대한수학교육학회지 수학교육학연구 제 17 권 제 2 호
Journal of Educational Research in Mathematics
Vol. 17, No. 2, 147-162. May 2007

Middle School Mathematics Teachers' Understanding of Division by Fractions

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This paper reports an analysis of 19 Chinese and Korean middles school mathematics teachers' understanding of division by fractions. The study analyzes the teachers' responses to the teaching task of generating a real-world situation representing the meaning of division by fractions. The findings of this study suggests that the teachers' conceptual models of division are dominated by the partitive model of division with whole numbers as equal sharing. The dominance of partitive model of division constraints the teachers' ability to generate real-world representations of the meaning of division by fractions, such that they are able to teach only the rule-based algorithm (invert-and-multiply) for handling division by fractions.

I. Introduction

Since the early 1980s, mathematics education researchers have confirmed the importance of analyses of specific mathematical content domains for developing effective instructional strategies (Good, Grouws, & Ebmeier, 1983). Along with the emphasis of analyzing specific mathematical content for effective teaching, mathematics education researchers have shifted their attentions in that effective teaching practices much depend on teachers' understanding of mathematical topics. For example, McGalliard (1983) investigated teachers' conceptual system of geometry and its relationship with their instructional behavior.

Since the realization of the importance of teachers' understanding of mathematical topics in effective teaching, a number of studies on teaching in mathematics education (Ball, 1988, 1990a, 1990b Ball & Bass, 2000; Ma, 1999) have addressed the nature of teachers' knowledge of mathematics. However, despite the realization of the importance of teachers' sufficient subject matter knowledge of mathematics for effective teaching, little research have provided the feature of subject matter knowledge of mathematics for teaching and the specific examples of teachers' sufficient subject matter knowledge of mathematical topics.

The analysis reported in this article is part of a larger study examining the reality of middle school mathematics teachers' subject matter knowledge for teaching (Kim, 2007). This article focuses on 19 Chinese and Korean middle school teachers' understanding of division by fractions, by

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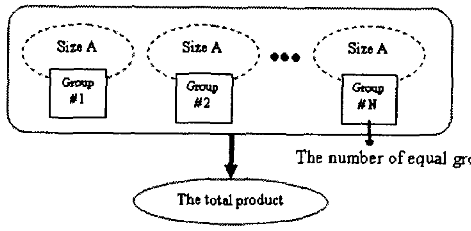
asking them to represent the meaning of division by fractions in a real-world situation. The analysis of the teachers' responses to the teaching task provides appropriate conceptual models for representing the meaning of division by fractions, and the teachers' misconceptions concerning the topic. In particular, this article illustrates the teachers' well-developed conceptual structures of real-world situations involving division by fractions. Prior to reporting those findings, the theoretical conceptual structures and models involving multiplication and division situations are illustrated in the next section that are used for categorizing and analyzing the participants' conceptual models of division by fractions.

II. The Structures of Real-world Situations Involving Multiplication and Division and The Models

There are many structures of real-world situation problems involving multiplication and division such as *equal groups, multiplicative comparison, Cartesian product (i.g., Combinations), part-and-whole, and rectangular area* (Greer, 1992; Van De Walle, 2001). Within these multiplicative structures, three number factors play distinct roles. In multiplication problems, there are two known number factors and one unknown factor. One known factor represents how many sets, groups, or parts of equal size there are, and the other known factor tells the size of each set, group, or part. These two factors have been referred to as the *multiplier* and the *multiplicand* respectively. The third unknown factor which is the calculation of the total of all of the sets, groups, or parts is referred to as the *whole* or *product*. Conversely, when the product is known, but either the number or the size of the sets, groups, or parts is unknown, the resulting situation is division: division by multiplier (the partitive model of division) or division by multiplicand (the measurement model of division). Multiplication and division problems can therefore be seen as special cases of these three number factors.

1. The Partitive and Measurement Models of Division in Equal-groups Structure and Multiplicative Comparison Structure

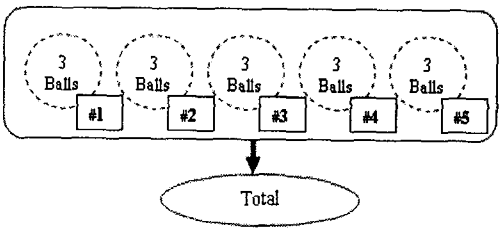
The most common structures of multiplication and division problems in school mathematics are equal groups and multiplicative comparison. In situations of equal-groups structure, the comparison among the equal groups is merely a difference in the quantity of equal groups, so equal-groups multiplication and division problems are termed *repeated-addition* problems and *repeated-subtraction* problems respectively (Van De Walle, 2001). [Figure II-1] represents the structure of equal groups.



[Figure II-1] The structure of equal-groups multiplication and division situations

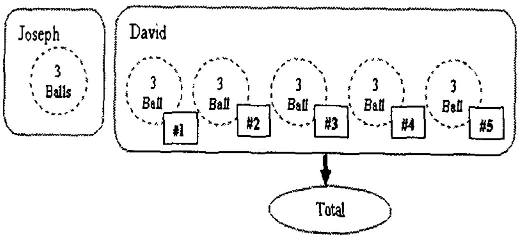
Although multiplicative comparison is similar to equal groups in structure, in multiplicative comparison there are two distinct sets. One set consists of multiple copies of the other. The comparison is based on one set's size being a particular multiple of the other. To illustrate the difference, consider how the multiplication 5×3 would be represented in the equal-group structure and the multiplicative comparison structure as shown in the following examples:

Multiplication in the equal-groups structure: 5 bags have 3 tennis balls each. How many tennis balls do the bags have altogether?



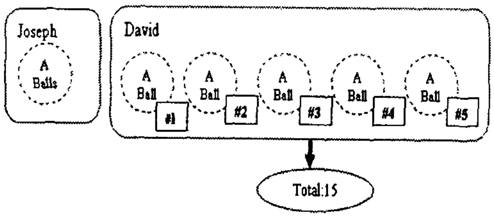
[Figure II-2] Multiplication in the equal-groups structure

Multiplication in the multiplicative comparison structure: Joseph has 3 tennis balls. David has 5 times as many balls as Joseph. How many tennis balls David does have?



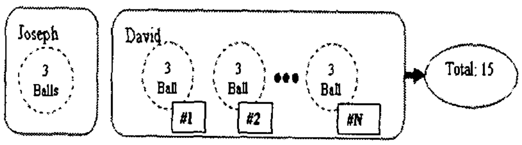
[Figure II-3] Multiplication having the multiplicative comparison structure

If which of the quantities are known in these situations changes, the multiplication situations become division problems. For example, consider the situation where it is known that David has 15 tennis balls, which is 5 times as many as Joseph has. This situation of finding the number of tennis balls Joseph has represents 15÷5.



[Figure II-4] Partitive model of division in the multiplicative comparison structure

This type of problem where the size of each group is unknown is represented by the *partitive* model of division. Problems where the number of groups is unknown are represented by the *measurement* model of division (also called *quotitive* division). For example, consider the situation where Joseph has 3 balls and David has 15. How many times more balls does David have compared to Joseph?



[Figure II-5] Measurement model of division in the multiplicative comparison structure

Although both equal groups and multiplicative comparison structures are common in school mathematics, Ball (1988) and Ma (1999) found that teachers' models of division focus on equal groups structures. Like multiplicative comparison structures, equal groups can involve either partitive division or measurements division. In equal groups, partitive division involves dividing the total by the number of groups to find the size of each group, which corresponds to the practice of *equal sharing*. Measurement division involves dividing the total by the size in each group to find the number of groups, which corresponds to the practice of *equal measuring*. In other words, the situation of finding the size of equal groups when the number of groups is known coincides with the conception of partitive division, and the situation of finding the number of equal groups when the size of the groups is known coincides with the conception of measurement division. The following examples represent changing the original equal-groups multiplication problem into partitive and measurement division problems.

Partitive division (equal sharing): There are 15 tennis balls. If you divide the tennis balls equally into 5 bags, how many balls does each bag have?
Measurement division (equal measuring): How many bags do you need if you are dividing up a group of 15 tennis balls and putting 3 balls in each bag?

2. The Rate Partitive Model and Measurement Model of Division in the Rate Structure

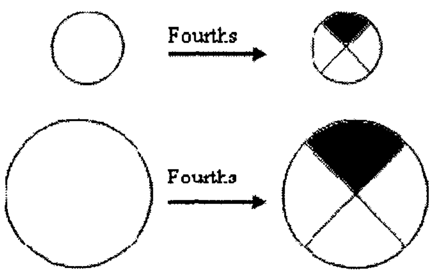
There is an alternative type of equal-groups structure, rate structure, but most teachers are unfamiliar with it. The alternative type is generated by conceptualizing the equal groups in terms of a rate. The following examples represent changing the equal-groups multiplication and division problems presented above into rate structure multiplication and division problems.

Multiplication in rate structure: If there are 3 tennis balls per bag, how many tennis balls are there in 5 bags?
Partitive division in rate structure: There are 15 tennis balls for every 5 bags. How many balls are there per bag?
Measurement division in rate structure: How many bags do you need to divide up a group of 15 tennis balls at the rate of 3 balls per bag?

In rate structure, there are two models for the meaning of division, similar to equal-groups structure: partitive division and measurement division. To distinguish the partitive and measurement models in equal-groups structure from these two models in rate structure, I call the former two models of division the *standard partitive model* and the *standard measurement model*, and the latter models the *rate partitive model* and the *rate measurement model*.

3. The Construct-the-Unit Model of Division in the Part-and-Whole Relationships

In the development of the concept of fraction, it is important for pupils to construct the idea of *fractional parts of the whole* that result when the whole or unit has been partitioned into equal-sized portions or fair shares. Fractional parts are expressed in terms of halves, thirds, fourths, fifths, and so on. Notice that the size of a fractional part depends on the size of the whole. For instance, wholes make different fractional parts in the same model even if they have the same written or oral names for the fractional parts. For example, $\frac{1}{4}$ of a large pizza is greater $\frac{1}{4}$ of a small pizza. Figure II-6 illustrates this difference.



[Figure II-6] $\frac{1}{4}$ versus $\frac{1}{4}$: differing quantities

The numerator and denominator of a fraction are based on the idea of counting fractional parts. Numerator tells how many fractional parts have been counted, and denominator tells what fractional part is being counted. In Figure II-6 above, the pieces in the circles are counted as: one-fourth, two-fourths, three-fourths, and four-fourths. Four-fourths is equal to the whole circle. That is, the fractional names represent the relationship between the parts and wholes. Part-and-whole tasks involve determining any of the 3 following values when the other two are known: the fraction expressing the relations between part and whole ($\frac{a}{b}$), the value of the whole(x),and the value of the part (y).

$$\frac{a}{b} \times x = y$$

($\frac{a}{b}$: fractional number x, y: rational numbers)

Behr and Post (1992, p. 209) coined the term construct-the-unit model to apply to the task of finding a quantity x, given that y is a/b of x. The construct-the-unit model requires the problem solver to construct the unit-whole from a given fractional part. It is the reversal of the problem of finding a fractional part of a unit-whole. Thus, the construct-the-unit model can be used to generate a real-world situation representing the meaning of division by fractions in *part-and-whole* relationships. However, the construct-the-unit model is not well known among elementary teachers because textbooks typically represent multiplication by fractions as finding a fractional part of a unit-whole.

4. The Categorization of The Conceptual Models Representing The Meaning of Division

According to the broad version of the partitive model of division, when the multiplicand and the multiplier can be distinguished, division by the multiplier can be classified under the partitive model of division because division problems of this kind are all fundamentally asking the size of a unit (the multiplicand) when the number of units (the multiplier) and the total product (certain amount of the unit) are known. For example,

> *There are 15 tennis balls (total product/certain amount of the unit) for every 5 bags (the multiplier). How many balls (multiplicand) are there per bag?*

This division problem has the rate partitive model of division, but it can be called in terms of the partitive model of division according to the broad version of the partitive model of division.

The present study pre-selected five types of structures involving division situations for analyzing the teachers' responses: *equal groups, multiplicative comparison, rate, part-and-whole, and rectangular area.* Each division situation in these structures can be classified under the broad version of the partitive model or measurement model – depending on whether the situation involves division by multiplier or multiplicand, with the exception of the rectangular area structure. The rectangular area structure was not discussed above because the conceptual structure was not found from the teachers in this study, but it is described in the <Table II-1> summarizing the models of division under the five structures.

III. Methods and Procedures

1. Participants

The ultimate purpose of author's dissertation research was to reveal the picture of middle school mathematics teachers' subject matter knowledge of mathematics for teaching. The sample of middle school mathematics teachers was selected by convenience sampling choosing teachers arbitrarily. The participants in this study are 19 middle school mathematics teachers in China and South Korea. In China, 9 middle school mathematics teachers were interviewed from three

<Table II-1> *The Framework Categorizing Teachers' Representations for the Meaning of Division*

Structures	Partitive model (Division by multiplier)	Measurement model (Division by multiplicand)
Equal groups	Standard partitive model (equal sharing) *finding the size of equal groups when the product and the number of equal groups are known*	Standard measurement model (equal measuring) *finding the number of equal groups when the product and the size of equal groups are known*
Multiplicative comparison(MC)	MC partitive model *finding the size of a reference set when the product and its multiplier are known*	MC measurement model *finding the number of reference sets when the size of the reference set and the total product are known*
Rate	Rate partitive model *finding the rate of two measurements' quantities*	Rate measurement model *finding one measurement's quantity when another measurement's quantity and the rate of the two quantities*
Part-and-whole relationship	Construct-the-unit model *finding a unit-whole when its fractional part is known*	Find-the-fraction model *finding a fraction when the factional part and whole are known*
Rectangular area*	Rectangular model *finding a number factor that represents a width or length when one of the two factors and the area of the rectangle are known*	

* Ma (1999) called this "factors and products".

urban middle schools in ChangSha, Hunan. The other 10 teachers came from nine Korean middle schools: four located in Seoul, and six in southern Korea. <Table III-1> summarizes demographic information about the participating teachers.

2. Instrumentation

Participating teachers were presented with a division by fractions teaching task to assess their understanding of division by fractions.

Scenario

I. Division by fractions is often confusing. People seem to have different approaches to solving problems involving division with fractions. Do you remember how you were taught to divide fractions? How do you solve a problem like this one?

$$1\frac{3}{4} \div \frac{1}{2}$$

II. Something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story problems to show the application of some particular piece of content. Sometimes it is pretty challenging to do this. What would you say would be a good situation, story, or model for $1\frac{3}{4} \div \frac{1}{2}$ (that is, something real for which $1\frac{3}{4} \div \frac{1}{2}$ is the appropriate mathematical formulation)?

The teaching task presented above is one of the Teacher Education and Learning to Teach Study (TELT) (Kennedy et al., 1993) mathematics interview questions developed by the National Center for Research on Teacher Education (NCRTE) at Michigan State University. The teaching task was developed to examine elementary and secondary teachers' understanding of division by fractions.

<Table III-1> Demographic Summary of Participating Teachers

	Chinese Teachers (N=9)	Korean Teachers (N=10)
Gender	Female (3) Male (6)	Female (7) Male (3)
Educational Background	High school graduate (1) Bachelor's degree in mathematics education (8)	Bachelor's degree in mathematics education (4) Master's degree in mathematics education (5) Bachelor's degree in mathematics and Master's degree in mathematics education (1)
Average teaching experience	12 years	8 years

3. Procedure

Teacher interviews were conducted outside of normal class time. Data collection in China was conducted with a professional Korean-Chinese translator. The interviews consisted of two sessions, which together lasted about thirty minutes. The first session of the interview included a *brief questionnaire* and *general questions*. The brief questionnaire was designed to elicit respondents'demographic and background information. The general questions concerned participants' personal and academic histories and their views on some general issues about teaching and learning mathematics. The purpose of the general questions was to establish rapport between the interviewer and the respondent by demonstrating the researcher's interest in the respondents and removing tension. The scenario problem that comprised the teaching task was conducted in the second session, after participants had completed the questionnaire and general questions. Participants were not allowed to use any resources while completing the scenario problem. Participants were asked follow-up questions that were specific to particular situations that arose in the interviews. The interviews were audiotaped and transcribed.

Ⅳ. Results

While all of the Chinese and Korean teachers easily completed the computation of the operation and obtained the correct answer, almost all failed to come up with a conceptually correct representation for the meaning of division by fractions. <Table Ⅳ-1> shows the distribution of responses by the Chinese and Korean middle school teachers. Among the 19 teachers, only one Chinese teacher and one Korean teacher generated appropriate real-world models. Three Chinese teachers and one Korean teacher created stories with misconceptions, and one Chinese teacher and five Korean teachers tried to come up with a representation of division by fractions, but they failed to complete their representations. Interestingly, four Chinese teachers and three Korean teachers did not even bother getting into the problem, offering the following explanations.

<Table Ⅳ-1> Distribution of Responses by Chinese and Korean Middle School Mathematics Teachers to the Task of Generating a Real-world Story Problem or Model for $1\frac{3}{4} \div \frac{1}{2}$

		Chinese Teachers (n=9)	Korean Teachers (n=10)
Offered Representation	Appropriate	1	1
	Inappropriate	3	1
	Incomplete	1	5
Did not offer Representation	Unable to generate	2	2
	Claim task unfeasible	2	1

1. Appropriate Representation

A. Construct-the-Unit model

One Chinese and one Korean middle school mathematics teacher generated appropriate representations for the meaning of $1\frac{3}{4} \div \frac{1}{2}$. The Chinese teacher's story problem was based on the *construct-the-unit model* of finding a unit-whole when its fractional part is known. His response is presented below.

Teacher : I can teach this problem in the context of going shopping. "When you were supposed to buy cake, you wanted to buy a half (*yi1ban4*:一半) of the cake because you cannot eat the whole cake. You bought half of the cake for $1\frac{3}{4}$ Yuan." After saying it like this, based on the price of the half of the cake··· "How much money do you have to spend to buy the whole cake?"

Interviewer : How did you come up with this idea?

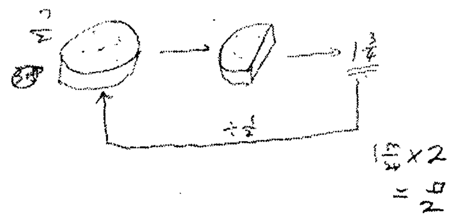
Teacher : By considering students' psychology··· they love to eat.

Interviewer : Would you mind drawing some pictures to show what you said?

Teacher : Yes.[While drawing some pictures, he repeated what he said again].

Interviewer : Many teachers find it hard to represent the part "dividing by $\frac{1}{2}$ ". Could you explain how you represented that part in your story problem in more detail?

Teacher : "The price of a something's half" means $\frac{1}{2}$ of the original price for the one whole thing. So, the cake you took is a *half* and its price is also a *half* (of the original price). Therefore, how much is one whole cake?... That is what I am saying. So, you must multiply by 2.



[Figure IV-1] A construct-the-unit model representing the meaning of $1\frac{3}{4} \div \frac{1}{2}$

When this Chinese teacher was asked to explain how the part "dividing by $\frac{1}{2}$" was represented in his story problem, he explained that dividing one whole thing's price in half is the same as $\frac{1}{2}$ *of* the original price for the whole thing. At the same time, dividing one whole cake and its price in half was represented by arrows pointing right in Figure IV-1 above, and the long arrow pointing left represented the inverse process of calculating the original price for the one whole cake.

B. The rate model of division

One Korean middle school mathematics teacher conceptualized the division $1\frac{3}{4} \div \frac{1}{2}$ as a ratio and generated a rate model representing its meaning. The conception of division as a rate is a new piece of knowledge supporting the understanding of the meaning for division by fractions.

[Wrote down $2 \div 1$, $4 \div 2$, and $4 \div \frac{1}{2}$ on the task paper] The meaning of division is···the quantity per unit···I think it has much to do with the concept of a unit. For example, the unit of speed···we call that Kilometers per hour. Speed is the distance per hour··· It seem to focus on the point: how much can one go per one hour?

The explanation of dividing 8 by 2 is represented in "if (one) eats 8 bread rolls in two days, how much is eaten per day?" The answer is four bread rolls per day. Correspondingly, "if (the person) eats four bread rolls in half of the day, how much is eaten in the whole day?" The answer is 8 bread rolls per day because it has to be 2 *times to become a day···by multiplying* by 2 times.

Suppose that there are two numbers represented as □and☆. With the division □ ÷ ☆, in order to convert the number represented by ☆ to "1", you should multiply the number corresponding to☆ by its reciprocal. At the same time, you would multiply the reciprocal with the number corresponding to□ as well, which results in an equal value to the computational value of the original division. For example, in $5 \div 2$, to convert 2 to 1, $\frac{1}{2}$ has to be multiplied with 2, and $\frac{1}{2}$ also has to be multiplied with 5. In $5 \div \frac{1}{4}$, $\frac{1}{4}$ has to be multiplied by 4 to be one, and 5 times 4 is equal to 20. In $5 \div \frac{3}{4}$, $\frac{3}{4}$ has to be multiplied by $\frac{4}{3}$ to be one because when we suppose that $\frac{3}{4}$ times a certain number is equal to one that number is $\frac{4}{3}$. So, 5 times $\frac{4}{3}$ is equal to $\frac{20}{3}$.

Therefore, the division $1\frac{3}{4} \div \frac{1}{2}$ is interpreted in the same manner. If (one) eats $1\frac{3}{4}$ bread rolls for half of the day, how much will one eat per day? Again, to change $\frac{1}{2}$ into one, multiply it by 2, and then multiply $\frac{7}{4}$ by 2, that results in $\frac{14}{4}$ which is equal to $\frac{7}{2}$. This result is the same as the value computed by the algorithm.

This Korean teacher's meaning of the unit was refined through analysis of her statement about the concept of speed. When she explained the meaning of speed, she defined it as "Kilometers per hour" and "distance per hour," and she called it "the unit of speed." That is, she expressed the concept of rate in terms of a unit, and she conceptualized division problems as real-world situations involving finding the unit (rate). To justify her idea, she actually provided two story problems of finding rates, which represent the meaning of $8 \div 2$ and $4 \div \frac{1}{2}$ in Step 2 of the interview transcript. However, this Korean teacher didn't recognize that the units as results of division problems are identical with the concept of rate.

2. Inappropriate Representation

Three Chinese teachers and one Korean teacher generated story problems which did not correspond to the meaning of the problem. These teachers' errors were categorized into: confounding dividing by $\frac{1}{2}$ with dividing by 2, representing multiplying by 2 instead of dividing by $\frac{1}{2}$, and converting the number factors of $1\frac{3}{4} \div \frac{1}{2}$ to natural numbers.

A. Confusing dividing by $\frac{1}{2}$ with dividing by 2: one Chinese teacher generated a real-world model for the meaning of division by 2 instead of division by $\frac{1}{2}$. The model was represented in sharing $1\frac{3}{4}$ watermelons between two persons equally. His response is presented below.

> When we suppose that there are two watermelons, one watermelon can be considered as "one." After dividing another watermelon into four equal parts, take three parts among the four parts.
> When we suppose that we are going to divide the watermelons evenly between two people, how much can one person eat?
> Divide one whole watermelon in half, and then give a half portion to each person. With $\frac{3}{4}$ watermelon portions, split each quarter part in half, and then share them equally among the two persons.

From analyzing this Chinese teacher's response, the confusion resulted from the linguistic characteristic of *"half"* in everyday life. When we are given the story problem-- if you take $\frac{3}{4}$ of five gallons of oil, how much oil do you have-- we may represent the story problem as the multiplication $5\times\frac{3}{4}$. In the same manner, when we suppose that we want to find a half portion of a certain quantity, we express the situation as multiplying the quantity by $\frac{1}{2}$ instead of dividing it by 2. Converting dividing something in half to multiplying it by $\frac{1}{2}$ is common in everyday life.

B. Representing multiplying by 2 instead of dividing by $\frac{1}{2}$: two Chinese teachers generated story problems for the meaning of multiplying by 2 instead of dividing by $\frac{1}{2}$. The following conversation demonstrates one teacher's response of the two Chinese teachers

Interviewer: Why do you think that pupils have difficulty generating a real-world story problem for the meaning of $1\frac{3}{4}\div\frac{1}{2}$ when they do not have similar difficulty with $1\frac{3}{4}\div 2$?

Teacher : Umm··· it seems to be hard because they have no experience seeing a certain real-life situation corresponding to division by $\frac{1}{2}$, while dividing by 2 is easy to represent in a real-world situation because it means dividing in half.

C. Converting the number factors of $1\frac{3}{4}\div\frac{1}{2}$ to natural numbers: One Korean teacher predetermined that $1\frac{3}{4}\div\frac{1}{2}$ could not be represented in a real world situation or model, and thus he devoted all his efforts to converting the two fractional factors of $1\frac{3}{4}\div\frac{1}{2}$ to natural numbers.

Teacher: It is supposed that there are 28 bread rolls and two groups of people in terms of A and B. In the situation of sharing the bread rolls among two groups equally···Meanwhile, within group A, a grandfather, grandmother, mother, and me···these four people are in group A. If we are supposed to share the seven bread rolls with the people in the two groups equally, how much bread can I have? … The answer is the same… exactly…

Interviewer: How did you come up with the number 28?

Teacher: divide…14… [He was concentrating on reviewing his story problem]… Ah!...because thinking of anything "real" could be represented by natural number s…so…by converting these ($1\frac{3}{4}$ and $\frac{1}{2}$) to natural numbers … [Again, by continuing to concentrate on reviewing of the story problem]…this is $\frac{7}{4}$… Ah!...I multiplied by the bigger number although it was fine to multiply by 4… it could be 7….

3. Unable to generate a representation

A. Incomplete Response

One out of the nine Chinese teachers and five of the Korean teachers tried to find a story problem or model for the meaning of $1\frac{3}{4} \div \frac{1}{2}$ based on their own understanding of division and fractions, but they failed to provide a compete model or situation. Most of these teachers realized the difference between dividing by 2 and dividing by $\frac{1}{2}$, and they were very clear about what made performing the task difficult. For example, one Korean teacher said that "representing the divisor, $\frac{1}{2}$ makes it hard because we are familiar with integer-number divisors. It does not seem to make sense to share the $1\frac{3}{4}$ amount of pizza with one-half people." Another Korean teacher explained that students are confused about the meaning of dividing by $\frac{1}{2}$ and dividing by 2.

B. Refusal to Respond to The Task

Four out of the nine Chinese teachers and three of the Korean teachers refused to respond to the task of representing the meaning of $1\frac{3}{4} \div \frac{1}{2}$. There were two types of refusals. The first type of refusal resulted from the teachers' lack of self-confidence in the exploration of representing mathematical concepts in real-world situations or models. Two Chinese and two Korean teachers presented this type of refusal they believed that they might not be able to achieve a good result because they do not have teaching experience in elementary mathematics, so they have little knowledge about how to teach elementary mathematics. The second type of refusal was related to the teachers' fundamental attitudes in regard to teaching and learning school mathematics topics in real-world contexts. These teachers believed that not all school mathematics topics can be learned or taught by connecting to real-world situations and models. Some school mathematics topics can be taught in relation to realistic situations or models, but some of them do not need to be taught in that way.

V. Discussion and Implications

The present study revealed that the 19 Chinese and Korean middle school mathematics teachers' ability to teach division by fractions is largely limited to teaching the invert-and-multiple algorithm. Most of the teachers' conceptions of division are dominated by the standard partitive model of division with whole numbers as equal sharing as shown in Table V-1 below, and the dominance of this model constrained the teachers' ability to represent the meaning of division by fractions in a real-world situation. The standard partitive model of division with whole numbers as equal sharing breaks down when the divisors are fractions, so they were able to teach only the rule-based algorithm for handling division by fractions.

According to follow-up interviews with the teachers in this study who could not complete the teaching task because of the dominance of the standard partitive model, the dominance of the model was a result of their learning experiences at the elementary school level. The teachers testified that their conceptions of division as equal sharing were mostly achieved during elementary school mathematics learning, and they were taught the meaning of division with whole numbers using only the standard partitive model by their teachers. After graduating elementary school, they did need to think of how the meaning of division can be represented in a real-world situation because secondary mathematics teachers did not ask them to think about that. Correspondently, when they became middle school mathematics teachers, they did not need to teach the topic of division with rational numbers in a real-world context, and thus they did not have an opportunity to think about the other ways of generating the meaning of division in a real-world situation when the primitive model is not feasible.

There is another important finding from this study. Although the teachers' primitive model of division was dominated by the standard partitive model, if the teachers deeply understand the conceptual structure of the partitive model, rather than merely memorizing the model mechanically, they could represent the meaning of division by fractions in a real-world situation.

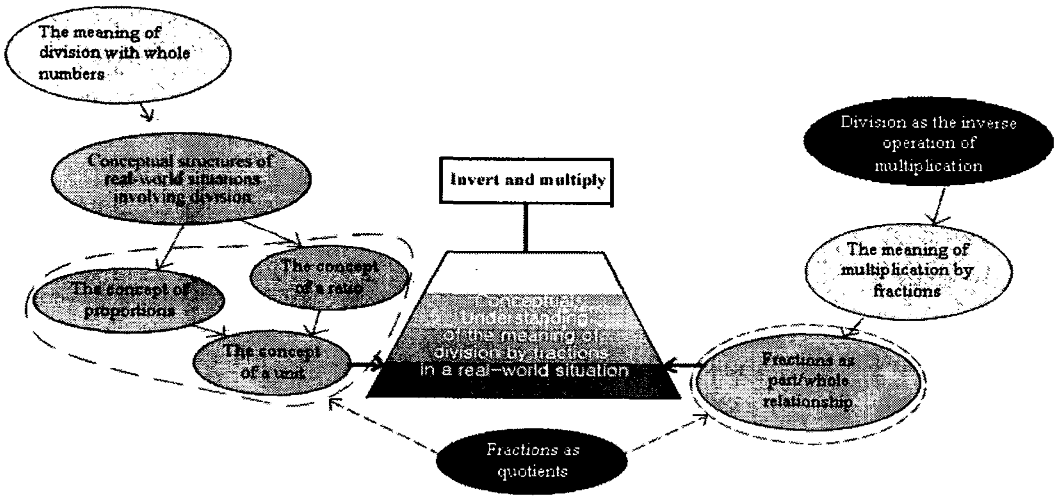
Figure V-1 illustrates examples of teachers' well-developed conceptual structures representing meaning of division by fractions in real-world models. The knowledge pieces on the left side of the conceptual structure were drawn from the Korean teacher's response, and the knowledge pieces on the right side of the structure were drawn from the Chinese teacher's response who provided appropriate representations in this study.

<Table V-1> The Frequency of Types of Division Models Used by Chinese and Korean Teachers

Types of division models	Appropriate	Inappropriate	Unable to generate
Standard partitive model		4	13
Construct-the-unit mode	1		
Rate partitive model	1		

In Figure V-1 above, the Korean teacher's conceptual structure shows that, in initially addressing the meaning of division by fractions, she referred to the meaning of division with whole numbers in real-world situations, whereas the Chinese teacher came up with the basic principle of division as the inverse operation of multiplication. The Korean teacher's approach led her to come up with the key idea of conceptualizing the computational value of a division problem as "a certain quantity per unit," and this key idea contributed to generating the rate partitive model of division for representing the meaning of division by fractions. On the other hand, the Chinese teacher's approach to thinking of division as the inverse operation of multiplication was connected to the topic of representing the meaning of multiplication with fractions. By thinking of the meaning of multiplication by fractions based on the part-and-whole relationships of fractions, the Chinese teacher generated the construct-the-unit model of division by fractions. Therefore, the deep understanding of the conceptual structure of division with whole numbers allowed the teachers to be able to conceptually extend the structure to the other structures of situations involving division, such as the rate structure.

The findings above suggest that mathematics teachers and mathematics educators should encourage students and prospective teachers to conceptually understand the structures of division models rather than merely memorizing those models mechanically. Although achieving this level of conceptual understanding might seem difficult at first, the teachers will later realize that it is an efficient step toward understanding the meaning of division in real-world situations, regardless of divisor type.



[Figure V-1] Two types of conceptual structures for generating a real-world model representing the meaning of division by fractions

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중학교 수학 교사들의 분수나눗셈에 대한 이해

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본 논문은 교수를 위한 중학교 수학교사들의 수학적 지식을 조사한 저자의 학위논문의 일부분으로써, 19명의 한국 및 중국 중학교 수학 교사들의 분수 나눗셈(division by fractions)에 대한 개념적 실생활 모델을 조사, 분석 하였다. 분수 나눗셈에 대한 이론적 배경을 제공함과 동시에, 실제 현장 교사들이 가지고 있는 분수 나눗셈에 대한 개념적 이해를 조사, 분석함으로써 분수 나눗셈을 효과적으로 가르치기 위한 교사 지식의 구체적 예들을 제공하고 있다. 본 연구에서는, 연구에 참가한 교사들 대부분이 분수 나눗셈을 "역수 곱하기(invert and multiply)"와 같은 전통적 알고리즘에 기초하여 이해하고 있었으며, 분수 나눗셈의 의미를 실생활 모델로 나타내는 교수과제를 성공적으로 수행한 교사는 단 두 명에 뿐이었다. 이러한 현상은 그 교사들 대부분이 가지고 있는 범자연수 나눗셈 모델이 분할 모델 (partitive model)로 제한되어 있기 때문이었다. 하지만, 또 다른 흥미로운 연구 결과는, 교사가 분할모델 만을 가지고 있더라도, 그 모델의 개념적 구조(conceptual structure)를 깊이 이해하고 있을 때는, 그 기본적 개념 구조를 변형하여 분수 나눗셈의 실생활 모델을 응용해 내는 사고의 융통성을 보였다. 본 논문에서는 이러한 교사들의 성공적 사례뿐만 아니라, 주어진 교수 과제를 수행하는데 실패한 교사들의 인터뷰 결과들도 분석, 해석하여 제공하였다.

* **Key words** : division by fractions(분수 나눗셈), conceptual models for division(나눗셈의 개념적 모델), partitive model of division(나눗셈의 분할모델), conceptual structure of division by fractions(분수 나눗셈의 개념적 구조)

논문 접수: 2007. 4. 2
심사 완료: 2007. 5. 1