A proposed estimator $\tilde{\beta}$ has less MSE than the MLE $\hat{\beta}$.

Since $\frac{\beta}{4}\sum_{i=1}^{m} X_i^2$ follows a χ^2 -distribution with m degree of freedom(df), we can obtain the following interval: a $(1-\alpha)100\%$ confidence interval of β is given by

$$\left(\chi^2_{\alpha/2}(m) \cdot 4/\sum_{i=1}^{m} X_i^2,\ \chi^2_{1-\alpha/2}(m) \cdot 4/\sum_{i=1}^{m} X_i^2 \right) ,$$

where $\int_0^{c(\gamma)} \chi^2(m)(t)dt = \gamma,\ c(\gamma) \equiv \chi^2_\gamma(m),$
$\chi^2(m)(t) is\, the\ density\ of\ \chi^2 - distribution\, with\, df\ m.$

3. Reliability and UMVUE of the right-tail probability

From the density (1.1) and the formula 3.381(3) in Gradshteyn & Ryzhik (1965, p.317), the right-tail probability of the half-normal random variable X can be obtained by:

$$R(t) = P(X > t) = \frac{1}{\sqrt{\pi}} \cdot \Gamma(\frac{1}{2}, \frac{t}{\sqrt{2\beta}}), \quad t > 0 \ ,$$

where $\Gamma(a,x)$ is the incomplete gamma function.

As the classical method of deriving the UMVUE and the Lehman-Scheffe Theorem in Rohatgi(1976, p.356), assume $X_1, X_2, \ldots, X_m$ be iid with the density (1.1). The following statistics $u(X_1)$ is an unbiased estimator of R(t)=P(X>t):

$$u(X_1) = \begin{cases} 1, & \text{if } X_1 > t \\ 0, & else \end{cases} \quad \text{, if } t>0.$$

Since $S \equiv \sum_{i=1}^{m} X_i^2/2$ is a complete sufficient statistics for β . from Lehmann-Scheffe Theorem in Rohatgi(1976, p.375), the conditional expectation $E(u(X_1) \mid S)$ is a UMVUE of the right-tail probability in the half-normal random variable with the density (1.1). At first we derive the conditional pdf $f(x \mid s)$ of X_1 given S=s as following:

$$f(x \mid s) = \frac{\sqrt{2}}{B(\frac{m-1}{2}, \frac{1}{2})} \cdot s^{-\frac{m}{2}+1} \cdot (s - \frac{1}{2}x^2)^{(m-3)/2}, \text{if } s > \frac{x^2}{2} \tag{3.1}$$

From the conditional density (3.1) and the formula 6.6.2 in Abramowitz & Stegun(1972, p.263), the conditional expectation $E(u(X_1) \mid S)$ can be obtained by:

$$E(u(X_1) \mid S) = I_{1-\frac{t^2}{2S}}(\frac{1}{2}, \frac{m-1}{2}) \text{ , if } S > \frac{t^2}{2} \quad S \equiv \frac{1}{2}\sum_{i=1}^{m} X_i^2,$$

where $I_x(a,b)$ is the incomplete beta function.

Therefore, we can obtain the following:

Fact 2. $I_{1-\frac{t^2}{2S}}(\frac{1}{2}, \frac{m-1}{2})$ is a UMVUE of the right tail probability in the half-normal variable, if $S > t^2/2$.

Assume X and Y be independent half-normal random variables each having density (1.1) with parameters $\beta_1 \text{and} \beta_2$,respectively. Then from the formulas 3.381(1) and 6.455(1) in Gradshteyn & Ryzhik (1965, p.317 & p.663) and the formula 15.2.3 in Abramowitz & Stegun(1972. p.557), we can obtain the reliability P(Y<X):

Fact 3. Assume X and Y be independent half-normal random variable each having density (1.1) with parameters $\beta_1 \text{and} \beta_2$,respectively. Then the reliability $P(Y < X) = \frac{2}{\pi} \cdot \frac{\rho^2}{1+\rho} \cdot F(1,2\,;\,\frac{3}{2}\,;\,\frac{1}{1+\rho})$ is a monotone decreasing function of $\rho \equiv \beta_1/\beta_2, \quad \text{if } \rho > 1$,
where F(a,b;c;x) is the hypergeometric function.

If $\rho > 1$, since P(Y<X)) is a monotone decreasing function of ρ , inference on the reliability is equivalent to inference on ρ (see McCool(1991)).

Remark 1. When $0 < \rho < 1$, since P(Y>X) =1-P(Y<X) is a monotone increasing function of ρ , we can consider an inference on P(Y>X), instead of inferencing on P(Y<X).

We only consider inference on $\rho \equiv \beta_1/\beta_2$ when the $\beta_i's$, $i=1,2$ are parameters in the density (1.1), instead of estimating R=P(X<Y),

Assume $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be two independent samples from the density (1.1) with different β_1 and β_2, respectively.
From the MLE $\hat{\beta}_i's$ and a proposed estimator $\tilde{\beta}_i's$ of $\beta_i's$, the following estimators are given by:

the MLE $\hat{\rho}$ of ρ : $\hat{\rho} = n\sum_{i=1}^{m} X_i^2 / m\sum_{i=1}^{n} Y_i^2$.

A proposed estimator $\tilde{\rho}$ of ρ : $\tilde{\rho} = (n+2)\sum_{i=1}^{m} X_i^2 / ((m+2)\sum_{i=1}^{n} Y_i^2)$.
The expectations of $\hat{\rho}$ and $\tilde{\rho}$ can be obtained:

$$E(\hat{\rho}) = \frac{n}{n-2}\rho \text{ and } E(\tilde{\rho}) = \frac{m(n+2)}{(m+2)(n-2)}\rho \qquad . \tag{3.2}$$

From the expectation (3.2) of $\hat{\rho}$, we define an unbiased estimator $\bar{\rho}$ of ρ :
An unbiased estimator $\bar{\rho}$ of ρ : $\bar{\rho} = (n-2)\sum_{i=1}^{m} X_i^2 / (m\sum_{i=1}^{n} Y_i^2)$.

From three estimators $\hat{\rho}$, $\tilde{\rho}$, and $\bar{\rho}$ and (3.2), we obtain their variances;

Fact 4.(a) $Var(\hat{\rho}) = \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}\rho^2$.

(b) $Var(\tilde{\rho}) = \frac{2m(n+2)^2(m+n-2)}{(m+2)^2(n-2)^2(n-4)}\rho^2$.

(3) $Var(\bar{\rho}) = \frac{m+n-2}{m(n-4)}\rho^2$.

From expectations (3.2) and variances in Fact 4, Table 1 shows numerical values of mean squared errors(MSE's) of $\hat{\rho}$, $\tilde{\rho}$, and $\bar{\rho}$ when m and n are 10, 20, and 30.

<Table 1> Mean squared errors of $\hat{\rho}$, $\tilde{\rho}$, and $\bar{\rho}$ (units: ρ^2)

m	n	$\hat{\rho}$	$\tilde{\rho}$	$\bar{\rho}$
10	10	1.00000	1.00000	0.30000
	20	0.44444	0.36343	0.17500
	30	0.34066	0.26740	0.14615
20	10	0.79167	1.00000	0.23333
	20	0.30556	0.30556	0.11875
	30	0.21703	0.20080	0.09231
30	10	0.72222	1.00000	0.21111
	20	0.25926	0.28385	0.10000
	30	0.17582	0.17582	0.07436

From Table 1, we observe an unbiased estimator $\bar{\rho}$ performs better than other two estimators in the sense of MSE.

By a pivot quantity we consider a confidence interval of ρ .We can know that $Q \equiv \rho \cdot m \sum_{i=1}^{n} Y_i^2 / (n \sum_{i=1}^{m} X_i^2)$ follows a F-distribution with (n,m) degree of freedom, and hence an $(1-\alpha)100\%$ confidence interval of ρ is given by:

$$\left(\frac{n}{m \cdot F_{\alpha/2}(m.n)} \cdot \frac{\sum_{i=1}^{m} X_i^2}{\sum_{i=1}^{n} Y_i^2} , \frac{n}{m} \cdot F_{\alpha/2}(n,m) \cdot \frac{\sum_{i=1}^{m} X_i^2}{\sum_{i=1}^{n} Y_i^2} \right) .$$

4. Distributions of product, quotient, and ratio

Assume X and Y be independent half-normal random variables each having densities (1.1) with parameters β_1 and β_2 ,respectively. Then, we derive the densities of product V=XY and quotient W=X/Y.

From the product density in Rohatgi(1976, p.141) and the formulas 3.471(9) in Gradshteyn & Ryzhik (1965, p.340), the density of V=XY can be obtained by:

$$f_V(x) = \frac{2}{\pi \sqrt{\beta_1 \beta_2}} \cdot K_0\left(\frac{1}{\sqrt{\beta_1 \beta_2}} x\right), \quad x > 0 \ , \tag{4.1}$$

where $K_0(x)$ is the modified Bessel function of order 0.

Remark 2, From (4.1) and the formula 11.1 in Oberhettinger(1974, p.115), integral of $f_V(x)$ over $(0,\infty)$ is one.

From the quotient density in Rohatgi(1976, p.141) and the formula 3.381(4) in Gradshteyn & Ryzhik (1965, p.317), the density of W=X/Y can be obtained by:

$$f_W(x) = \frac{2}{\pi\sqrt{\beta_1\beta_2}} \cdot \frac{1}{\frac{1}{\beta_2}+\frac{1}{\beta_1}x^2}, \quad x > 0 \ . \tag{4.2}$$

Remark. 3. From (4.2) and the formula 3.241(2) in Gradshteyn & Ryzhik (1965, p.292), integral of $f_W(x)$ over $(0,\infty)$ is one.

From the density (4.1), the formula 15.14 in Oberhettinger & Badii(1973, p.365) and the formula 3.11 in Oberhettinger(1974, p.27), we can obtain moment generating functions $m_V(t)$ of V as follows:

$$m_V(t) = \frac{1}{\sqrt{\beta_1\beta_2}}(t^2 - \frac{1}{\beta_1\beta_2})^{-1/2}, \ \text{if}\, t < \frac{1}{\sqrt{\beta_1\beta_2}} \ .$$

From the density (4.1), and the formula 11.1 in Oberhettinger(1974, p.115), we can obtain k-th moment m_k of V=XY:

$$m_k = E(V^k) = \frac{2^k}{\pi} \cdot \Gamma^2(\frac{k+1}{2}) \cdot (\beta_1\beta_2)^{(k+1)/2}. \quad k = 1,2,3...$$

By transformation of variable W, R=Y/(X+Y)=1/(1+W) and the quotient density (4.2), we can derive the density of ratio R=Y/(X+Y):

$$f_R(x) = \frac{2}{\pi}\sqrt{\frac{\beta_2}{\beta_1}} \cdot \frac{1}{x^2 + \frac{\beta_2}{\beta_1}(1-x)^2}, \quad \text{if}\, 0 < x < 1 \ . \tag{4.3}$$

From the density (4.3) of ratio R=Y/(X+Y), and the power series of binomials (1.110) & the formula 3.194(1) in Gradshteyn & Ryzhik (1965, p.21 & p.284), we can obtain the following k-th moment of ratio R=Y/(X+Y):

$$E(R^k) = \frac{2}{\pi}\sqrt{\frac{\beta_2}{\beta_1}} \cdot \sum_{i=0}^{\infty}\frac{(-k)_i}{i!}[\frac{1}{i} \cdot F(1, \frac{i+1}{2}; \frac{i+3}{2}; -\frac{\beta_2}{\beta_1}) + \frac{1}{k+i+1} \cdot \frac{\beta_1}{\beta_2} \cdot F(1, \frac{k+i+1}{2}; \frac{k+i+3}{2}; -\frac{\beta_1}{\beta_2})], \quad (4.4)$$

where F(a,b;c;x) is the hypergeometric function, $(a)_0 \equiv 1$, and $(a)_i = a \cdot (a-1) \cdot (a-2) \cdot \cdot \cdot (a-i+1)$.

From the k-th moment (4.4) of ratio R=Y/(X+Y), Table 2 shows numerical values of k-th moment of ratio R=Y/(X+Y) when $\beta_0 \equiv \beta_2/\beta_1 = 1/\rho$ is 1/10, 1/5. 1/2. 2, 5, 10.

<Table 2> Numerical values of k-th moment of the ratio R=Y/(X+Y)

β_0 / k	1(mean)	2	3	4	variance
1/10	0.30162	0.14696	0.09082	0.06417	0.05599
1/5	0.35760	0.18980	0.12230	0.08823	0.06192
1/2	0.43736	0.25836	0.17652	0.13161	0.06707
2	0.56270	0.38369	0.28653	0.22629	0.06706
5	0.64248	0.47468	0.37436	0.30747	0.06190
10	0.69849	0.54382	0.44529	0.37625	0.05593

From Table 2, we observe that every k-th moment of the ratio is an increasing function of $\beta_0 \equiv \beta_2/\beta_1 = 1/\rho$, and variance of the ratio is larger slightly as $\beta_0 \equiv \beta_2/\beta_1 = 1/\rho$ approaches near to 1.

References

1. Abramowitz, M. and Stegun, I. A.(1972), *Handbook of Mathematical Functions*, Dover Publications, Inc., New York.
2. Ali, M. M and Woo, J.(2005a), Inference on Reliability P(Y<X) in the Levy Dis- tribution, *Mathematical and Computer Modelling* 41, 965-971.
3. Ali, M. M., Woo, J. and Nadarajah, S. (2005b), On the ratio X/(X+Y) for the Power Function Distribution, *Pakistan J. Statistics* 21-2, 131-138.
4. Ali, M. M., Woo, J. and Pal, M.(2006), Distribution of the Ratio of Generalized Uniform Variates, *Pakistan J. Statistics* 22-1, 11-19.
5. Gradshteyn, I. S. and Ryzhik, I. M.(1965), *Tables of Integral, Series, and Products*, Academic Press, New York.
6. Johnson, N. L., Kotz, S. and Balakrishnan, N.(1994), *Continous Univariate Distribution-1*, Houghton Mifflin Com., Boston.
7. Kim, J.(2006a), Truncated Point and Reliability in a Right Truncated

Rayleigh Distribution, *J. of the Korean Data & Information Science Society* 17-4, 1343- 1348.

8. Kim, J.(2006b), The UMVUE of $[P(Y > X)]^k$ in a Two Parameter Expon- ential Distribution, *J. of the Korean Data & Information Science Society* 17-2, 493-498.
9. Lee, C. and Won, H.(2006), Inference on Reliability in Exponentiated Uniform Distribution, *J. of the Korean Data & Information Science Society* 17-2, 507- 514.
10. Lee. J. (2006), Reliability and Ratio of two Independent Exponential Distribution, *J. of the Korean Data & Information Science Society* 17-2, 515-520.
11. McCool, J. I.(1991), Inference on P(X<Y) in the Weibull Case, *Commun. Statist. -Simula.*, 20(1), 129-148.
12. Oberhettinger, F. (1974), *Tables of Mellin Transforms*, Springer-Verlag, New York.
13, Oberhettinger, F. and Badii, L. (1973), *Tables of Laplace Transforms*, Springer- Verlag, New York.
14. Rohatgi, V. K.(1976), *An Introduction to Probability Theory and Mathematical Statistics*, John Wiley & Sons, New York.

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