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Sufficient Conditions for Stationarity of Smooth Transition ARMA/GARCH Models[1)]

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Abstract

Nonlinear asymmetric time series models have the growing interest in econometrics and finance. Threshold model is one of the successful asymmetric model. We consider a smooth transition ARMA model which converges a.s. to a threshold ARMA model and show that the smooth transition ARMA model admits a stationary measure, provided a suitable condition on the coefficients of the autoregressive parts of the different regimes is satisfied. Stationarity of a smooth transition GARCH model is also obtained.

Keywords : Smooth Transition ARMA Model, Smooth Transition GARCH Model, Stationarity, Threshold Model

1. Introduction

In the recent literature, nonlinear behaviors of economic and financial variables have attracted much attention. One frequently encountered type of nonlinearity is asymmetry. Threshold ARMA model suggested by Tong(1983) is one of the most popular and fruitful asymmetric models. Basically, threshold ARMA models are piecewise linear ARMA models in which the linear relationship varies over regimes delineated by the threshold values.

$$y_t = \phi_0^{(j)} + \sum_{i=1}^{p} \phi_i^{(j)} y_{t-i} + \sum_{i=1}^{q} \theta_i^{(j)} e_{t-j} + e_t, \quad a_{j-1} < y_{t-d} \le a_j \tag{1.1}$$

where $\phi_i^{(j)}$, $\theta_k^{(j)}$ $(i=0,1,\cdots,p,\ k=1,\cdots,q,\ j=1,\cdots,l)$ are constants, $d \in \{1,\cdots,p\}$,

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$-\infty = a_0 < a_1 < \cdots < a_l = \infty$, and $\{e_t\}$ is a sequence of independent and identically distributed random variables whose distribution is absolutely continuous with $E|e_t| < \infty$.

ARCH/GARCH models introduced by Engle(1982) and Bollerslev(1986) have been applied widely in the econometrics and finance literature to model volatility. Primary feature of ARCH/GARCH model is that conditional variances change over time. In GARCH model, the conditional variance is a linear function of squared past disturbances and past observations. Given the success of Tong's threshold model in nonlinear time series, it is natural to consider threshold structures for the conditional variance specification. A process ϵ_t is a threshold GARCH model of order (p,q) and a delayed parameter d if it is a solution of the equations:

$$\epsilon_t = \sqrt{h_t}\, e_t, \tag{1.2}$$

$$h_t = \alpha_0^{(j)} + \sum_{i=1}^{p} \alpha_i^{(j)} \epsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_i^{(j)} h_{t-i}, \qquad c_{j-1} < \epsilon_{t-d} \le c_j, \tag{1.3}$$

where $\alpha_0^{(j)} > 0, \alpha_i^{(j)} \ge 0, \beta_k^{(j)} \ge 0$ ($i = 1,\cdots,p,\ k = 1,\cdots,q,\ j = i,\cdots,l$) are constants, $d \in \{1,\cdots,p\}$, and $-\infty = c_0 < c_1 < \cdots < c_l = \infty$.

Threshold asymmetric modeling rather than symmetric ARCH/GARCH has frequently provided a better fitting especially in the field of financial time series (Rabemananjara and Zakoïan(1993), Liu et al (1997), Ling(1999)).

When a time series model is proposed, a basic question concerns the conditions under which the model will be stationary. There are various stability results for nonlinear time series models, for example, Tweedie(1988), Tong(1990), Meyn and Tweedie(1993), Bhattacharya and Lee(1995), Cline and Pu(1999) and references therein.

One way to study the stationarity/ergodicity of time series is to use the following Foster-Lyapunov's drift condition under some irreducibility, Feller continuity or T-chain continuity.

Drift Condition : There exists a nonnegative measurable function g with $g(x) < \infty$ for at least one x, such that for some constants $0 < \lambda < 1$, $b < \infty$ and a compact set K in $B(R^k)$,

$$\int P(x,dy) g(y) \le \lambda g(x) + b I_K(x). \tag{1.4}$$

To use the drift condition to prove the stationarity, we need the associated Markov chain to be a Feller chain (that is, $\int P(x,dy) g(y)$ is continuous whenever g is continuous and bounded), which is not true for some nonlinear models nor easy to check in most other cases.

Without assuming Feller continuity, we may establish the stationarity by using the negative Lyapunov exponent, but the condition is very difficult to verify and can only be estimated by Monte Carlo simulation (Brandt(1986), Bougerol and Picard(1992)). Uniform countable additivity condition provides another way to prove stationarity, but it requires rather strong additional assumptions (Liu and Susko(1992), Fonseca and Tweedie(2002)).

For this reason, we consider a smooth transition ARMA/GARCH model which has Feller continuity and converges a.s. to the corresponding threshold ARMA/GARCH model.

The aim of this paper is to study the stationarity of a smooth transition ARMA/GARCH model which is itself an interesting model (see, e.g., Dijk et al (2002)).

In Section 2, we concentrate on smooth transition ARMA process. Smooth transition GARCH process is observed in Section 3.

2. Smooth Transition ARMA(p,q) Process

Define

$$\Phi^{(j)} = (\phi_0^{(j)}, \phi_1^{(j)}, \cdots, \phi_p^{(j)}, \theta_1^{(j)}, \cdots, \theta_q^{(j)}) \quad (j = 1, \cdots, l),$$

and

$$Y_t = (1, y_t, \cdots, y_{t-p+1}, e_t, \cdots, e_{t-q+1})' \quad (t \geq 0).$$

Then y_t in (1.1) can be rewritten as

$$y_t = \sum_{j=1}^{l} \Phi^{(j)} Y_{t-1} I_{\{a_{j-1} < y_{t-d} < a_j\}} + e_t.$$

We consider the following smooth transition function:

$$G(\gamma, x, c) = [1 + \exp\{-\gamma(x-c)\}]^{-1}.$$

For each $\gamma > 0$, we define

$$y_t^{(\gamma)} = \sum_{j=1}^{l} (\Phi^{(j)} - \Phi^{(j-1)}) Y_{t-1}^{(\gamma)} G(\gamma, y_{t-d}^{(\gamma)}, a_{j-1}) + e_t, \tag{2.1}$$

where $\Phi^{(0)} = 0$, $Y_t^{(\gamma)} = (1, y_t^{(\gamma)}, \cdots, y_{t-q+1}^{(\gamma)}, e_t, \cdots, e_{t-q+1})'$ $(t \geq 0)$ and $Y_0^{(\gamma)}$ is prescribable random variable independent on $\{e_t : t \geq 1\}$.

Note that for each $\gamma > 0$, $G(\gamma, y_{t-d}^{(\gamma)}, a_j)$ is continuous and by absolute continuity of the distribution of e_t, $G(\gamma, y_{t-d}^{(\gamma)}, a_j)$ converges to $I_{\{y_{t-d} > a_j\}}$ a.s. as $\gamma \rightarrow \infty$, and hence

$$G(\gamma, y_{t-d}^{(\gamma)}, a_{j-1}) - G(\gamma, y_{t-d}^{(\gamma)}, a_j) \rightarrow I_{\{a_{j-1} < y_{t-d} \leqq a_j\}}, \quad a.s. \tag{2.2}$$

For $x = (1, x_1, \cdots, x_p, z_1, \cdots, z_q)'$, let

$$r(x) = \sum_{j=1}^{l} (\Phi^{(j)} - \Phi^{(j-1)}) x G_{j-1}, \quad (j = 1, \cdots, l-1)$$

where $G_0 = 1$ and $G_j = G(\gamma, x_d, a_j)$.

We first establish the strict stationarity of $y_t^{(\gamma)}$in (2.1) employing Tweedie(1988).

Theorem 1. $y_t^{(\gamma)}$ in (2.1) has a strictly stationary solution if

$$\max_j \sum_{i=1}^{p} |\phi_i^{(j)}| < 1.$$

Proof. According to Theorem 2 in Tweedie(1988), we need to construct a proper test function g satisfying drift condition.

Notice that

$$\begin{aligned} r(x) &= \sum_{j=1}^{l} \Phi^{(j)} - \Phi^{(j-1)}) x G_{j-1} \\ &= \sum_{j=1}^{l-1} \Phi^{(j)} x (G_{j-1} - G_j) + \Phi^{(l)} x G_{l-1}. \end{aligned} \tag{2.3}$$

Since $0 \leq G_j \leq 1$ and $G_{j-1} - G_j \geq 0$ $(j = 1, \cdots, l)$,

$$|r(x)| \leq \max_j \left\{ \sum_{i=1}^{p} |\phi_i^{(j)}| |x_i| \right\} + \max_j \left\{ \sum_{i=1}^{q} |\theta_i^{(j)}| |z_i| \right\} + \max_j \{ |\phi_0^{(j)}| \}. \tag{2.4}$$

The remaining part of the proof follows essentially the same line as that of the threshold ARMA model. For convenience of readers, we sketch the proof.

Define a test function $g:\{1\}\times R^{p+q}\rightarrow R^{+}$ by

$$g(1,x_1,\cdots,x_p,z_1,\cdots .z_q)=\max_{1\le i\le p}\{u_i|x_i|\}+\sum_{i=1}^{q}w_i|z_i|+1,$$

where u_i and w_i are strictly positive constants which are specified below.

It is obvious that

$$\begin{aligned}&E[g(Y_t^{(\gamma)})|\,Y_{t-1}^{(\gamma)}=(1,x_1,\cdots,x_p,z_1,\cdots,z_q)] \qquad (2.5)\\ &=E[g(1,r(x)+e_t,x_1,\cdots .x_{p-1},e_t,z_1,\cdots,z_{q-1})]\\ &\le \max\{u_1|r(x)|,u_2|x_1|,\cdots,u_p|x_{p-1}|\}+\sum_{i=2}^{q}w_i|z_{i-1}|+(u_1+w_1)E|e_t|+1\end{aligned}$$

From assumption $\max_j\sum_{i=1}^{p}|\phi_i^{(j)}|<1,$ we may choose $\rho>0$ and $\delta>0$ such that

$$\max_j\sum_{i=1}^{p}|\phi_i^{(j)}|<\rho<\rho^{\frac{1}{p}}<\delta<1.$$

Choose $u_1<0$ arbitrarily and fix. Take

$$u_k=\rho^{\frac{1}{p}}u_{k-1},\quad k=2,3,\cdots,p.$$

Then we have that

$$\frac{u_{i+1}}{u_i}<\delta \text{ and } \max_j\sum_{i=1}^{p}|\phi_i^{(j)}|\frac{u_1}{u_i}<\delta<1,$$

and hence

$$\max\left\{u_1\max_j\left\{\sum_{i=1}^{p}|\phi_i^{(j)}||x_i|\right\},u_2|x_1|,\cdots,u_p|x_{p-1}|\right\}\le\delta\max_i\{u_i|x_i|\}. \qquad (2.6)$$

Let $\theta_i=\max_j|\theta_i^{(j)}|,\ \ (i=1,\cdots,q)$ and define

$$w_{q-i}=u_1\left(\frac{\theta_{q-i}}{\delta}+\frac{\theta_{q-i+1}}{\delta^2}+\cdots+\frac{\theta_q}{\delta^{i+1}}\right)\quad(i=0,1,\cdots,q-1). \qquad (2.7)$$

Some calculations and (2.7) reveal that

$$u_1 \max_j \left\{ \sum_{i=1}^{q} |\theta_i^{(j)}||z_i| \right\} + \sum_{i=2}^{q} w_i |z_{i-1}| \le \delta \sum_{i=1}^{q} w_i |z_i|. \tag{2.8}$$

Inequalities (2.4)-(2.6) and (2.8) imply that

$$E[g(Y_t^{(\gamma)}) | Y_{t-1}^{(\gamma)} = x] \le \delta g(x) + K, \tag{2.9}$$

where $K = (u_1 + w_1) E|e_t| + u_1 \max_j |\phi_0^{(j)}| + 1 < \infty$.

(2.9) together with the Feller continuity of $Y_t^{(\gamma)}$ implies the existence of a strictly stationary solution $y_t^{(\gamma)}$ of (2.1).

Remark (1) Above theorem tells us that the moving average part does not affect the stationarity of the smooth transition ARMA model. (2) Note that in Theorem 1 it is not assumed that a probability density function of e_t is positive a.s.

3. Smooth Transition GARCH(p,q) Process

We consider the following smooth transition GARCH model given by the equations :

$$\epsilon_t^{(\gamma)} = \sqrt{h_t^{(\gamma)}}\, e_t \tag{3.1}$$

$$h_t^{(\gamma)} = \sum_{j=1}^{l} (\alpha_0^{(j)} + \sum_{i=1}^{p} \alpha_i^{(j)} \epsilon_{t-i}^{(\gamma)2} + \sum_{i=1}^{q} \beta_i^{(j)} h_{t-i}^{(\gamma)}) G(\gamma, \epsilon_{t-d}, c_j), \tag{3.2}$$

where $G(\gamma, x, c_j) = [1 + \exp\{-\gamma(x - c_j)\}]^{-1}$ and $\{e_t\}$ is a sequence of independent and identically distributed random variables with mean zero and unit variance. The distribution of e_t is absolutely continuous.

Theorem 2. If $\sum_{i=1}^{p} \max_j \alpha_i^{(j)} + \sum_{i=1}^{q} \max_j \beta_i^{(j)} < 1$, then the process given by (3.1) and (3.2) has a strictly stationary solution.

Proof. Define

$$H_t^{(\gamma)} = (1, \epsilon_t^{(\gamma)}, \cdots, \epsilon_{t-p+1}^{(\gamma)}, h_t^{(\gamma)}, \cdots, h_{t-q+1}^{(\gamma)}).$$

We first show that the process $H_t^{(\gamma)}$ has a strictly stationary solution. Since for each $\lambda > 0$, $H_t^{(\gamma)}$is a Feller Markov chain by continuity of G, it suffices to show that $H_t^{(\gamma)}$ satisfies the drift condition in (1.4).

For given $x = (1, x_1, \cdots, x_p, z_1, \cdots, z_q) \in R^{p+q+1}$, let

$$s(x) = \sum_{j=1}^{l} (A^{(j)} - A^{(j-1)}) x^* G_{j-1}(x),$$

where $x^* = (1, x_1^2, \cdots, x_p^2, z_1, \cdots, z_q)$, $A^{(j)} = (\alpha_0^{(j)}, \cdots, \alpha_p^{(j)}, \beta_1^{(j)}, \cdots, \beta_q^{(j)})$ and $G_j(x) = G(\gamma, x_d, c_j)$.

Now, we define a test function $v: \{1\} \times R^{p+q} \to R$ by

$$v(1, x_1, \cdots, x_p, z_1, \cdots, z_q) = \sum_{i=1}^{p} a_i x_i^2 + \sum_{i=1}^{q} b_i |z_i| + 1,$$

where $a_i > 0 (i = 1, \cdots, p)$ and $b_k > 0 (k = 1, \cdots, q)$ are to be defined later.

We can easily show that

$$|s(x)| \le \max_j \left\{ \sum_{i=1}^{p} \alpha_i^{(j)} x_i^2 + \sum_{i=1}^{q} \beta_i^{(j)} |z_i| \right\} + \max_j \left\{ \alpha_0^{(j)} \right\},$$

and

$$E[v(H_t^{(\gamma)}) | H_{t-1}^{(\gamma)} = (1, x_1, \cdots, x_p, z_1, \cdots, z_q)] \tag{3.3}$$

$$\le (a_1 + b_1) \max_j \left(\sum_{i=1}^{p} \alpha_i^{(j)} x_i^2 + \sum_{i=1}^{q} \beta_i |z_i| \right) + \sum_{i=1}^{p-1} a_{i+1} x_i^2 + \sum_{i=1}^{q-1} b_{i+1} |z_i| + \max_j \left\{ \alpha_0^{(j)} \right\} + 1.$$

For simplicity of notations, let $\alpha^* = \sum_{i=1}^{p} \max_j \alpha_i^{(j)}$, $\beta^* = \sum_{i=1}^{q} \max_j \beta_i^{(j)}$.

Choose $a_1 > 0$ arbitrary and take $b_1 = a_1 \frac{\beta^*}{\alpha^*} > 0$. Since $\alpha^* + \beta^* < 1$, we may pick up $\xi > 0$ such that $\alpha^* + \beta^* + \xi = 1$.

Now, define $a_{p+1} = b_{q+1} = 0$ and

$$a_k = a_1 \left\{ 1 - \left(\frac{1-\xi}{\alpha^*} \right) \sum_{i=1}^{k-1} \max_j \alpha_i^{(j)} - \frac{(k-1)\xi}{p} \right\} > 0, \ k = 2, \cdots, p$$

$$b_k = b_1 \left\{ 1 - \left(\frac{1-\xi}{\beta^*} \right) \sum_{i=1}^{k-1} \max_j \beta_i^{(j)} - \frac{(k-1)\xi}{q} \right\} > 0, \ k = 2, \cdots, q.$$

We thus have that

$$a_1 \geq a_2 \geq \cdots \geq a_p > 0, \quad b_1 \geq b_2 \geq \cdots \geq b_q > 0,$$

and for each $j = 1, \cdots, l,$,

$$(a_1 + b_1)\alpha_k^{(j)} + a_{k+1} \leq a_k(1 - \frac{\xi}{p}), \ \ k = 1, \cdots, p, \tag{3.4}$$

$$(a_1 + b_1)\beta_k^{(j)} + b_{k+1} \leq b_k(1 - \frac{\xi}{q}), \ \ k = 1, \cdots, q. \tag{3.5}$$

Combining (3.3)-(3.5) yields that

$$E[v(H_t^{(\gamma)}) | H_{t-1}^{(\gamma)} = x] \leq (1 - \frac{\xi}{p})v(x) + K,$$

for some constant $K < \infty$ which implies that the drift condition (1.4) holds and consequently, a strictly stationary solution of (3.1)-(3.2) exists. This completes the proof.

Remark (1) Observe that $H_t = (1, \epsilon_t, \cdots, \epsilon_{t-p+1}, h_t, \cdots, h_{t-q+1})'$ given by (1.2)-(1.3) is a Feller chain if $\alpha_0^{(j)} + \sum_{i=1}^{p} \alpha_i^{(j)} \epsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_t^{(j)} h_{t-i} = \alpha_0^{(j+1)} + \sum_{i=1}^{p} \alpha_i^{(j+1)} \epsilon_{t-i}^2 + \sum_{i=i}^{q} \beta_t^{(j+1)} h_{t-i}$ holds whenever $\epsilon_{t-d} = c_j$, $j = 1, \cdots, l$.
(2) The process $H_t = (\epsilon_t, \cdots, \epsilon_{t-l+1})'$ generated by (1.2) and $h_t = \alpha_0 + \alpha_{11}(\epsilon_{t-1}^+)^2 + \alpha_{21}(\epsilon_{t-1}^-)^2 + \alpha_{12}(\epsilon_{t-2}^+)^2 + \alpha_{22}(\epsilon_{t-2}^-)^2 + \cdots + \alpha_{1p}(\epsilon_{t-p}^+)^2 + \alpha_{2p}(\epsilon_{t-p}^-)^2$ is a Feller chain, where $\epsilon_t^+ = \max(\epsilon_t, 0)$ and $\epsilon_t^- = \max(-\epsilon_t, 0)$.

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